BRIEF NOTE ON HEAT FLOW WITH ARBITRARY HEATING RATES IN A HOLLOW CYLINDER

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In this paper the temperature distribution is determined through a hollow cylinder under an arbitrary time dependent heat flux at the outer surface and zero heat flux at the internal boundary due to internal heat generation within it. To develop the analysis of the temperature field, we introduce the method of integral transform. The results are obtained in a series form in-terms of Bessel's functions.

Key words: *heat conduction problem, homogeneous heat conduction problem, non-homogeneous problem, heat generation*

Introduction

Nowacki [1] has determined the temperature distribution on the upper face, with zero temperature on the lower face and the circular edge thermally insulated. Roy Choudhury [2] has determined the transient temperature along the circumference of a circle over the upper face with lower face at zero temperature and the fix circular edge of the plate thermally insulated. Wankhede [3] determined the arbitrary temperature on the upper face with the lower face at zero temperature and the fix circular edge thermally insulated. Deshmukh *et al.* [4] determined the temperature distribution subjected to arbitrary initial temperature on the lower face with the upper face at zero temperature and fix circular edge thermally insulated. Phythian [5] studied the cylindrical heat flow with arbitrary heating rates at the outer surface and zero heat flux at the internal boundary.

In this paper we consider a non-homogeneous hollow cylinder and determine the temperature distribution under an arbitrary time dependent heat flux at the outer surface and zero heat flux in the internal surface with the help of integral transform technique. The results are obtained in a series form in terms of Bessel's functions. We also discuss the limiting case of homogeneous boundary value problem in a solid cylinder.

The solution should be useful in current aerospace problems, for stations of a missile body not influenced by nose tapering. The missiles skill material is assumed to have physical properties independent of temperature, so that the temperature $T(r, t)$ is a function of radius r and time t only.

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Formulation of the problem

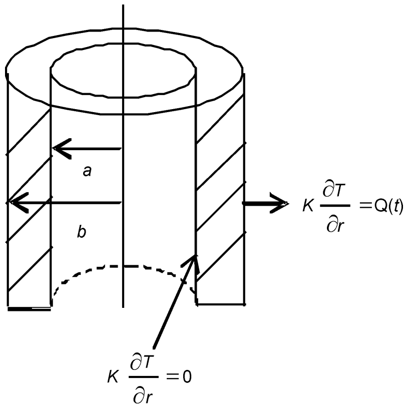

Figure 1. The geometry of the problem

Consider a hollow cylinder of radius $a \le r \le b$, which is initially at temperature $F(r)$. For time $t > 0$ heat is generated within the solid at a rate of $g(r, t)$ [$\text{Btuh}^{-1}\text{ft}^{-3}$]*. The inner circular boundary surface at $r = a$ is insulated, while the arbitrary time dependent heat flux $Q(t)$ is applied at outer circular boundary $r = b$ (fig. 1). The arbitrary time dependent heat flux means radial heat flux prescribed on boundary surface, which is function of time only (for all $t \ge 0$) and converges rapidly for large values of t. Often a rapidly convergent series is required for small values of time.

The boundary value problem of the heat conduction is given as:

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{g(r,t)}{K} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \quad a \le r \le b, \quad t > 0 \tag{1}$$

subject to the boundary conditions

$$K\frac{\partial T}{\partial r} = 0 \quad \text{at} \quad r = a, \quad t > 0 \tag{2}$$

$$K\frac{\partial T}{\partial r} = Q(t) \quad \text{at} \quad r = b, \quad t > 0 \tag{3}$$

and initial condition

$$T(r, t) = F(r) \quad \text{when} \quad t = 0, \; a \le r \le b \tag{4}$$

where, K and α are the thermal conductivity and thermal diffusivity of the material of the hollow cylinder, respectively.

Equations (1) to (4) constitute the mathematical formulation of the heat conduction problem in a hollow cylinder.

The solution

Following the general procedure of Ozisik [6], we develop the finite Hankel transform and its inversion to the above stated problem.

