PERISTALTIC FLOW OF A FRACTIONAL SECOND GRADE FLUID THROUGH A CYLINDRICAL TUBE

by

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The investigation is to explore the transportation of a viscoelastic fluid with fractional second grade model by peristalsis through a cylindrical tube under the assumptions of long wavelength and low Reynolds number. Analytical solution of problem is obtained by using Caputo's definition. It is assumed that the cross-section of the tube varies sinusoidally along the length of tube. The effects of fractional parameter, material constant and amplitude on the pressure and friction force across one wavelength are discussed numerically with the help of illustrations. It is found that pressure decreases with increase in fractional parameter whereas increases with increase in magnitude of material constant or time. The pressure for the flow of second grade fluid is more than that for the flow of Newtonian fluid.

Keywords: *Peristalsis; Fractional second grade model; Pressure; Friction force; Caputo's fractional derivative.*

1. Introduction

Peristalsis is a form of fluid transport generated by a progressive area of contraction or expansion along the walls of a distensible tube containing fluid. It occurs in many biological and biomechanical systems, such as urine transport from kidney to bladder through the ureter, movement of chyme in the gastrointestinal tract, the movement of spermatozoa in the ducts afferents of the male reproductive tract and the ovum in the female fallopian tube, the locomotion of some warms, transport of lymph in the lymphatic vessels and vasomotion of small blood vessels such as arterioles, venules and capillaries are the examples of physiology and finger, roller pumps and heart lung machine are few examples of biomechanical system. In the mechanical point of view, the idea of peristaltic transport was firstly investigated by Latham [1]. Since then, other workers [2-5] studied peristaltic flow theoretically and they used perturbation techniques and long wavelength and low Reynolds approximation to obtain the solution of problem. They considered two dimensional and axi-symmetric flows.

Fractional calculus has encountered much success in the description of viscoelastic characteristics. The starting point of the fractional derivative model of non-Newtonian model is usually a classical differential equation which is modified by replacing the time derivative of an integer order by the so-called Riemann–Liouville fractional calculus operators. This generalization allows one to define precisely non-integer order integrals or derivatives. Fractional second grade model is the model of viscoelastic fluid. In general, fractional second grade model is derived from well known second grade model by replacing the ordinary time derivatives to fractional order time derivatives and this plays an important role to study the valuable tool of viscoelastic properties. Some authors [6-17] have investigated unsteady flows of viscoelastic fluids with fractional Maxwell model, fractional generalized Maxwell model fractional, second grade fluid, fractional Oldroyed-B model, fractional Burgers' model and fractional generalized Burgers' model through channel/ annulus/ tube and solutions for velocity field and the associated shear stress are obtained by using Laplace transform, Fourier transform, Weber transform, Hankel transform and discrete Laplace transform. Some important works [18-21] such as; the flow of viscoelastic fluids, the effects of heat transfer on flow, thermal and hydrodynamic characteristics, and hydromagnetic flows, through sine, triangular, arc-shaped channels, and vertical channel have been studied.

Recently, Tripathi et al. [22] have studied the peristaltic flow of fractional Maxwell fluids through a channel under long wavelength and low Reynolds number approximations by using homotopy perturbation method and Adomian decomposition methods. Further, Tripathi et al. [23] have reported the slip effects on peristaltic transport of fractional Burger's fluids through a channel and solution is obtained by homotopy analysis method. In this paper, we study the peristaltic transport of viscoelastic fluid with fractional second grade model through a cylindrical tube under the assumptions of long wavelength and low Reynolds number. Caputo's definition is used to find fractional differentiation and numerical results of problem for different cases are discussed graphically. The effects of fractional parameter, material constant, and time on the pressure rise and friction force across one wavelength are discussed. This model is applied to study of movement of chyme through the small intestine and also applicable in mechanical point of view.

