

FAILURE CRITERIA OF FIBRE REINFORCED COMPOSITES IN HOMOGENEOUS TEMPERATURE FIELD

by

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The present paper examines the failure criteria of layered composites with orthotropic properties in the homogeneous temperature field. The composite has modeled by two mechanically equivalent families of fibres. The paper formulates constitutive equations in terms of intrinsic "preferred" directions, which are defined by the orientation of fibers at any point of the composite. A uniformly heated, thermoelastic solid undergoes distortion as well as volume change because it experiences differential expansions in different directions. This effect is more complicated if, in addition of being anisotropic, the material is inhomogeneous, as in the case with laminated materials.

In order to illustrate the influence of temperature on the failure of this group of materials constitutive equations are derived and adopted for use in failure criteria, without the influence of temperatures, and with the influence of increased temperature.

Key words: failure, composite, thermo-elasticity, fibres, reinforcement.

Introduction

When an isotropic, thermo-elastic material is uniformly heated, it undergoes a uniform expansion without change of shape. By contrast, an anisotropic material has, in general, different thermal expansion coefficients in different directions, and so exhibits distortion as well as volume change when it is subjected to temperature change. This distortion effect is compounded when the material is inhomogeneous as well as anisotropic, because then the preferred direction for expansion vary with position, and this gives rise to further distortion. Laminated materials, which are extensively and increasingly used in advanced materials applications, are extreme cases of inhomogeneous materials. The anisotropic materials types to be considered here will be taken to have orthotropic symmetry

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built up by lamination with laminas reinforced with one family of fibres, which have transversely isotropic symmetry [1]. Depending on stacking sequences of lamination, one obtains different kinds of anisotropy. Perhaps the best known example is that of aligned fiber composite materials, but there are many other examples. A further condition will be taken is such that the degree of anisotropy is large [2]. This is in line with the interests here in high stiffness and high strength fibre composite materials as typified by carbon fibre, polymeric matrix systems.

It is necessary to deduce the proper scale for the corresponding idealization of homogeneity for this class of materials failure problems. There are three obvious choices. The so called micromechanics level takes the individual fibres and the separate matrix phase in between them as the size scale for homogeneity. The next level up is the aligned fibre, lamina level, which then is much larger than the size of the individual filament or fibre. Finally, at yet a still much larger scale, the homogenization could be taken at the laminate level, involving the stacking of various laminas in various directions. It is the intermediate scale, the lamina level that is seen as having the proper balance between small scale details, but large enough scale to include all the possible failure mechanisms which could be operative.

An example of the importance of the scale of the failure mode will be given later. Thus all idealizations to follow are taken at the aligned fibre, lamina scale of homogenization [3, 4]. This is the same scale as that at which the volume averaged elastic properties for fibre composites are normally rationalized.

The main purpose here is to develop the highly anisotropic failure criterion for carbon-polymer systems [5], which is the companion piece to those of the well known cases given in [6] that include thermal effects.

Basic governing equations and their formulation

Here we develop a continuum theory, in which a material particle contains many atoms, and represents their average behaviour. We label each material particle by its coordinates (x_1, x_2, x_3) in the reference configuration. At time t , the material particle (x_1, x_2, x_3) has the displacement $u_1(x_1, x_2, x_3, t)$ in the x_1 -direction, $u_2(x_1, x_2, x_3, t)$ in the x_2 -direction, and $u_3(x_1, x_2, x_3, t)$ in the x_3 -direction. A function of spatial coordinates is known as a field. *The displacement field is a time-dependent vector field.* If we place markers on a body, the motion of the markers visualizes the displacement field and its variation with time. With this definition, one can write the six linearized strain-displacement relations neatly as:

$$\varepsilon_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \equiv \frac{1}{2} (u_{i,j} + u_{j,i}) \quad (1)$$

We adopt the convention that a repeated index implies a summation over 1, 2 and 3. Thus, $\varepsilon_{kk} = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}$. One also may write the stress tensor $\sigma_{ij}(x_1, x_2, x_3, t)$, which, because of symmetry, has six independent components.