On applying the finite Hankel transform and its inverse transform to the eqs. (1) to (4), one obtains, the expression for the temperature function of a non-homogeneous boundary value problem of heat conduction in a hollow cylinder as:

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* Btu = 1055.0.5585J; ft = 0.3048 m; 1° F = (9/5) °C + 32

$$T(r,t)=\sum_{m=1}^{\infty}K_0(\lambda_m,r)e^{-\alpha\lambda_m^2 t}\left\{\int_a^b r'K_0(\lambda_m,r')F(r')dr'+\right.$$
$$\left.+\frac{\alpha}{K}\int_{t'=0}^{t}e^{\alpha\lambda_m^2 t'}\left[\int_a^b r'K_0(\lambda_m,r')g(r',t')dr'+bK_0(\lambda_m,b)Q(t')\right]dt'\right\} \tag{5}$$

where

$$K_0(\lambda_m,r)=\frac{\pi}{\sqrt{2}}\frac{\lambda_m J_0'(\lambda_m b)Y_0'(\lambda_m b)}{\sqrt{1-\frac{J_0'^2(\lambda_m b)}{J_0'^2(\lambda_m b)}}}\left[\frac{J_0(\lambda_m r)}{J_0'(\lambda_m b)}-\frac{Y_0(\lambda_m r)}{Y_0'(\lambda_m b)}\right] \tag{6}$$

and $\lambda_m's$ are the positive roots of transcendental of equation:

$$\frac{J_0'(\lambda_m a)}{J_0'(\lambda_m b)}-\frac{Y_0'(\lambda_m a)}{Y_0'(\lambda_m b)}=0 \tag{7}$$

Special cases and numerical calculations

Dimension

- Inner radius of a thin hollow cylinder a = 1 ft,
- Outer radius of a thin hollow cylinder b = 2 ft, and
- Central circular path of hollow cylinder r_1 = 1.5 ft.

Material properties

The numerical calculation has been carried out for an aluminum (pure) hollow cylinder with the material properties as:
- thermal conductivity K = 117 Btu/hft°F,
- thermal diffusivity α = 3.33 ft^2/h,

Roots of transcendental equation

Let λ_1 = 3.1965, λ_2 = 6.3123, λ_3 = 9.4445, λ_4 = 12.5812, and λ_5 = 15.7199 are the positive roots of transcendental equation:

$$\left(\frac{J_0'(\lambda_m)}{J_0'(2\lambda_m)}-\frac{Y_0'(\lambda_m)}{Y_0'(2\lambda_m)}\right)=0$$

Case 1: If $F(r)$ = 0
$g(r,t)$ = 0, $Q(t)$ = $e^{-\omega t}$, where δ is the Dirac-delta function, ω > 0, and a = 0, *i. e.* zero initial temperature with no internal heat generation, only arbitrary time dependent heat flux $Q(t)$ is applied at outer circular boundary $r = b$.

Using in eq. (5), then temperature distribution is obtained as:

$$T(r,t)=\frac{\alpha}{K}\sum_{m=1}^{\infty}K_0(\lambda_m,r)e^{-\alpha\lambda_m^2 t}bK_0(\lambda_m,b)\int_{t'=0}^{t}e^{\alpha\lambda_m^2 t'}Q(t')dt' \tag{8}$$

where

$$K_0(\lambda_\mathrm{m}, r) = \frac{\sqrt{2}}{b} \frac{J_0(\lambda_\mathrm{m} r)}{J_0(\lambda_\mathrm{m} b)} \tag{9}$$

and $\lambda_m^{'s}$ are positive roots of the transcendental equation $J_0'(\lambda b) = 0$. (10)

Let λ_1 = 3.8317, λ_2 = 7.0156, λ_3 = 10.1735, λ_4 = 13.3237, λ_5 = 16.470, λ_6 = 19.6159, λ_7 = 22.7601, λ_8 = 25.9037, λ_9 = 29.0468, and λ_{10} = 32.18 are the roots of transcendental equation $J_1(\lambda b) = 0$.

Case 2: If $F(r) = 0$
$g(r,t) = [(g_{\mathrm{cyl,i}})/(2\pi r')]\delta(r' - r_1)\delta(t' - \tau)$, $Q(t) = \mathrm{e}^{-\omega t}$ where δ is the Dirac-delta function, $\omega > 0$, *i. e.* zero initial temperature with internal heat generation $g(r,t)$ and arbitrary time dependent heat flux $Q(t)$ is applied at outer circular boundary $r = b$.