2. Caputo's definition

Caputo's definition [24-27] of the fractional –order derivative is defined as

$$D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{b}^{t} \frac{f^{n}(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau \qquad (n-1 < \mathrm{Re}(\alpha) \le n, \;\; n \in N),$$

where, α is the order of derivative and is allowed to be real or even complex, b is the initial value of function f. For the Caputo's derivative we have

$$D^{\alpha} t^{\beta} = \begin{cases} 0 & (\beta \le \alpha - 1), \\ \dfrac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} t^{\beta-\alpha} & (\beta > \alpha - 1). \end{cases}$$

3. Mathematical formulation

The constitutive equation for viscoelastic fluid with fractional second grade model is given by

$$\tilde{S} = \mu (1 + \tilde{\lambda}_1^{\alpha} \frac{\partial^{\alpha}}{\partial \tilde{t}^{\alpha}}) \dot{\gamma}, \tag{1}$$

where, $\tilde{t}$, $\tilde{S}$, $\dot{\gamma}$ and $\tilde{\lambda}_1$, is the time, shear stress, rate of shear strain and material constants, μ is viscosity, and α is the fractional time derivative parameters such that $0 < \alpha \le 1$. This model reduces to second grade models with $\alpha = 1$, and Classical Navier Stokes model is obtained by substituting $\tilde{\lambda}_1 = 0$.

The governing equations of the motion of viscoelastic fluid with fractional second grade model for axi-symmetric flow are given by

$$\left. \begin{aligned} &\rho \frac{D\tilde{u}}{D\tilde{t}} = -\frac{\partial \tilde{p}}{\partial \tilde{x}} + \mu \left(1 + \tilde{\lambda}_1^{\alpha} \frac{\partial^{\alpha}}{\partial \tilde{t}^{\alpha}} \right) \left\{ \frac{1}{\tilde{r}} \frac{\partial}{\partial \tilde{r}} \left(\tilde{r} \frac{\partial \tilde{u}}{\partial \tilde{r}} \right) + \frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2} \right\}, \\ &\rho \frac{D\tilde{v}}{D\tilde{t}} = -\frac{\partial \tilde{p}}{\partial \tilde{r}} + \mu \left(1 + \tilde{\lambda}_1^{\alpha} \frac{\partial^{\alpha}}{\partial \tilde{t}^{\alpha}} \right) \left\{ \frac{\partial}{\partial \tilde{r}} \left(\frac{1}{\tilde{r}} \frac{\partial}{\partial \tilde{r}} (\tilde{r} \tilde{v}) \right) + \frac{\partial^2 \tilde{v}}{\partial \tilde{x}^2} \right\}, \\ &\frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{1}{r} \frac{\partial (\tilde{r} \tilde{v})}{\partial \tilde{r}} = 0, \end{aligned} \right\} \tag{2}$$

where, $\dfrac{D}{D\tilde{t}} \equiv \dfrac{\partial}{\partial \tilde{t}} + \tilde{u} \dfrac{\partial}{\partial \tilde{x}} + \tilde{v} \dfrac{\partial}{\partial \tilde{r}}$. For carrying out further analysis, we introduce the following non-dimensional parameters:

$$
\left.
\begin{aligned}
&x=\frac{\tilde{x}}{\lambda},\quad r=\frac{\tilde{r}}{a},\quad t=\frac{c\tilde{t}}{\lambda},\quad \lambda_1^{\alpha}=\frac{c\tilde{\lambda}_1^{\alpha}}{\lambda},\ u=\frac{\tilde{u}}{c},\quad v=\frac{\tilde{v}}{c\delta},\\
&\delta=\frac{a}{\lambda},\ \phi=\frac{\tilde{\phi}}{a},\quad p=\frac{\tilde{p}a^2}{\mu c\lambda},\quad Q=\frac{\tilde{Q}}{\pi a^2 c},\quad \mathrm{Re}=\rho c a\delta/\mu.
\end{aligned}
\right\} \tag{3}
$$

where ρ is fluid density, δ is defined as wave number; $\lambda, r, t, u, v, c, \phi, p$ and Q stand for wavelength, radial coordinate, time, axial velocities, radial velocities, wave velocity, amplitude, pressure, and volume flow rate respectively in non-dimensional form, and $\tilde{x}, \tilde{r}, \tilde{t}, \tilde{u}, \tilde{v}, \tilde{\phi}, \tilde{p}$ and $\tilde{Q}$ represent the corresponding physical parameters in the dimensional form.