When talking about homogeneity, one should think about at least two length scales: a large (macro) length scale, and a small (micro) length scale. A material is said to be

homogeneous if the macro-scale of interest is much larger than the scale of microstructures. A fibre-reinforced material is regarded as homogeneous when used as a component of an airplane, but should be thought of as *heterogeneous* when its fracture mechanism is of interest. Steel is usually thought of as a homogeneous material, but really contains numerous voids, particles and grains.

A material is isotropic when response in one direction is the same as in any other direction. Metals and ceramics in polycrystalline form are isotropic at macro-scale, even though their constituents—grains of single crystals—are anisotropic. Woods, single crystals, uniaxially fibre reinforced composites are *anisotropic* materials.

Isotropic case

According to Hooke's law, for an isotropic, homogeneous solid, only two independent constants are needed to describe its elastic property: Young's modulus E and Poisson's ratio ν . In addition, a thermal expansion coefficient α characterizes strains due to temperature change. When temperature changes by $\Delta T = T - T_{\text{ref}}$, where T current and T_{ref} reference temperature, thermal expansion causes a strain $\alpha\Delta T$ in all three directions. The combination of multi-axial stresses and a temperature change causes uniform strains. The six stress-strain relation may be written as:

$$\varepsilon_{ij} = \frac{1+\nu}{E} \sigma_{ij} - \frac{\nu}{E} \sigma_{kk} \delta_{ij} + \alpha\Delta T \delta_{ij} \quad (2)$$

The Kronecker's delta symbol δ_{ij} stands for 0, when $i \neq j$, and for 1, when $i = j$. The inverse relation is:

$$\sigma_{ij} = \frac{E}{1+\nu} \left\{ \varepsilon_{ij} + \frac{\nu}{1-2\nu} \varepsilon_{kk} \delta_{ij} \right\} - \frac{E\alpha\Delta T}{1-2\nu} \delta_{ij} \quad (3)$$

Imagine a plane inside a material. The plane has the unit normal vector \mathbf{n} , with three components n_1 , n_2 and n_3 . The force per area on the plane is called the **traction**. The traction is a vector, with three components, which can be written collectively in the tensor form as

$$t_{ij} = \sigma_{ij} n_j \quad (4)$$

In an elastic material, the work done during loading is stored as recoverable strain energy in the solid. The work done to deform a specimen depends only on the state of strain at the end of the test. Based on these observations, the strain energy density of a solid can be defined as the work done per unit volume to deform a material from a stress free reference state to a loaded state. To write down an expression for the strain energy density, it is convenient to separate the strain into two parts:

$$\varepsilon_{ij} = \varepsilon_{ij}^e + \varepsilon_{ij}^T \quad (5)$$

where, for an isotropic solid:

$$\varepsilon_{ij}^T = \alpha \Delta T \delta_{ij} \quad (6)$$

represents the strain due to thermal expansion, known as thermal strain, and:

$$\varepsilon_{ij}^e = \frac{1+\nu}{E} \sigma_{ij} - \frac{\nu}{E} \sigma_{kk} \delta_{ij} \quad (7)$$

is the strain due to mechanical loading, known as elastic strain. Work is done on the specimen only during mechanical loading. It is straightforward to show that the strain energy density is:

$$W = \frac{1}{2} \sigma_{ij} \varepsilon_{ij}^e \quad (8)$$

Denote the distributed external force per unit volume by $b_j(x_1, x_2, x_3, t)$. Using the summation convention, one can write the three equations of momentum balance as:

$$\frac{\partial \sigma_{ij}}{\partial x_j} + b_j = \rho \frac{\partial^2 u_i}{\partial t^2}, \quad i, j = 1, 2, 3 \quad (9)$$

where ρ represent mass density of considered material. When the body is in equilibrium, we drop the acceleration terms from the above equations.

Anisotropic case

The simple isotropic model described in the preceding section is unable to describe the response of some materials accurately, even though the material may deform elastically. This is because some materials do have a characteristic orientation. For example, in a block of wood, the grain is oriented in a particular direction in the specimen. The block will be stiffer if it is loaded parallel to the grain than if it is loaded perpendicular to the grain. The same observation applies to fibre reinforced composite materials. Generally, single crystal specimens of a material will also be anisotropic. This is important when modeling stress effects in small structures such as microelectronic circuits. Even polycrystalline metals may be anisotropic, because a preferred texture may form in the specimen during manufacture.

