MATHEMATICAL AND NUMERICAL INVESTIGATIONS OF FRACTIONAL STOCHASTIC EPIDEMIC MODEL

by

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Sexually transmitted diseases are infectious diseases and a significant threat to human health. In this work, a standard integer-order model of Chlamydia is transformed into a fractional-order stochastic mathematical model. The steady-state of the continuous system is determined and considered for disease forecasting and stability analysis. The fractional stochastic system is tested for stability at both equilibrium states by following the classical Jacobian matrix theory. It is investigated the underlying epidemic model has a unique solution. The non-negative and bounded solutions of the model also provide a deeper understanding of the disease propagation. Then, a finite difference numerical algorithm is constructed for approximating the solution. To assess the efficiency of the algorithm, non-negativity and boundedness of the numerical method are investigated. Furthermore, the algorithm is applied to a test example to obtain the simulated graphs. Ultimately, the study's outcomes are summarized in the form of conclusions.

Key words*: fractional differential equation, stochastic process, chlamydia disease, Lyapunov function, GL-NSFD scheme, stability*

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Introduction

Sexually transmitted diseases (STD) are propagated by some pathogens. Chlamydia trachomatis (CT) infection is the most prevalent STD [1]. Chlamydia infection is caused by the bacterium named Chlamydia trachoma. Sexual activity with an infected individual and an unhygienic environment are two key reasons for transmission of Chlamydia. Approximately 90 million new cases of Chlamydia are reported each year, making it a serious public health concern. An infected person can transmit the infection to other persons through intercourse, anal sex, or oral sex. Other transmission possibilities may include handshakes, sharing beds, towels, and clothes. In exceptional cases, a person may acquire conjunctivitis if vaginal fluid comes into contact with the eyes. The CT is also the primary cause of blindness worldwide [2]. The CT affects both genders, with approximately 4.2% of females and 2.7% of males in the whole population worldwide [3, 4]. Teenagers are mainly infected with Chlamydia. Young girls aged 15 to 24 have a higher risk of contracting the infection [5]. Chlamydia infection damages the rectum and cervix and severely affects the reproductive system. The most common STD include chlamydia, gonorrhea, syphilis, herpes, human papillomavirus (HPV), and human immunodeficiency virus (HIV) [6-16].

Stochastic epidemic models serve as mathematical frameworks for revealing the transmission behavior of infectious diseases, including STD. Stochastic models account for the inherent unpredictability in transmission dynamics, in contrast to deterministic models, which rely on fixed parameters and continuous variables [10]. When a disease spreads mainly due to how people interact one-on-one, or when there aren't many people around, randomness becomes extremely important. These models, which incorporate randomness, help us better understand how infectious diseases spread in unpredictable ways, especially when they involve one-on-one interactions. They use these fancy stochastic epidemic models to check out STD, looking at all kinds of stuff that messes with how diseases spread in a group of people. These models typically incorporate factors such as who's hooking up with whom, the likelihood of someone contracting the disease, and behaviors that accelerate its spread. Those stochastic epidemic models give us a good handle on how all sorts of things come together to spread STD in a bunch of people [13]. These stochastic epidemic models help us understand how different factors affect the spread of STD among people. They combine various elements to get the complete picture. Experts use these stochastic epidemic models to dig into STD, looking at all kinds of factors that mess with how diseases spread in a population. These models typically include factors such as who's hooking up with whom, how easily the disease spreads, and what people are doing that accelerates its spread [17-19].

The purpose of this article is to improve and expand the work on [20] by considering an accurate CT infection model with the appropriate non-linear frequency rate to execute a complete analysis of the resultant model.