Introducing the non-dimensional parameters and taking long wavelength and low Reynolds number approximations, Eqs.(2) reduce to

$$
\left.
\begin{aligned}
&\frac{\partial p}{\partial x}=\left(1+\lambda_1^{\alpha}\frac{\partial^{\alpha}}{\partial t^{\alpha}}\right)\left\{\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}\right\},\\
&\frac{\partial p}{\partial r}=0,\\
&\frac{\partial u}{\partial x}+\frac{1}{r}\frac{\partial (r v)}{\partial r}=0.
\end{aligned}
\right\} \tag{4}
$$

Boundary conditions are given by

$$
\frac{\partial u}{\partial r}=0 \quad at \quad r=0,\ u=0 \quad at \quad r=h. \tag{5}
$$

Integrating Eq.(4) with respect to r, and using first condition of Eq.(5), the velocity gradient is obtained as

$$
(1+\lambda_1^{\alpha}\frac{\partial^{\alpha}}{\partial t^{\alpha}})\frac{\partial u}{\partial r}=\frac{r}{2}\frac{\partial p}{\partial x}. \tag{6}
$$

Further integrating Eq.(6) from 0 to r, we get the axial velocity as

$$
(1+\lambda_1^{\alpha}\frac{\partial^{\alpha}}{\partial t^{\alpha}})u=\frac{1}{4}\frac{\partial p}{\partial x}\left(r^2-h^2\right). \tag{7}
$$

The volume flow rate is defined as $Q=\int_0^h 2r\,u\,dr$, which, by virtue of Eq.(7), reduces to

$$(1+\lambda_1^{\alpha}\frac{\partial^{\alpha}}{\partial t^{\alpha}})Q=-\frac{h^4}{8}\frac{\partial p}{\partial x}. \tag{8}$$

The transformations between the wave and the laboratory frames, in the dimensionless form, are given by

$$X = x - 1,\ R = r,\ U = u - 1,\ V = v,\ q = Q - h^2, \tag{9}$$

where, the left side parameters are in the wave frame and the right side parameters are in the laboratory frame.

We further assume that the wall undergoes contraction and relaxation is mathematically formulated as

$$h=1-\phi\cos^2(\pi X). \tag{10}$$

The following are the existing relations between the averaged flow rate, the flow rate in the wave frame and that in the laboratory frame:

$$\overline{Q}=q+1-\phi+\frac{3\phi^2}{8}=Q-h^2+1-\phi+\frac{3\phi^2}{8}. \tag{11}$$

Eq.(8), in view of Eq.(11) gives

$$\frac{\partial p}{\partial X}=-\frac{8}{h^4}(1+\lambda_1^{\alpha}\frac{\partial^{\alpha}}{\partial t^{\alpha}})(\overline{Q}+h^2-1+\phi-3\phi^2/8). \tag{12}$$

Using Caputo's definition in Eq.(12), we get

$$\frac{\partial p}{\partial X}=-\frac{3(\overline{Q}+h^2-1-\phi+3\phi^2/8)}{h^4}\left(1+\lambda_1^{\alpha}\frac{t^{\alpha}}{\Gamma(1-\alpha)}\right). \tag{13}$$

The pressure difference and friction force across one wavelength are given by

$$\Delta p=\int_0^1\frac{\partial p}{\partial X}dX, \tag{14}$$

$$F=\int_0^1\left(-h^2\frac{\partial p}{\partial X}\right)dX. \tag{15}$$

4. Numerical results and discussion

The purpose of this section is to discuss the effects of various emerging parameters such as fractional parameter (α), material constant (λ_1), time (t), and amplitude (ϕ) on pressure difference across one wavelength (Δp) and friction force across the one wavelength (F) with the help of graphical illustrations. Mathematica 5.2 version is used to plot the figures.

Figs.1-4 depict the variation of pressure (Δp) with averaged flow rate ($\overline{Q}$) for various values of α, λ_1, t, and ϕ. It is observed that there is a linear relation between pressure and averaged flow rate, also an increase in the average flow rate reduces the pressure and thus, maximum averaged flow rate is achieved at zero pressure and maximum pressure occurs at zero averaged flow rate.