A more general stress – strain – temperature relation is needed to describe anisotropic solids. The most general linear stress – strain – temperature relation has the form:

$$\sigma_{ij} = c_{ijkl}^T (\varepsilon_{kl} - \alpha_{kl} \Delta T) = c_{ijkl} \varepsilon_{kl} - \beta_{ij} \Delta T \quad (10)$$

where c_{ijkl}^T is fourth order tensor at constant temperature, known as the elastic stiffness tensor, $\alpha_{kl} = \alpha_{lk}$ is the thermal expansion coefficient tensor, and $\beta_{ij} = \beta_{ji}$ are thermal modules.

Energy equation may be written as:

$$K_{ij}(\Delta T)_{,ij} - \rho C_e (\dot{T}) + \tau_0 (\ddot{T}) = T_0 \beta_{ij} (\dot{u}_{i,j} + \tau_0 \ddot{u}_{i,j}) \quad (11)$$

where K_{ij} are the thermal conductivities, C_e is the specific heat at constant strain, τ_0 is thermal relaxation time, $T(x_1, x_2, x_3, t)$ is current temperature, and superimposed dot is differentiation with respect to time.

At first sight it appears that the stiffness tensor has 81 components. However, it must satisfy the Green symmetry conditions $c_{ijkl}^T = c_{klij}^T = c_{ijlk}^T = c_{jikl}^T$. This reduces the number of material constants to 21. We can also write the constitutive equations in another form. The state of strain is specified by the six components labeled as $e_1 \equiv \varepsilon_{11}, e_2 \equiv \varepsilon_{22}, e_3 \equiv \varepsilon_{33}, e_4 \equiv \varepsilon_{23}, e_5 \equiv \varepsilon_{13}, e_6 \equiv \varepsilon_{12}$. It may be noted that the first and last indices, “ i ” and “ j ”, in c_{ij} are related to first two and last two indices, “ ij ” and “ kl ”, in c_{ijkl} by the following rule 1→11, 2→22, 3→33, 4→23, 5→13, 6→12.

The six strain components can vary independently. The elastic energy per unit volume is a function of all six strain components, $W(e_1, e_2, e_3, e_4, e_5, e_6)$. Each stress component $\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6$ is the differential coefficient of the energy density function:

$$\sigma_i = \frac{\partial W}{\partial e_i} \quad (12)$$

If the function $W(e_1, e_2, e_3, e_4, e_5, e_6)$ is known, we can determine the six stress-strain relations by the differentiations. Consequently, by introducing the energy density function, we only need to specify one function, rather than six functions, to determine the stress-strain relations.

The above considerations apply to solids with linear or non-linear stress-strain relations. We now examine **linear elastic solids**. For the stress components to be linear in the strain components, the energy density function must be a *quadratic form* of the strain components:

$$w = \frac{1}{2} c_{ij} e_i e_j, \quad i, j = 1, 2, \dots, 6 \quad (13)$$

The matrix c_{ij} is symmetric, with 21 independent elements. Consequently, 21 constants are needed to specify the elasticity of a linear anisotropic elastic solid. Because the elastic energy is positive for any nonzero strain state, the matrix c_{ij} is positive definite.

Recall that each stress component is the differential coefficient of the energy density function, $\sigma_i = \partial w / \partial e_i$. The stress relation becomes:

$$\sigma_i = c_{ij} e_j, \quad i, j = 1, 2, \dots, 6 \quad (14)$$

The physical significance of the constants c_{ij} is now evident. For example, when the solid is under a *uniaxial strain state*, $e_1 \neq 0, e_2 = e_3 = e_4 = e_5 = e_6 = 0$, the six stress components on the solid are $\sigma_1 = c_{11}e_1, \sigma_2 = c_{21}e_1, \dots$. The matrix c_{ij} is known as the stiffness matrix.