Model description

The $S(t)$, $E(t)$, $I_S(t)$, $I_U(t)$, and $R(t)$ are the state variables, and the fractional order chlamydia model [20] is presented by a system of equations given in:

$$\begin{gathered} {}_0^cD_t^\theta S(t) = B - b_1^\theta SE + b_6^\theta R - \mu^\theta S,\ \ {}_0^cD_t^\theta E(t) = b_1^\theta SE - b_2^\theta E - b_3^\theta - \mu^\theta E \\ {}_0^cD_t^\theta I_S(t) = b_2^\theta E - b_4^\theta I_S - \mu^\theta I_S,\ \ {}_0^cD_t^\theta I_U(t) = b_3^\theta E - b_5^\theta I_U - \mu^\theta I_U \\ {}_0^cD_t^\theta R(t) = b_4^\theta I_S + b_5^\theta I_U - b_6^\theta R - \mu^\theta R \end{gathered} \tag{1}$$

With initial conditions $S(0) = S_0 \square 0$, $E(0) = E_0 \square 0$, $I_S(0) = I_{S0} \square 0$, $I_U(0) = I_{U0} \square 0$, $R(0) = R_0 \square 0$. The basic reproduction number for system (1), denoted by R_0, is:

$$R_0 = \frac{b_1^\theta B}{\left(\mu^\theta\right)\left(b_2^\theta + b_3^\theta + \mu^\theta\right)} \tag{2}$$

System (1) possesses two equilibrium points, specifically the disease-free steady-state, G_0, and endemic steady-state, G_1, given by

$$G_0 = \left(S^0, E^0, I_S^0, I_U^0, R^0\right) = \left(\frac{B}{\mu^\theta}, 0, 0, 0, 0\right)$$

and

$$G1 = \left(S^*, E^*, I_S^*, I_U^*, R^*\right)$$

where

$$S^* = \frac{b_3^\theta + b_2^\theta + \mu^\theta}{b_1^\theta}$$

$$E^* = \frac{\left(b_5^\theta + \mu^\theta\right)\left(b_6^\theta + \mu^\theta\right)\left(b_4^\theta + \mu^\theta\right)\left[-\mu^{2\theta} + \left(-b_3^\theta - b_2^\theta\right)\mu^\theta + Bb_1^\theta\right]}{\left(b_1^\theta \mu^\theta \left\{\mu^{3\theta} + \left(A_1\right)\mu^{2\theta} + \left[\left(A_2\right)b_4^\theta + \left(A_3\right)b_5^\theta + \left(b_3^\theta + b_2^\theta\right)b_6^\theta\right]\mu^\theta + \left[\left(A_3\right)b_5^\theta + b_6^\theta b_5^\theta\right]b_4^\theta + b_6^\theta b_5^\theta b_2^\theta\right\}\right)}$$

$$I_S^* = \frac{\left(b_5^\theta + \mu^\theta\right)\left(b_6^\theta + \mu^\theta\right)\left(b_2^\theta\right)\left[-\mu^{2\theta} + \left(-b_3^\theta - b_2^\theta\right)\mu^\theta + Bb_1^\theta\right]}{\left(b_1^\theta \mu^\theta \left\{\mu^{3\theta} + \left(A_1\right)\mu^{2\theta} + \left[\left(A_2\right)b_4^\theta + \left(A_3\right)b_5^\theta + \left(b_3^\theta + b_2^\theta\right)b_6^\theta\right]\mu^\theta + \left[\left(A_3\right)b_5^\theta + b_6^\theta b_5^\theta\right]b_4^\theta + b_6^\theta b_5^\theta b_2^\theta\right\}\right)}$$

$$I_U^* = \frac{\left(b_4^\theta + \mu^\theta\right)\left(b_6^\theta + \mu^\theta\right)\left(b_3^\theta\right)\left[-\mu^{2\theta} + \left(-b_3^\theta - b_2^\theta\right)\mu^\theta + Bb_1^\theta\right]}{\left(b_1^\theta \mu^\theta \left\{\mu^{3\theta} + \left(A_1\right)\mu^{2\theta} + \left[\left(A_2\right)b_4^\theta + \left(A_3\right)b_5^\theta + \left(b_3^\theta + b_2^\theta\right)b_6^\theta\right]\mu^\theta + \left[\left(A_3\right)b_5^\theta + b_6^\theta b_5^\theta\right]b_4^\theta + b_6^\theta b_5^\theta b_2^\theta\right\}\right)}$$