Fig.1 shows that the pressure rise vs. averaged flow rate for various values of α at $\phi = 0.4$, $t = 0.5$, $\lambda_1 = 1.0$. It is evident that the pressure decreases with increasing in α. It is physically interpreted that the fractional behavior of second grade fluids increases, the pressure for flow diminishes. The variation of Δp with $\overline{Q}$ for various values of λ_1 at $\phi = 0.4$, $t = 0.5$, $\alpha = 1/5$ is presented in Fig.2. It is revealed that the pressure increases with increasing λ_1. This means the viscoelastic behavior (in the sense of λ_1) of fluids increases, the pressure for flow of fluids decreases i.e. the flow for second grade fluid is required more pressure than that for the flow of Newtonian fluids. Figs.4 depicts the variation of Δp with $\overline{Q}$ for various values of ϕ at $\lambda_1 = 1.0$, $\alpha = 1/5$, $t = 0.5$. It is found that the pressure increases with increasing ϕ. Fig.5 shows that the graph between Δp and $\overline{Q}$ for various values t at $\phi = 0.4$, $\lambda_1 = 1.0$, $\alpha = 1/5$ and this figure reveals that, the effect of time on pressure is similar to that of amplitude. Maximum flow rate are unique for various values of α, λ_1, t, but it is different for ϕ.

Figs.5-8 show the variations of friction force (F) with the averaged flow rate ($\overline{Q}$) under the influences of all emerging parameters such as α, λ_1, t and ϕ. From figures, It is observed that the effects of all parameters on friction force are opposite to the effects on pressure with averaged flow rate.

5. Conclusions

Fractional models of viscoelastic fluids play important role in physics of polymers and rheology. One of the fractional models of viscoelastic fluids named as fractional second grade model has been taken to study the peristaltic flow behavior through the cylindrical tube. The Caputo's definition is used for differentiating the fractional derivatives. It is evident that less pressure is required to flow of the fractional second grade fluid ($0 < \alpha < 1$) in compare to the flow of second grade fluid ($\alpha = 1$). It is revealed that the flow of Newtonian fluid ($\lambda_1 = 0$) is taken less effort than that of the second grade fluid ($\lambda_1 > 0$). It is also found that the pressure increases by increasing the amplitude or time. The characteristics of Δp with $\overline{Q}$, and F with $\overline{Q}$ at various parameters, are found to be opposite in nature.

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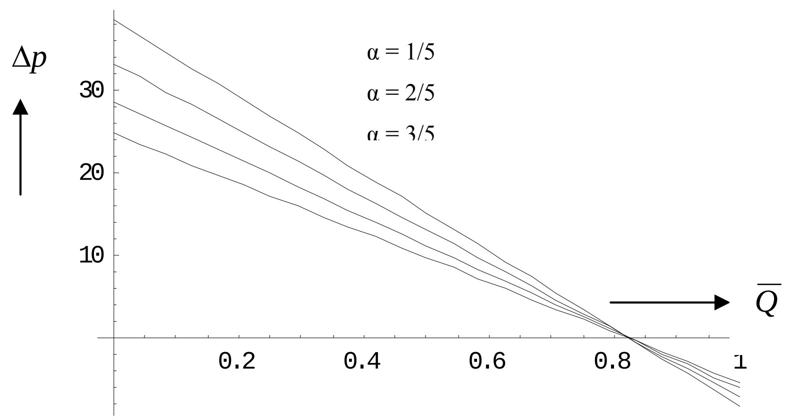

Fig.1. Pressure vs. averaged flow rate for various values of α at $\phi = 0.4$, $t = 0.5$, $\lambda_1 = 1$