Ferroelastic phase transition

This part goes beyond linear elasticity. Suppose we have the following experimental observations. A crystal has a rectangular symmetry at a high temperature. When the temperature drops below a critical value, T_c , the crystal undergoes a phase transition. The crystal at a low temperature acquires a spontaneous strain in shear. Because of the symmetry, the shear strain can go both directions. We model this crystal with a free energy density:

$$W(\gamma, T) = \frac{1}{2} A(T - T_c)\gamma^2 + \frac{1}{4} B\gamma^4 \quad (15)$$

where A and B are positive constants, and γ is the strain. Due to symmetry, the crystal is equally likely to shear in two directions, so that we keep the even powers in the strain γ .

When $T > T_c$, the coefficient of the γ^2 term is positive, so that the crystal behaves like usual elastic solid, with the shear modulus $A(T - T_c)$. The γ^4 term is unnecessary to describe the behavior of the crystal.

When $T < T_c$, the coefficient of the γ^2 term is negative, and the energy is no longer minimal at $\gamma = 0$. Instead, the energy is minimal at two nonzero strains, known as the spontaneous strains, $\pm\gamma_s$. In this case, the γ^4 term will ensure that energy goes up again when the strain is large enough.

The stress-strain relation is:

$$\sigma = \frac{\partial W(\gamma, T)}{\partial \gamma} = A(T - T_c)\gamma + B\gamma^3 \quad (16)$$

Setting $\sigma = 0$, we find the spontaneous strains:

$$\gamma_s = \pm\sqrt{A(T_c - T)/B} \quad (17)$$

Because the material is non-linear, the shear modulus is no longer a constant, and is given by:

$$\mu = \frac{\partial \sigma(\gamma, T)}{\partial \gamma} = A(T - T_c) + 3B\gamma^2 \quad (18)$$

At the spontaneous strain, the shear modulus is given by:

$$\mu = 2A(T_c - T) \quad (19)$$

Constitutive relations of fibre reinforced materials

Consideration in elastic range is used to model materials that exhibit non-linear, but reversible, stress strain behavior even at small strains. Its most common application is in the

so-called “deformation theory of plasticity”, which is a crude approximation of the behavior of metals loaded beyond the elastic limit.

Materials, we are going to model has the properties described in what is follow. The solid has a preferred shape, the specimen deforms reversibly, the strain in the specimen depends only on the stress applied to it doesn't depend on the rate of loading, or the history of loading, the stress is a non-linear function of strain, even when the strains are small.

When we develop constitutive equations for non-linear elastic materials, it is usually best to find an equation for the *strain energy density* of the material as a function of the strain, instead of trying to write down stress-strain laws directly. This has several advantages, such as that we can work with a scalar function and the existence of a strain energy density guarantees that deformations of the material are perfectly reversible.

If the material is isotropic, the strain energy density can only be a function strain measures that do not depend on the direction of loading with respect to the material. One can show that this means that the strain energy can only be a function of *invariants* of the strain tensor that is to say, combinations of strain components that have the same value in any basis. The strain tensor always has three independent invariants, which could be the three principal strains, for example. In practice it is usually more convenient to use the three fundamental scalar invariants. However, if material has anisotropic characteristics as fibre reinforced materials, it is necessary to construct strain energy density by consideration of specific preferred directions.

When we construct strain energy density function it is easy to obtain fourth order elastic stiffness tensor c_{ijkl} , which has different values depending on thermal considerations. This difference arises for all materials between adiabatic elastic constants, which apply to rapid or dynamic loading, and isothermal constants, which apply to very slow or static loading. Using a superscript ε to denote constant entropy for adiabatic stiffness, and superscript T to denote constant temperature for isothermal stiffness the difference between two is:

$$c_{ijkl}^{\varepsilon} - c_{ijkl}^T = \frac{\lambda_{ij}\lambda_{kl}\Delta T}{C_V} \quad (20)$$

where λ_{ij} and λ_{kl} are temperature coefficients of stress at constant strain, and C_V is the heat capacity per unit volume at constant volume. Most of the differences are small as little if any greater than combined deviations due to errors of measurements and variability in the material. In some cases, particularly when comparing static and dynamic values of elastic constants, it may be desirable to take difference into account, but it is not necessary to do so.