$$R^* = \frac{\left[\left(b_2^\theta b_4^\theta + b_3^\theta b_5^\theta\right)\mu^\theta + b_4^\theta b_5^\theta\left(b_2^\theta + b_3^\theta\right)\right]\left[-\mu^{2\theta} + \left(-b_3^\theta - b_2^\theta\right)\mu^\theta + Bb_1^\theta\right]}{\left(b_1^\theta \mu^\theta \left\{\mu^{3\theta} + \left(A_1\right)\mu^{2\theta} + \left[\left(A_2\right)b_4^\theta + \left(A_3\right)b_5^\theta + \left(b_3^\theta + b_2^\theta\right)b_6^\theta\right]\mu^\theta + \left[\left(A_3\right)b_5^\theta + b_6^\theta b_5^\theta\right]b_4^\theta + b_6^\theta b_5^\theta b_2^\theta\right\}\right)}$$

where

$$A_1 = b_6^\theta + b_5^\theta + b_4^\theta + b_3^\theta + b_2^\theta, \quad A_2 = b_6^\theta + b_5^\theta + b_3^\theta + b_2^\theta, \quad A_3 = b_6^\theta + b_3^\theta + b_2^\theta$$

Analysis of model

In this section, we will analyze the CT model by investigating the positivity as well as the boundedness of the model. To this end, we construct some benchmark results.

Theorem 1. For the given initial condition at $t = 0$, $S(t) = S_0 > 0$, $E(t) = E_0 > 0$, $I_S(t) = I_{S0} > 0$, $I_U(t) = I_{U0} > 0$, $R(t) = R_0 > 0$, the solution of $(S(t),E(t),I_S(t),I_U(t),R(t)) \in R^5$ is non-negative [21].

Proof. For the fractional differential equation model. We define the norm as

$$f_\infty = \sup_{t \in D_f}^{|f(t)|}$$

Let's consider

$$ {}_0^cD_t^\theta S(t) = B - b_1^\theta SE + b_6^\theta R - \mu^\theta S,\ \ {}_0^cD_t^\theta S(t) + M_1 S(t) \geq 0 $$

where

$$ M1 = b_1^\theta E_\infty + \mu^\theta $$

Then, by Laplace Transformation, we get

$$ L\{S(t)\} \geq \frac{s^{\theta-1}S(0)}{\left(s^\theta + M_1\right)} $$

inverse Laplace transformation gives

$$ S(t) \geq S(0)E_{\theta,1}\left(-M_1 t^\theta\right),\ \text{so}\ S(t) \geq 0,\ \forall t \geq 0 $$

Similarly, proceed with the rest of the equations. We conclude that the model holds positivity.

Theorem 2. For the initial condition, $S(t) = S_0 > 0$, $E(t) = E_0 > 0$, $I_S(t) = I_{S0} > 0$, $I_U(t) = I_{U0} > 0$, $R(t) = R_0 > 0$, the solution of system (1) is uniformly bounded.

Proof. By adding all the equations of system (1), we have

$$ {}_0^cD_t^\theta N(t) = B - \mu^\theta N(t) $$

By applying the Laplace and Laplace inverse Transformation, we get

$$ N(t) \leq M\left[-\mu^\theta t^\theta E_{\theta,\theta+1}\left(-\mu^\theta t^\theta\right) + \frac{1}{\Gamma(1)} + \mu^\theta t^\theta E_{\theta,1+\theta}\left(-\mu^\theta t^\theta\right)\right] $$

where

$$ M = \max\left(N(0), \frac{B}{\mu^\theta}\right) $$

Which is required.

Stability analysis

In this section, we will investigate both the local stability of the model at the disease-free state and the global stability at both steady-states.

Local stability

Using the Jacobian matrix approach, we examine the local stability of the model at the disease-free equilibrium.

Theorem 3. The disease-free steady-state exhibits local asymptotic stability if $0 < R_0 < 1$.

Proof. By using Jacobian matrix theory, the proof is straightforward [20].

Global stability

Here, we will present the global stability of both equilibrium states.

Lemma 1. Let $y : R^+ \cup \{0\} \to R^+$ be a continuously defined function and let t be non-negative, $\alpha \in R^+$ the following inequality is satisfied:

$$ D^\alpha(t)\left[y(t) - y^* - y^* \ln\frac{y(t)}{y^*}\right] \leq D^\alpha y(t)\left(1 - \frac{y^*}{y(t)}\right) $$

Theorem 4. The system (1) shows global asymptotically stable behavior at DFSS if $R_0 < 1$.

