NEW ANALYTICAL SOLUTION FOR NATURAL CONVECTION OF DARCIAN FLUID IN POROUS MEDIA PRESCRIBED SURFACE HEAT FLUX

by

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Original scientific paper
UDC: 532.546:536.25:517.957
DOI: 10.2298/TSCI100424001G

A new analytical method called He's variational iteration method is introduced to be applied to solve non-linear equations. In this method, general Lagrange multipliers are introduced to construct correction functional for the problems. It is strongly and simply capable of solving a large class of linear or non-linear differential equations without the tangible restriction of sensitivity to the degree of the non-linear term and also is very user friend because it reduces the size of calculations besides; its iterations are direct and straightforward. In this paper the powerful method called variational iteration method is used to obtain the solution for a non-linear ordinary differential equations that often appear in boundary layers problems arising in heat and mass transfer which these kinds of the equations contain infinity boundary condition. The boundary layer approximations of fluid flow and heat transfer of vertical full cone embedded in porous media give us the similarity solution for full cone subjected to surface heat flux boundary conditions. The obtained variational iteration method solution in comparison with the numerical ones represents a remarkable accuracy.

Key words: *porous media, variational iteration method, ordinary differential equations*

Introduction

Most scientific problems and physical phenomena occur non-linearly. Except in a limited number of these problems, we have difficulty in finding their exact analytical solutions. Therefore, there have been attempts to develop new techniques for obtaining analytical solutions which reasonably approximate the exact solutions [1]. In recent decades, numerical calculation methods were good means of analyzing the non-linear equations; but as the numerical calculation methods improved, semi-exact analytical methods did, too. Most scientists believe that the combination of numerical and semi-exact analytical methods can also end with useful results. The investigation of convective heat transfer in fluid-saturated porous media has many important applications in technology geothermal energy recovery such as oil recovery, food processing, fiber and granular insulation, porous burner and heater, combustion of low-calorific fuels to Diesel engines and design of packed bed reactors. Many problems have been investigated about external natural convection in a porous medium adjacent to

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heated bodies [2, 3]. In these analyses, the boundary layer approximations are applied to the problem of external natural convection and also it is assumed that Darcy law approximations are applicable. The coupled set of governing equations was solved by numerical methods or analytical solution. A great deal of information is available on heat and fluid flow about such cones as reviewed in [2-4]. In this paper the same approximations are applied to the problem of natural convection about an inverted heated cone embedded in a porous medium of infinite extent. No similarity solution exists for the truncated cone, but for the case of full cone similarity solutions exist if the prescribed surface heat flux or surface heat flux is a power function of distance from the vertex of the inverted cone [2, 5]. As we know the perturbation method is one of the well-known methods to solve non-linear problems; it is based on the existence of small/large parameters, the so-called perturbation quantity [6, 7]. Many non-linear problems do not contain such kind of perturbation quantity, and we cannot use perturbation methods, such as the artificial small parameter method [8], the δ-expansion method [9], the Adomian's decomposition method [10], the homotopy perturbation method (HPM) [11-29], and the variational iteration method (VIM) [30-33]. In this study, we have applied VIM to find the analytical solutions of non-linear ordinary differential equations arising from similarity solution of natural convection of Darcian fluid about a vertical full cone embedded in porous media, and have made a comparison with numerical solution, which is solved by MATLAB software [34] command ODE45 with the shooting procedures for the systematically guessing of the missing initial conditions.

Mathematical formulation

Consider an inverted cone with semi angle γ and take axes in the manner indicated in fig. 1. The boundary layer develops over the heated frustum $x = x_0$. In terms of the stream function ψ defined by:

$$u = \frac{1}{r}\frac{\partial \psi}{\partial y}, \quad v = \frac{1}{r}\frac{\partial \psi}{\partial x} \tag{1}$$

The boundary layer equations are:

$$\frac{1}{r}\frac{\partial^2 \psi}{\partial y^2} = \frac{g\beta K}{\nu}\frac{\partial T}{\partial y}$$
$$\frac{1}{r}\left(\frac{\partial \psi}{\partial y}\frac{\partial T}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial T}{\partial y}\right) = \alpha_m \frac{\partial^2 T}{\partial y^2} \tag{2}$$

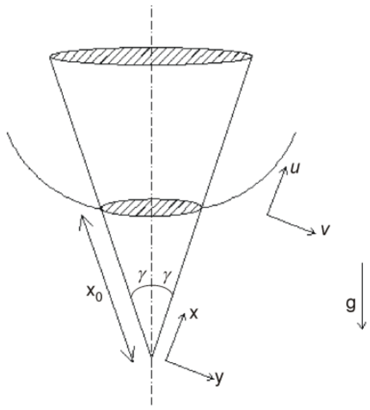

Figure 1. Co-ordinate system for a full cone

For a thin boundary layer we have approximately $r = x\sin\gamma$. We suppose that either a power law of temperature or a power law of heat flux is prescribed on the frustum. Accordingly, the boundary conditions are:

$$\begin{aligned} &u = 0, \quad T = T_\infty && y \to \infty \\ &u = 0 && y = 0, \quad x_0 \le x < \infty \\ &q''_m = -k_m \left.\frac{\partial T}{\partial y}\right|_{y=0} = A(x - x_0)^\lambda && y = 0, \quad x_0 \le x < \infty \end{aligned} \tag{3}$$

For the case of a full cone $x_0 = 0$, fig. 1, a similarity solution exists.