Classical thermostatic relations connecting equilibrium states have a firm position in the theory of solids. It seems most appropriate [7] to state them as fundamental constitutive assumptions, because their validity may reach beyond any other special set of assumptions one may use to derive them. Proceeding directly we make the following three constitutive assumptions for homogeneous solids:

- If the strain e_{ij} and internal energy density W are maintained constant in time and space for long enough time, the solid approaches a static equilibrium state;

- The entropy per unit reference volume S in the static equilibrium state is a function of W , e_{ij} , and preferred direction, and can be inverted to obtain W as a function:

$$W = W(e_{ij}, a_k, S) \quad (21)$$

- The static equilibrium temperature T and static equilibrium Piola-Kirchhoff stress σ_{ij} satisfy the classical thermostatic relations:

$$T = \left(\frac{\partial W}{\partial S} \right)_e \quad \text{and} \quad \sigma_{ij} = \left(\frac{\partial W}{\partial e_{ij}} \right)_T \quad (22)$$

and subscripts e and T , again, denote conditions under constant strain and constant temperature, respectively.

One family of fibres

Composite material is considered as continuum made of matrix material reinforced by one family of stiff fibres, which has preferred direction along the fibre described with the unit vectors field \mathbf{a} , with components a_i . That is transversely isotropic material and to determine the form of stiffness tensor we first note that, for given deformation, strain energy W has to be function of e_{ij} , a_k and temperature ΔT in form given in eq. (21).

If the only anisotropic properties of the material are those which arise from the presence of the fibres, then W is an isotropic invariant of e_{ij} , and a_k . Sense of \mathbf{a} is not significant and, therefore, W must be an even function of \mathbf{a} , and \mathbf{a} may be represented by dyadic product $\mathbf{a} \otimes \mathbf{a}$, which is second order tensor with Cartesian components $a_k a_l$. List of matrix products whose traces represents invariants on which W depends is, following [3, 4, 8 and 9], given as:

$$e, \quad e^2, \quad e^3, \quad \mathbf{a} \otimes \mathbf{a}, \quad (\mathbf{a} \otimes \mathbf{a})^2, \quad (\mathbf{a} \otimes \mathbf{a})^3, \quad e \cdot (\mathbf{a} \otimes \mathbf{a}), \\ e \cdot (\mathbf{a} \otimes \mathbf{a})^2, \quad e^2 \cdot (\mathbf{a} \otimes \mathbf{a}), \quad e^2 \cdot (\mathbf{a} \otimes \mathbf{a})^2 \quad (23)$$

Having in minds that \mathbf{a} is the unit vector it is easy to show validity of relations:

$$\mathbf{a} \otimes \mathbf{a} = (\mathbf{a} \otimes \mathbf{a})^2 = (\mathbf{a} \otimes \mathbf{a})^3 = \dots \quad (24) \\ tr(\mathbf{a} \otimes \mathbf{a}) = 1, \quad tre \cdot (\mathbf{a} \otimes \mathbf{a}) = \mathbf{a} \cdot e \cdot \mathbf{a}, \quad tre^2 \cdot (\mathbf{a} \otimes \mathbf{a}) = \mathbf{a} \cdot e^2 \cdot \mathbf{a}$$

which reduce set of invariants to:

$$tre, \quad tre^2, \quad tre^3, \quad \mathbf{a} \cdot e \cdot \mathbf{a}, \quad \mathbf{a} \cdot e^2 \cdot \mathbf{a} \quad (25)$$