Proof. We construct a candidate Lyapunov function as:

$$L_1 = \left(S - S^0 - S^0 \ln \frac{S}{S^0} \right) + E + I_S + I_U + R$$

Using *Lemma 1*, we have

$$D_t^\alpha \left(L_1 \right) \le \left(1 - \frac{S}{S^0} \right) D_t^\alpha S + D_t^\alpha E + D_t^\alpha I_S + D_t^\alpha I_U + D_t^\alpha R,\ D_t^\alpha \left(L_1 \right) \le \frac{-B\left(S^0 - S \right)}{SS^0}$$

$D_t^\alpha (L_1) < 0$ if $R_0 < 1$. It is concluded that the system exhibits global asymptotic stability at the disease-free equilibrium point when $R_0 < 1$.

Theorem 5. The system is GAS (globally asymptotically stable) at ESS if $R_0 > 1$.

Proof. Consider a candidate Lyapunov function, we have

$$D_t^\alpha \left(L_2 \right) \le \left(\frac{S - S^*}{S} \right) D_t^\alpha S + \left(\frac{E - E^*}{E} \right) D_t^\alpha E + \left(\frac{I_S - I_S^*}{I_S} \right) D_t^\alpha I_S + \left(\frac{I_U - I_U^*}{I_U} \right) D_t^\alpha I_U + \left(\frac{R - R^*}{R} \right) D_t^\alpha R$$

$$D_t^\alpha \left(L_2 \right) \le -\frac{\left(S - S^* \right)^2}{SS^*} \left(B + b_6^\theta R \right) - \frac{\left(E - E^* \right)^2}{EE^*} \left(b_1^\theta \right) - \frac{\left(I_S - I_S^* \right)^2}{I_S I_S^*} \left(b_2^\theta E \right) - \frac{\left(I_U - I_U^* \right)^2}{I_U I_U^*} \left(b_3^\theta E \right) -$$

$$- \frac{\left(R - R^* \right)^2}{RR^*} \left(b_4^\theta I_S + b_5^\theta I_U \right)$$

$D_t^\alpha (L_2) < 0$ if $R_0 > 1$. It is concluded that the system exhibits global asymptotic stability at the endemic equilibrium when $R_0 > 1$.

Chlamydia model with Caputo derivative and stochastic components

Finally, we explore a stochastic expansion of fractional epidemic models, employing diverse stochastic methodologies found in existing literature. We examine the subsequent set of stochastic differential equations, expanding upon our fractional epidemic model:

$$\begin{gathered} {}_0^c D_t^\theta S(t) = B - b_1^\theta SE + b_6^\theta R - \mu^\theta S + \sigma_1 S(t) dB_1,\ {}_0^c D_t^\theta E(t) = b_1^\theta SE - b_2^\theta E - b_3^\theta - \mu^\theta E + \sigma_2 S(t) dB_2 \\ {}_0^c D_t^\theta I_S(t) = b_2^\theta E - b_4^\theta I_S - \mu^\theta I_S + \sigma_3 I_S(t) dB_3,\ {}_0^c D_t^\theta I_U(t) = b_3^\theta E - b_5^\theta I_U - \mu^\theta I_U + \sigma_4 I_U(t) dB_4 \\ {}_0^c D_t^\theta R(t) = b_4^\theta I_S + b_5^\theta I_U - b_6^\theta R - \mu^\theta R + \sigma_5 R(t) dB_5 \end{gathered} \tag{3}$$