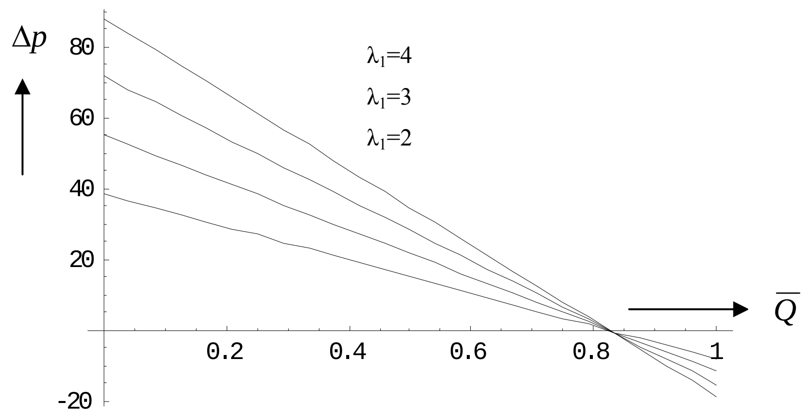

Fig.2. Pressure vs. averaged flow rate for various values of λ_1 at $\phi = 0.4$, $t = 0.5$, $\alpha = 1/5$

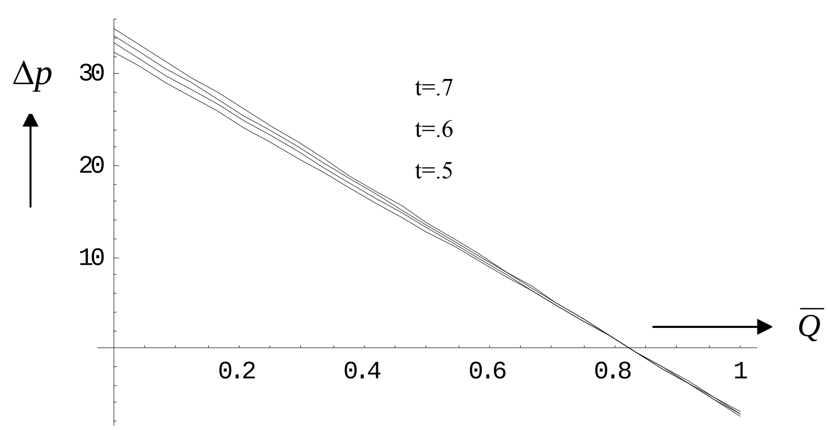

Fig.3. Pressure vs. averaged flow rate for various values of t at $\phi = 0.6$, $\alpha = 2/5$, $\lambda_1 = 1$

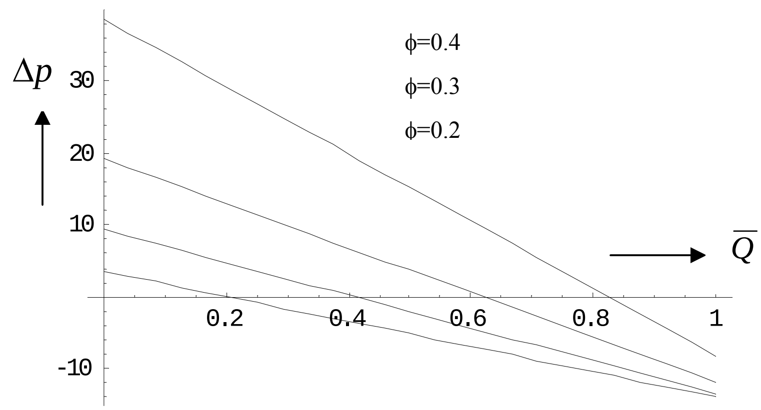

Fig.4. Pressure vs. averaged flow rate for various values of ϕ at $t = 0.5$, $\alpha = 1/5$, $\lambda_1 = 1$

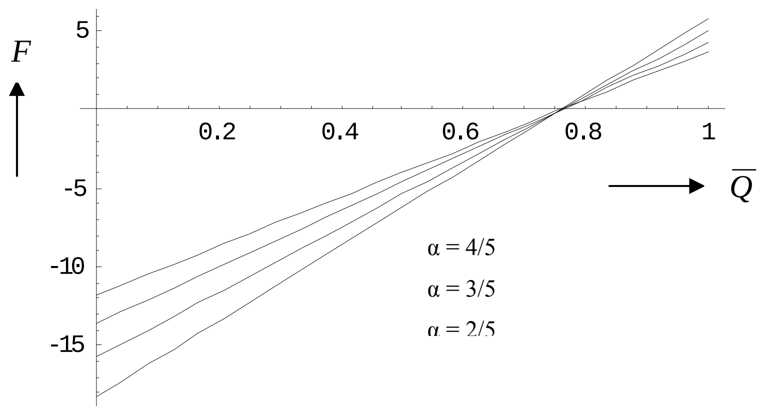

Fig.5. Friction force vs. averaged flow rate for various values of α at $\phi = 0.4$, $t = 0.5$, $\lambda_1 = 1$