The most general quadratic function in \mathbf{e} which can be formed from above list is:

$$W = \frac{1}{2} \lambda (tre)^2 + \mu_T tre^2 + \alpha (\mathbf{a} \cdot \mathbf{e} \cdot \mathbf{a}) tre + 2(\mu_L - \mu_T) \mathbf{a} \cdot \mathbf{e}^2 \cdot \mathbf{a} + \frac{1}{2} \beta (\mathbf{a} \cdot \mathbf{e} \cdot \mathbf{a})^2 \quad (26)$$

where λ , μ_L , μ_T , α , and β are elastic constants at certain level of temperature, which leads to stress strain relation in form:

$$\begin{aligned} \sigma_{ij} = \frac{\partial W}{\partial e_{ij}} = & \lambda e_{kk} \delta_{ij} + 2\mu_T e_{ij} + 2(\mu_L - \mu_T) (a_i e_{jn} a_n + a_l e_{li} a_j) \\ & + \alpha (e_{kk} a_i a_j + a_m e_{ml} a_l \delta_{ij}) + \beta (a_l e_{lm} a_m) a_i a_j \end{aligned} \quad (27)$$

This equation is equivalent to the conventional form of the constitutive equation for a transversely isotropic linearly elastic solid, but here explicit dependence on fibre direction is exposed.

The stiffness tensor can be calculated as:

$$\begin{aligned} c_{ijkl} = \frac{\partial W}{\partial e_{ij} \partial e_{kl}} = & \lambda \delta_{ij} \delta_{kl} + \mu_T (\delta_{ik} \delta_{jl} + \delta_{jk} \delta_{il}) \\ & + (\mu_L - \mu_T) (a_i a_k \delta_{jl} + a_l a_j \delta_{ik} + a_j a_k \delta_{il} + a_i a_l \delta_{jk}) + \\ & + \alpha (a_k a_l \delta_{ij} + a_l a_j \delta_{kl}) + \beta a_i a_j a_k a_l \end{aligned} \quad (28)$$

and it is clear direct dependence on preferred direction.

Two families of fibres

A material reinforced by two families of mechanically equivalent families of fibres, but with no other preferred directions, is locally orthotropic with respect to the plane of the fibres and the two planes which bisect the fibres and are orthogonal to the plane of the fibres. List of matrix products whose traces represents invariants on which W depends is, following [3, 4, 8 and 9], given as:

$$\begin{aligned} tre, \quad tre^2, \quad tre^3, \quad \mathbf{a} \cdot \mathbf{e} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{e} \cdot \mathbf{b}, \quad (\mathbf{a} \cdot \mathbf{e} \cdot \mathbf{a})(\mathbf{b} \cdot \mathbf{e} \cdot \mathbf{b}), \quad \mathbf{a} \cdot \mathbf{e}^2 \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{e}^2 \cdot \mathbf{b}, \\ (\mathbf{a} \cdot \mathbf{e}^2 \cdot \mathbf{a})(\mathbf{b} \cdot \mathbf{e}^2 \cdot \mathbf{b}), \quad \cos 2\phi (\mathbf{a} \cdot \mathbf{e} \cdot \mathbf{b}), \quad \cos^2 2\phi \end{aligned} \quad (29)$$

and, with substantial amount of algebra [3], can be proven identity which allows that one invariant, say $\cos 2\phi (\mathbf{a} \cdot \mathbf{e} \cdot \mathbf{b})$, can be omitted from above list. Thus the most general quadratic form for W is:

$$\begin{aligned}
W = & \frac{1}{2} \lambda (tr \mathbf{e})^2 + \mu tr \mathbf{e}^2 + \gamma_1 [(\mathbf{a} \cdot \mathbf{e} \cdot \mathbf{a})^2 + (\mathbf{b} \cdot \mathbf{e} \cdot \mathbf{b})^2] + \gamma_2 (\mathbf{a} \cdot \mathbf{e} \cdot \mathbf{b})^2 + \\
& \gamma_3 (\mathbf{a} \cdot \mathbf{e} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{e} \cdot \mathbf{b}) tr \mathbf{e} + \gamma_4 \cos 2\phi (\mathbf{a} \cdot \mathbf{e} \cdot \mathbf{b}) tr \mathbf{e} + \\
& + \gamma_5 \cos 2\phi (\mathbf{a} \cdot \mathbf{e} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{e} \cdot \mathbf{b}) (\mathbf{a} \cdot \mathbf{e} \cdot \mathbf{a}) + \\
& + \gamma_6 (\mathbf{a} \cdot \mathbf{e} \cdot \mathbf{a}) (\mathbf{b} \cdot \mathbf{e} \cdot \mathbf{b}) + \gamma_7 (\mathbf{a} \cdot \mathbf{e}^2 \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{e}^2 \cdot \mathbf{b})
\end{aligned} \tag{30}$$