In the case of prescribed wall temperature, we let:

$$\psi = \alpha_{\mathrm{m}} \sqrt[3]{\mathrm{Ra}_{\mathrm{x}}}\, f(\mathrm{g}),$$
$$T - T_{\infty} = \frac{q''_{\mathrm{w}} x}{k_{\mathrm{m}}} \mathrm{Ra}_{\mathrm{x}}^{-1/3} \theta(\mathrm{g}), \tag{4}$$
$$\mathrm{g}(x) = \frac{y}{x} \sqrt[3]{\mathrm{Ra}_{\mathrm{x}}},$$

where the Rayleigh number is based on heat flux:

$$\mathrm{Ra}_{\mathrm{x}} = \frac{\mathrm{g}\beta K \cos\gamma q''_{\mathrm{w}} x^2}{\upsilon \alpha_{\mathrm{m}} k_{\mathrm{m}}} \tag{5}$$

The governing equations become:

$$f' = h; \quad h'' + \frac{\lambda+5}{2} f h' - \frac{2\lambda+1}{3} f h = 0 \tag{6}$$

Subject to:

$$f(0) = 0, \quad h'(0) = -1, \quad h(\infty) = 0 \tag{7}$$

Finally from eqs. (6) and (7) we have:

$$f''' + \frac{\lambda+5}{2} f f'' - \frac{2\lambda+1}{3} (f')^2 = 0,$$
$$f(0) = 0, \quad f''(0) = -1, \quad f'(\infty) = 0 \tag{8}$$

It is of interest to obtain the value of the local Nusselt number which is defined as:

$$\mathrm{Nu}_{\mathrm{x}} = \sqrt[3]{\mathrm{Ra}_{\mathrm{x}}}\, h(0)^{-1} \tag{9}$$

From eqs. (4), (5), and (9) it follows that the local Nusselt number is given by:

$$\mathrm{Nu}_x = \sqrt[3]{\mathrm{Ra}_{\mathrm{x}}}\, [-h(0)] \tag{10}$$

Applications of VIM

Consider governing equation of fluid flow and heat transfer of full cone embedded in porous medium that is expressed by eq. (8) for wall temperature boundary condition. Consider equation that prescribed wall temperature case that is expressed by eq. (8).

To clarify the basic ideas of He's VIM, we consider the differential equation:

$$Lu + N\tilde{u} = \mathrm{g}(\tau) \tag{11}$$

where L is a linear operator, N a non-linear operator, and $\mathrm{g}(\tau)$ an inhomogeneous term.

According to VIM, we can write down a correction functional as follows:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda[Lu_n(\tau) + N\tilde{u}_n(\tau) - \mathrm{g}(\tau)]\mathrm{d}\tau \tag{12}$$

To solve eq. (8), using the VIM, we have the correction functional as:

$$f(\mathrm{g}) = 1 - \mathrm{e}^{-\mathrm{g}} + \int_0^{\mathrm{g}} \lambda \left[f'''(S) + \frac{1}{2}(\lambda+5) f(s) f''(s) - \frac{1}{3}(2\lambda+1)(f'(s) \right]^2 \mathrm{d}s \tag{13}$$

then

$$f(\mathrm{g}) = 1 - \mathrm{e}^{-\mathrm{g}} + \int_0^{\mathrm{g}} (s-\mathrm{g}) \left[f'''(s) + \frac{1}{2}(\lambda+5) f(s) f''(s) - \frac{1}{3}(2\lambda+1)(f'(s) \right]^2 \mathrm{d}s \tag{14}$$

So finally we have:

$$f(\mathrm{g}) = \frac{1}{24} + \frac{1}{2}\mathrm{e}^{-\mathrm{g}} - \frac{13}{24}\lambda + \frac{7}{12}\mathrm{g}\lambda + \frac{1}{2}\mathrm{e}^{-\mathrm{g}} + \frac{1}{24}\lambda\mathrm{e}^{-2\mathrm{g}} - \frac{13}{24}\mathrm{e}^{-2\mathrm{g}} \tag{15}$$

and also:

$$f(\mathrm{g}) = -\frac{1}{2}\mathrm{e}^{-\mathrm{g}} + \frac{5}{12} + \frac{7}{12}\lambda - \frac{1}{2}\lambda\mathrm{e}^{-\mathrm{g}} - \frac{1}{12}\lambda\mathrm{e}^{-2\mathrm{g}} + \frac{13}{12}\mathrm{e}^{-2\mathrm{g}} \tag{16}$$

So now for different amounts of λ and g, we can compare the answers (tab. 1).