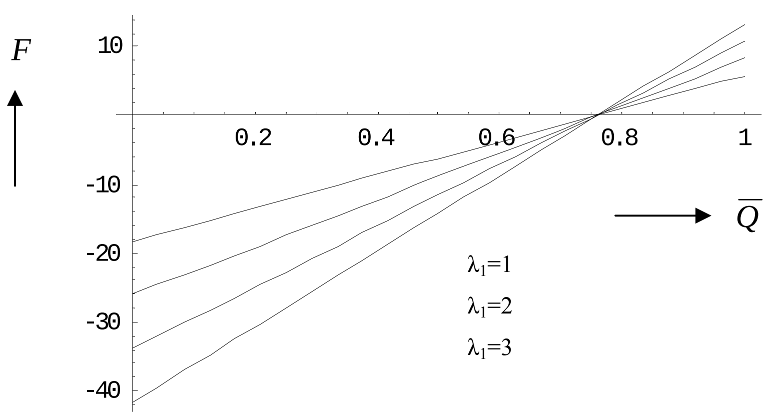

Fig.6. Friction force vs. averaged flow rate for various values of λ_1 at $\phi = 0.4$, $t = 0.5$, $\alpha = 1/5$

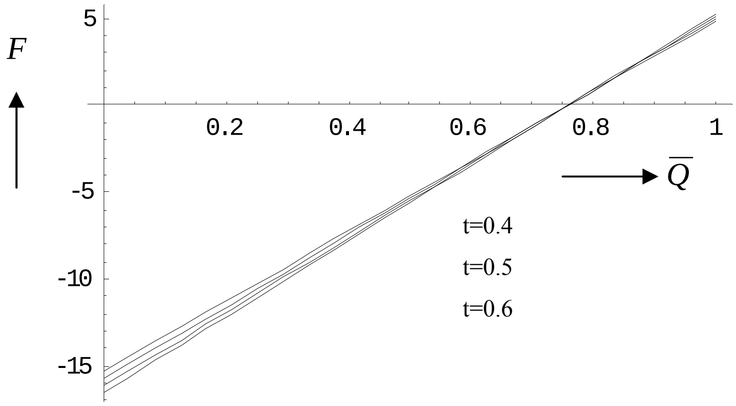

Fig.7. Friction force vs. averaged flow rate for various values of t at $\phi = 0.4$, $\alpha = 2/5$, $\lambda_1 = 1$

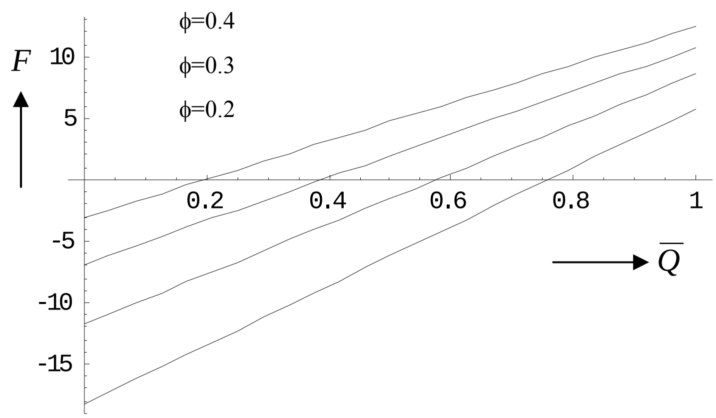

Fig.8. Friction force vs. averaged flow rate for various values of ϕ at $t = 0.5$, $\alpha = 1/5$, $\lambda_1 = 1$

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