Using eq. (5), then the temperature distribution is obtained as:

$$\begin{aligned} T(r,t) = {} & \frac{1}{2\pi} \frac{\alpha}{K} g_{\mathrm{cyl,i}} \sum_{m=1}^{\infty} \mathrm{e}^{-\alpha\lambda_m^2(t-\tau)} K_0(\lambda_\mathrm{m}, r) K_0(\lambda_\mathrm{m}, r_1) + \\ & + \frac{\alpha}{K} \sum_{m=1}^{\infty} K_0(\lambda_\mathrm{m}, r) \mathrm{e}^{-\alpha\lambda_m^2 t} b K_0(\lambda_\mathrm{m}, b) \int_{t'=0}^{t} \mathrm{e}^{\alpha\lambda_m^2 t'} Q(t') \mathrm{d}t' \end{aligned} \tag{11}$$

Case 3: If $F(r) = r^2$
$g(r,t) = g_\mathrm{i}\delta(r - r_1)\,\delta(t - \tau)$, $Q(t) = \mathrm{e}^{-\omega t}$, where r is the radius measured in feet, δ is the Dirac-delta function, $\omega > 0$.

The arbitrary initial temperature $F(r) = r^2$ is applied with internal heat source $g(r,t)$ is an instantaneous line heat source of strength g_i = 50 Btu/hft, situated at the centre of the circular cylinder along radial direction and releases its heat instantaneously at the time $t \to \tau =$ = 2 hours. Also time dependent heat flux $Q(t)$ is applied at outer circular boundary $r = b$.

$$\begin{aligned} T(r,t) = {} & \sum_{m=1}^{\infty} K_0(\lambda_\mathrm{m}, r) \mathrm{e}^{-\alpha\lambda_m^2 t} \int_a^b r' K_0(\lambda_\mathrm{m}, r') r'^2 \,\mathrm{d}r' + \\ & + \frac{\alpha}{K} g_i \sum_{m=1}^{\infty} \mathrm{e}^{-\alpha\lambda_m^2(t-\tau)} K_0(\lambda_\mathrm{m}, r) K_0(\lambda_\mathrm{m}, r_1) + \\ & + \frac{\alpha}{K} \sum_{m=1}^{\infty} K_0(\lambda_\mathrm{m}, r) \mathrm{e}^{-\alpha\lambda_m^2 t} b K_0(\lambda_\mathrm{m}, b) \int_{t'=0}^{t} \mathrm{e}^{\alpha\lambda_m^2 t'} Q(t') \mathrm{d}t' \end{aligned} \tag{12}$$

Conclusions

In this paper the work of Phythian [5] is extended to heat generation for non-homogeneous boundary value problem in a hollow cylinder. The temperature distribution under an arbitrary time dependent heat flux at $r = b$ is obtained with the help of integral transform technique.

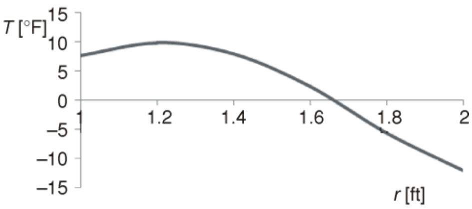

Figure 2. Temperature at t = 2 hours in radial direction

Three cases have been discussed in the given problem:

(a) F(r) = 0, g(r, t) = 0, and a = 0

In this case zero initial temperature with no internal heat generation, only arbitrary time dependent heat flux $Q(t)$ is applied at outer circular boundary $r = b$. So this is the limiting case of homogeneous boundary value problem in a solid cylinder. From fig. 2, one can observe heat flows from outer circular surface to inner circular surface of a hollow cylinder, *i. e.* in the direction of center.

(b) $F(r)=0,\ g(r,t)=\dfrac{g_{cyl,i}}{2\pi r'}\delta(r'-r_1)\delta(t'-\tau)$

In this case zero initial temperature with internal heat generation $g(r,t)$ and arbitrary time dependent heat flux $Q(t)$ is applied at outer circular boundary $r = b$. The temperature distribution is obtained as in eq. (11). This solution represents the case where the cylinder is initially at zero temperature. An instantaneous cylindrical surface heat source of radius $r = r_1$ (*i. e.* $a < r_1 < b$) and of strength $g_{\mathrm{cyl,\,i}}$ [Btuft^{-1}] and same of linear length as the cylinder, releases its heat spontaneously at time $t \rightarrow \tau$.