Analysis of existence and unique solution of the stochastic model

This section address the regularity-mapping properties of the existence of a unique solution within the framework of analytical solutions of the 1st order system of non-linear Stochastic dynamical system (3) together with its suitable initial data, namely, S_0, E_0, I_{S0}, I_{U0}, and R_0. System (3) with its corresponding initial conditions is equivalent to the following fixed-point operators. The system below is analogous to Volterra-type integral equations:

$$\hat{S} = S_0 + \int_0^t \frac{\left(B - b_1 SE - b_6 R - \mu S \right)}{\left(t - \tau \right)^{1-\alpha}} (\tau) \mathrm{d}\tau + \int_0^t \frac{\sigma_1 S dB_1(\tau)}{\left(1 - \tau \right)^{1-\alpha}} \mathrm{d}\tau \tag{4}$$

$$\hat{E} = E_0 + \int_0^t \frac{\left(b_1 SE - b_2 E - b_3 E - \mu E\right)}{\left(t-\tau\right)^{1-\alpha}}(\tau)\mathrm{d}\tau + \int_0^t \frac{\sigma_2 E dB_2(\tau)}{\left(1-\tau\right)^{1-\alpha}}\mathrm{d}\tau \tag{5}$$

$$I_S = I_{S_0} + \int_0^t \frac{\left(b_2 E - b_4 I_S - \mu I_S\right)}{\left(t-\tau\right)^{1-\alpha}}(\tau)\mathrm{d}\tau + \int_0^t \frac{\sigma_3 I_S dB_3(\tau)}{\left(1-\tau\right)^{1-\alpha}}\mathrm{d}\tau \tag{6}$$

$$I_U = I_{U_0} + \int_0^t \frac{\left(b_3 E - b_5 I_U - \mu I_U\right)}{\left(t-\tau\right)^{1-\alpha}}(\tau)\mathrm{d}\tau + \int_0^t \frac{\sigma_4 I_U dB_4(\tau)}{\left(1-\tau\right)^{1-\alpha}}\mathrm{d}\tau \tag{7}$$

$$\hat{R} = R_0 + \int_0^t \frac{\left(b_4 IS - b_5 I_U - b_6 R - \mu R\right)}{\left(t-\tau\right)^{1-\alpha}}(\tau)\mathrm{d}\tau + \int_0^t \frac{\sigma_5 R dB_5(\tau)}{\left(1-\tau\right)^{1-\alpha}}\mathrm{d}\tau \tag{8}$$

In each of the previous operators, the 1st integral is bounded. To estimate the stochastic integral

$$\int \frac{(f)dB(\tau)}{\left(1-\tau\right)^{1-\alpha}}\mathrm{d}\tau$$

We assume that the density (f) is only changed at a discrete time point, (t_i: i = 1,2,3,…,N –1) where we define the integral as in [22]. For and

$$\|S_0\| \le k_1,\ \|E_0\| \le k_2,\ \left\|I_{S_0}\right\| \le k_3,\ \left\|I_{U_0}\right\| \le k_4,\ \|R_0\| \le k_5$$

and

$$\|S\| \le \|E\| \le \|I_S\| \le \|I_U\| \le \|R\| \le r$$

in the ball B^*. Similarly, for contraction

$$\left\|\hat{S}_1 - \hat{S}_2\right\| \le \frac{\rho^\alpha}{\alpha}\left(b_1 r + \mu + \sigma_1 B_1[T]\right)\|S_1 - S_2\|$$

and the contraction estimates is

$$\rho_S < \left(\frac{\alpha}{b_1 r + \mu + \sigma_1 B_1[T]}\right)^{1/\alpha}$$

for other operators

$$\rho_E < \left(\frac{\alpha}{b_1 r + b_2 + \mu + \sigma_2 B_2[T]}\right)^{1/\alpha}, \quad \rho_{I_S} < \left(\frac{\alpha}{b_4 + \mu + \sigma_3 B_3[T]}\right)^{1/\alpha}$$

$$\rho_{I_U} < \left(\frac{\alpha}{b_5 + \mu + \sigma_4 B_4[T]}\right)^{1/\alpha}, \qquad \rho_R < \left(\frac{\alpha}{b_6 + \sigma_5 B_5[T]}\right)^{1/\alpha}$$

Now we are in a position to conduct IVP (3) is uniquely solvable, the *Contraction Mapping principle* in the Banach spaces in Ito's sense. Finally, summing up the above calculation, we prove the following result.

Theorem 6. Suppose the right-hand sides of system (9) are Lipchitz continuous in S, E, I_S, I_U, and R, respectively. Then the system is uniquely solvable in the sense of Ito.