where $\lambda, \mu, \gamma_1, \dots, \gamma_7$ are elastic constants at certain level of temperature, which leads to stress strain relation in form:

$$\begin{aligned}
\sigma_{ij} = & \frac{\partial W}{\partial e_{ij}} = [\lambda e_{rr} + \gamma_3 (a_r a_s e_{rs} + b_r b_s e_{rs}) + \gamma_4 a_r b_s e_{rs} \cos 2\phi] \delta_{ij} + \\
& 2\mu e_{ij} + (\gamma_3 e_{rr} + 2\gamma_1 a_r a_s e_{rs} + \gamma_6 b_r b_s e_{rs} + \gamma_5 a_r b_s e_{rs} \cos 2\phi) a_i a_j + \\
& (\gamma_3 e_{rr} + 2\gamma_1 b_r b_s e_{rs} + \gamma_6 a_r a_s e_{rs} + \gamma_5 a_r b_s e_{rs} \cos 2\phi) b_i b_j + \\
& \frac{1}{2} [\gamma_4 e_{rr} \cos 2\phi + \gamma_5 (a_r a_s e_{rs} + b_r b_s e_{rs}) \cos 2\phi + 2\gamma_2 a_r b_s e_{rs}] (a_i b_j + a_j b_i) \\
& \gamma_7 [a_r (a_j e_{ri} + a_i e_{rj}) + b_r (b_j e_{ri} + b_i e_{rj})]
\end{aligned} \tag{31}$$

Again constitutive relation is constructed with 9 material constants, which is in agreement with conventional form of the constitutive equation for a orthotropic linearly elastic solid, but here explicit dependence on two fibre directions is exposed.

Material constants introduced in this section are derived under assumption that temperature is constant. Thus, they can be regarded as function of temperature, and at various levels of temperature, under quasi static conditions, one may experimentally obtain set of material constants at each level.

Failure criteria consideration

Despite of widespread use of fibre reinforced composite materials, an outstanding problem is lack of experimentally validated theory of failure. For reason of simplicity, the most common approach is to use of limit theories such as maximum stress or maximum strain. These approaches are not appropriate because do not take into account interactive effects that have been observed for polymer based composites. A popular theory is the Tsai–Wu tensor polynomial theory. This strength criterion is a simplified version of the scalar relation involving expansion of the stress tensor. Polynomial strength criteria do provide for interactive stress effects and until now the interactive strength terms could only be determined by combined stress tests. The two interactive strength terms in the Tsai–Wu criterion can be derived from the commonly measured uniaxial strengths.

The Tsai–Wu failure criterion [5] is based on the expansion of the stress tensor, truncated at second order terms to give:

$$F_i \sigma_i + F_{ij} \sigma_i \sigma_j + F_{ijk} \sigma_i \sigma_j \sigma_k + \dots \leq 1, \quad i, j=1, 2, \dots, 6, \quad (32)$$

For this relation F_i , F_{ij} and F_{ijk} are the strength parameters and the contracted notation is used. When the theory is applied to a two-directional composite laminate, which is assumed to have two mechanically equivalent fibres, with the axes along bisectors of the fibres and perpendicular to plane of fibres, the expanded form of (32), with terms higher than quadratic in stress discarded, becomes

$$F_1 \sigma_1 + F_2 (\sigma_2 + \sigma_3) + F_{11} \sigma_1^2 + 2F_{12} \sigma_1 (\sigma_2 + \sigma_3) + F_{22} (\sigma_2^2 + \sigma_3^2) + F_{23} \sigma_2 \sigma_3 + F_{44} \sigma_4^2 + F_{55} (\sigma_5^2 + \sigma_6^2) \leq 1, \quad i, j=1, 2, \dots, 6 \quad (33)$$

where:

$$F_{44} = 2(F_{22} - F_{23}) \quad (34)$$

For the common case of plane stress, (33) is reduced to:

$$F_1 \sigma_1 + F_2 \sigma_2 + F_{11} \sigma_1^2 + 2F_{12} \sigma_1 \sigma_2 + F_{22} \sigma_2^2 + F_{55} \sigma_5^2 \leq 1, \quad i, j=1, 2, \dots, 6 \quad (35)$$