Table 1. *f´*(g) for different amounts of *λ*

G	$\lambda = 0$	$\lambda = 1/4$	$\lambda = 1/3$
0	1.0	1.0	1.0
0.1	0.8387675109	0.8476771213	0.8589823113
0.2	0.7136543428	0.7354009891	0.7587676332
0.3	0.6318779619	0.6588232114	0.6700892156
0.4	0.5456362289	0.6204533278	0.6376755403
0.5	0.5043329892	0.5299897623	0.5755492319
0.6	0.4283492333	0.5199657443	0.5432311958
0.7	0.4413998277	0.4978554362	0.5387835422
0.8	0.3894420811	0.4811675493	0.5167345686
0.9	0.3356865190	0.4687995055	0.4978623149
1	0.3143328787	0.4517877901	0.4874545130

Table 2. *f´* values obtained by numerical and VIM solution for *λ* = 1/4

g	$\lambda = 1/4$ (VIM)	$\lambda = 1/4$ (NUM)
0	1.0	0.911295
0.1	0.847677	0.813604
0.2	0.735400	0.721351
0.3	0.658823	0.635531
0.4	0.620453	0.556661
0.5	0.529989	0.484997

Results and discussion

The results for $f'(\mathrm{g})$ have been shown in fig. 2 with selected λ as 0, 1/4, and 1/3, and it is seen that increasing these amounts, makes more inclination from the main answer, and also the trend confirms the boundary condition which is applied by being near zero in infinity. Additionally there is a comparison between the VIM solution and numerical solution in tab. 2 and fig. 3.

Here non-linear differential equation arising from similarity solution of inverted cone embedded in porous medium has been studied using variational iteration method. The comparison with numerical results and convergence study shows that by increasing the number of order, the accuracy of the solution increases. Some of the advantage of VIM are that reduces the volume of calculations with the fewest number of iterations or even in some cases, once, it can converge to correct results. The proposed method is very simple and straightforward. In our work, we use the Maple package to calculate the functions obtained from the variational iteration method.

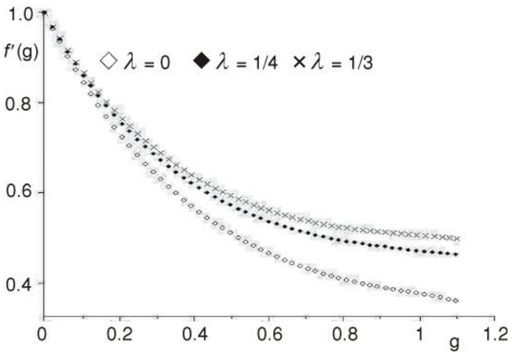

Figure 2. $f'(g)$ **for different amounts of** λ

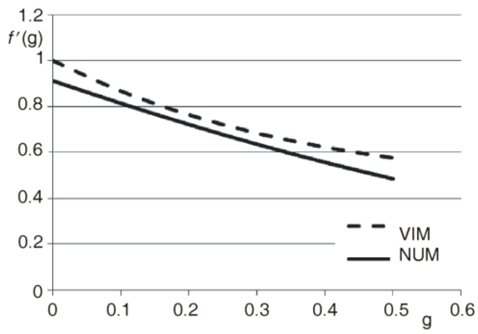

Figure 3. Comparison between VIM & Numerical solution for λ **=1/4**

Conclusions

The variational iteration method helps numerical solutions to get to the right answer more easily and it costs less, since it has fewer steps of iteration. Because this method is a plain and direct technique it can be offered for most numerical cases as a good choice.

Nomenclature

A – prescribed constant
f – similarity function for steam function
g – acceleration due to gravity
h, s, S – prescribed parameters
K – permeability of the fluid-saturated porous medium
k_m – poruous thermal coefficient
Nu_x – local Nusselt number
n – order of iteration (in eq. 12)
q_w – surface heat flux
q_m – porous heat flux
Ra_x – local Rayleigh number
r – local radius of the cone
T – temperature
T_∞ – ambient temperature
u, v – velocity vector along x, and y axis
x, y – Cartesian co-ordinate system
x_0 – distance of start point of cone from the vertex

Greeks symbols

α_m – thermal diffusivity of the fluid-saturated porous medium
β – expansion coefficient of the fluid
η – independent dimensionless parameter
θ – similarity function for temperature
λ – prescribed constants
υ – kinematic viscosity of the fluid
τ – prescribed parameters
Ψ – stream function

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Paper submitted: January 27, 2010
Paper revised: March 12, 2010
Paper accepted: March 12, 2011