Numerical schemes

This section presents a numerical scheme used for studying chlamydia disease. Using GL – NSFD inplace of ${}_0^C D_t^\theta$:

$$S_{n+1} - \sum_{v=1}^{n+1} c_v^\theta S_{n+1-v} - r_{n+1}^\theta S_0 = h^\theta \left[B - b_1^\theta SE + b_6^\theta R - \mu^\theta S \right]$$

$$S_{n+1} = \frac{\varphi\left(h^\theta\right) B + \varphi\left(h^\theta\right) \alpha_6^\theta R + \sum_{v=1}^{n+1} c_v^\theta S_{n+1-v} + r_{n+1}^\theta S_0}{\left[1 + \varphi\left(h^\theta\right)\left(\alpha_1^\theta E + \mu^\theta\right)\right]} \tag{14}$$

Doing it in the same manner, we have:

$$E_{n+1} = \frac{\varphi\left(h^\theta\right) b_1^\theta S_{n+1} E + \sum_{v=1}^{n+1} c_v^\theta E_{n+1-v} + r_{n+1}^\theta E_0}{\left[1 + \varphi\left(h^\theta\right)\left(b_2^\theta + b_3^\theta + \mu^\theta\right)\right]} \tag{15}$$

$$I_{S_{n+1}} = \frac{\varphi\left(h^\theta\right) b_3^\theta E + \sum_{v=1}^{n+1} c_v^\theta I_{S_{n+1-v}} - r_{n+1}^\theta I_{S_0}}{\left[1 + \varphi\left(h^\theta\right)\left(b_4^\theta + \mu^\theta\right)\right]} \tag{16}$$

$$I_{U_{n+1}} = \frac{\varphi\left(h^\theta\right) b_3^\theta E + \sum_{v=1}^{n+1} c_v^\theta I_{U_{n+1-v}} - r_{n+1}^\theta I_{U_0}}{\left[1 + \varphi\left(h^\theta\right)\left(b_5^\theta + \mu^\theta\right)\right]} \tag{17}$$

$$R_{n+1} = \frac{\varphi\left(h^\theta\right)\left(b_4^\theta I_S + b_5^\theta I_U\right) + \sum_{v=1}^{n+1} c_v^\theta R_{n+1-v} + r_{n+1}^\theta R_0}{\left[1 + \varphi\left(h^\theta\right)\left(b_6^\theta + \mu^\theta\right)\right]} \tag{18}$$

Theorem 7. *Positivity*. Suppose that every state variables and controlled parameter are positive or zero*, i.e.,* S_0, E_0, I_{S0}, I_{U0}, and $R_0 \geq 0$. Then S_{n+1}, E_{n+1}, I_{Sn+1}, I_{Un+1}, $R_{n+1} \geq 0$, $\forall\ n \in Z^+$.

Proof. Since,

$$S_{n+1} = \frac{\varphi\left(h^\theta\right) B + \varphi\left(h^\theta\right) b_6^\theta R + \sum_{v=1}^{n+1} c_v^\theta S_{n+1-v} + r_{n+1}^\theta S_0}{\left[1 + \varphi\left(h^\theta\right)\left(b_1^\theta E + \mu^\theta\right)\right]}$$

as all discrete state variables and parameters are positive. The principle of mathematical induction rigorously establishes the proof. Therefore S_{n+1}, E_{n+1}, I_{Sn+1}, I_{Un+1}, $R_{n+1} \geq 0$.

Theorem 8. Boundedness. Suppose that $S_0 + E_0 + I_{S0} + I_{U0} + R_0 = N(N_0,\theta)$, and all the parameters are positive for $\theta \in (0,1)$. Then there is a constant

$$N\left(N_{n+1},\theta\right) = \frac{\theta + \dfrac{1}{\Gamma(1-\theta)} + N\left(N_{n+1},\theta\right)\left[1 + \sigma dB\right]}{1 + \mu^\theta \varphi\left(h^\theta\right)}$$

also, $\sigma_1 dB_1 = \sigma_2 dB_2 = \sigma_3 dB_3 = \sigma_4 dB_4 = \sigma_5 dB_5 = \sigma dB$, such that S_{n+1}, E_{n+1}, I_{Sn+1}, I_{Un+1}, $R_{n+1} = N(N_{n+1},\theta)$, for $n = 0,1,2,3, \ldots, N_{n+1}$.