The instantaneous cylindrical surface heat source is related to the volume heat source by means of Dirac-delta function.

The strength of the instantaneous cylindrical surface source $g_{\mathrm{cyl,\,i}}$ is expressed as:

$$S_{\mathrm{cyl.i}}=\frac{\alpha}{K}g_{\mathrm{cyl,i}}\ [^{\circ}\mathrm{Fft}^2]$$

Substituting this into eq. (11), one obtains:

$$\begin{aligned}T(r,t)={}&\frac{S_{\mathrm{cyl,i}}}{2\pi}\sum_{m=1}^{\infty}\mathrm{e}^{-\alpha\lambda_m^2(t-\tau)}K_0(\lambda_{\mathrm{m}},r)K_0(\lambda_{\mathrm{m}},r_1)+\\&+\frac{\alpha}{K}\sum_{m=1}^{\infty}K_0(\lambda_{\mathrm{m}},r)\mathrm{e}^{-\alpha\lambda_m^2 t}\,bK_0(\lambda_{\mathrm{m}},b)\int_{t'=0}^{t}\mathrm{e}^{\alpha\lambda_m^2 t'}Q(t')\,\mathrm{d}t'\end{aligned}\tag{13}$$

The term

$$\frac{1}{2\pi}\sum_{m=1}^{\infty}\mathrm{e}^{-\alpha\lambda_m^2(t-\tau)}K_0(\lambda_{\mathrm{m}},r)K_0(\lambda_{\mathrm{m}},r_1)$$

represents the temperature at time t due to instantaneous cylindrical surface heat source of strength $S_{\mathrm{cyl,\,i}} = 1$ [°Fft^2] situated at $r = r_1$ and releasing its heat spontaneously at time $t = \tau$ in the region $a \le r \le b$.

(c) $F(r) = r^2$, $g(r,t) = g_{\mathrm{i}}\delta(r - r_1)\delta(t - \tau)$ [Btuh^{-1}ft^{-3}] and $Q(t) = \mathrm{e}^{-\omega t}$,

The heat source $g(r,t)$ is an instantaneous volume heat source of strength $g_{\mathrm{i}} = 50$ Btuh^{-1}ft^{-1}, situated at the center of the hollow cylinder along radial direction and releases its heat instantaneously at the time $t = \tau = 2$ hours.

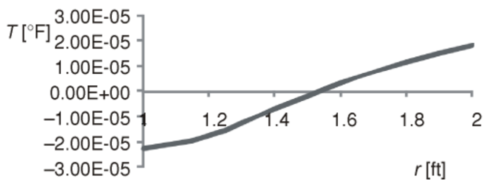

Figure 3. Temperature at t = 2 hours in radial direction

From figs. 3 and 4, it is observed that due to internal heat generation, the temperature function T increases non-uniformly from inner circular boundary to the outer circular boundary.

The solutions of the homogeneous boundary value problem are valid for all values of t (≥ 0), but converge rapidly for large values of t only. Often a rapidly convergent series is required for small values of time at the surface $r = b$.

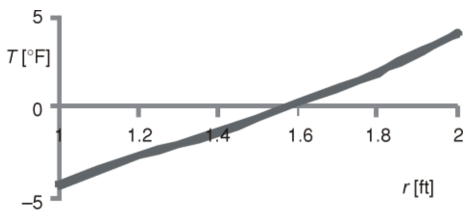

Figure 4. Temperature at *t* = 2 hours in radial direction

Physically, this shows that, for the initial temperature changes only, the reflection of the "temperature wave" at the inner boundary may be neglected and the temperature may be obtained as if the cylinder were solid.

The temperature obtained for the hollow cylinder and solid cylinder is particularly useful for evaluating the maximum temperature gradient and the thermal stresses.

Any particular case of special interest can be derived by assigning suitable values to the parameter and function in eqs. (8), (11), and (12).

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