The strength terms F_i and F_{ij} are determined directly from the uniaxial strengths of the lamina using

$$F_{11} = \frac{1}{X_T |X_C|}, \quad F_{22} = \frac{1}{Y_T |Y_C|}, \quad F_1 = \frac{1}{X_T} - \frac{1}{|X_C|}, \quad F_2 = \frac{1}{Y_T} - \frac{1}{|Y_C|}, \quad F_{44} = \frac{1}{S_T^2}, \quad F_{55} = \frac{1}{S_L^2} \quad (36)$$

where, X_T is the longitudinal tensile strength; X_C is the longitudinal compressive strength. Y_T is the transverse tensile strength; Y_C is the transverse compressive strength. S_L is the longitudinal shear strength and S_T is the transverse shear strength.

The remaining parameter F_{12} is the interactive strength terms. It must be determined experimentally using combined stress tests in order to apply the theory to either the full three-dimensional or the reduced plane stress cases. Very few materials have been characterized for this interactive parameter. Since combined stress tests are neither common nor straightforward, there has been a lack of data from which to calculate interactive term. In the absence of experimental results to calculate this interactive parameter, users of the theory have had to resort to making estimates for its value without a rational basis.

In the original description of the theory, Tsai and Wu placed significant emphasis on maintaining closed, ellipsoidal failure envelopes for all stress states. To avoid open-ended failure surfaces, "stability" conditions were proposed as:

$$F_{11} F_{22} - F_{12}^2 \geq 0, \quad F_{22}^2 - F_{23}^2 \geq 0 \quad (37)$$

The failure envelope for this theory is in general an ellipse in $\sigma_1 - \sigma_2$ space. The advantage of this theory is that there is interaction between the stress components and the theory does distinguish between the tensile and compressive strengths. A major disadvantage of this theory is that it is not simple to use.

Tsai-Hill failure theory is used to model damage in brittle laminated fibre – composite. The Tsai-Hill criterion assumes that a plane stress state exists in the solid. In this theory, failure is assumed to occur whenever the distortional yield energy equals or exceeds a certain value related to the strength of the lamina. In this theory, there is no distinction between the tensile and compressive strengths. Therefore, The Tsai-Hill failure theory is written mathematically for the lamina as follows:

$$\frac{\sigma_1^2}{(\sigma_1^F)^2} - \frac{\sigma_1\sigma_2}{(\sigma_1^F)^2} + \frac{\sigma_2^2}{(\sigma_2^F)^2} + \frac{\tau_{12}^2}{(\tau_{12}^F)^2} \leq 1 \quad (38)$$

where $\sigma_1^F, \sigma_2^F, \tau_{12}^F$ are strengths of the lamina in longitudinal and transverse directions, and shear strength, respectively.

There exists many others tensor polynomial criteria, which are extension of criteria used in isotropic materials.

Concluding remarks

Here we are developed constitutive relations for composite materials reinforced with one and two families of fibres. The constitutive relations are well known; the usual method of deriving them is to select appropriate coordinate system, and examine the restrictions on the strain energy functions which results from the requirements of invariance under rotations about coordinate axes. Here we proceed in a rather different, though equivalent way. The main reason for this is that, because fibre direction is dependent on position, it is convenient to have formulation which does not depend on particular choice of coordinate system. Such constitutive relations seem to be more satisfactory for use in failure criteria for anisotropic laminate structures. That make easier to set experiment for testing composite for strength. Although this is reasonably well satisfied for the various classes of isotropic materials, it is not usually available for evaluating failure criteria for anisotropic fibre composite materials. The latter are extremely difficult to test, at least partially because of the extreme anisotropy. The method of polynomial invariants also gives failure criteria for the more general cases of orthotropic laminates. It must be added, however, that definitive and determining experimental data are not yet available.

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