Proof. Adding the equations and rearranging the previous equations

$$\left(S_{n+1}+E_{n+1}+I_{S_{n+1}}+I_{U_{n+1}}+R_{n+1}\right)\left(1+\mu^{\theta}\varphi\left(h^{\theta}\right)\right)=$$

$$=\varphi\left(h^{\theta}\right)B\theta+\sum_{v=0}^{n+1}c_v^{\theta}\left(S_{n+1-v}+E_{n+1-v}+I_{S_{n+1-v}}+I_{U_{n+1-v}}+R_{n+1-v}\right)+\gamma_{n+1}^{\theta}+\sigma dB\left(S_n+E_n+I_{S_n}+I_{Un}+R_n\right)$$

The proof is confirmed through the application of mathematical induction, where $N(N_{n+1},\theta)$ represents the conclusion or final point in this series of linked identities and inequalities.

Numerical simulations

With the help of the given parametric values [20], the broadcasting of chlamydia has been simulated to demonstrate the actions of the virus.

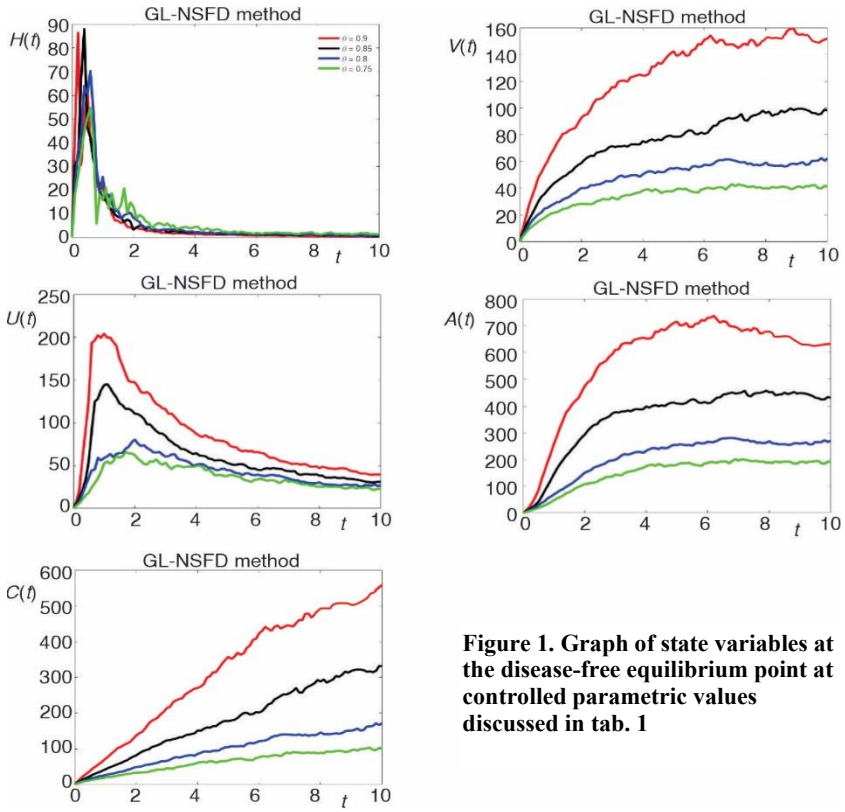

Figure 1. Graph of state variables at the disease-free equilibrium point at controlled parametric values discussed in tab. 1

The graphs in fig. 1 display the evolution of various state variables for the chlamydia disease. All the graphs in this figure indicate that they accurately hit the target point. Each graph illustrates a distinct convergence trajectory toward the target point, influenced by the specific value of θ. Similarly, all the graphs in fig. 2 demonstrate the progress behavior at the ESS point. It is noted that each graph attains its true convergence with some specific rate, depending upon the value of θ. So, the fractional order of the differential operator significantly influences the dynamical behavior, effectively regulating the evolution speed of the state variables. A key characteristic of the proposed scheme is its ability to converge accurately

to the fixed points corresponding to the carious values of θ. On these grounds, it can be said that the reliability of the scheme is marvelous.

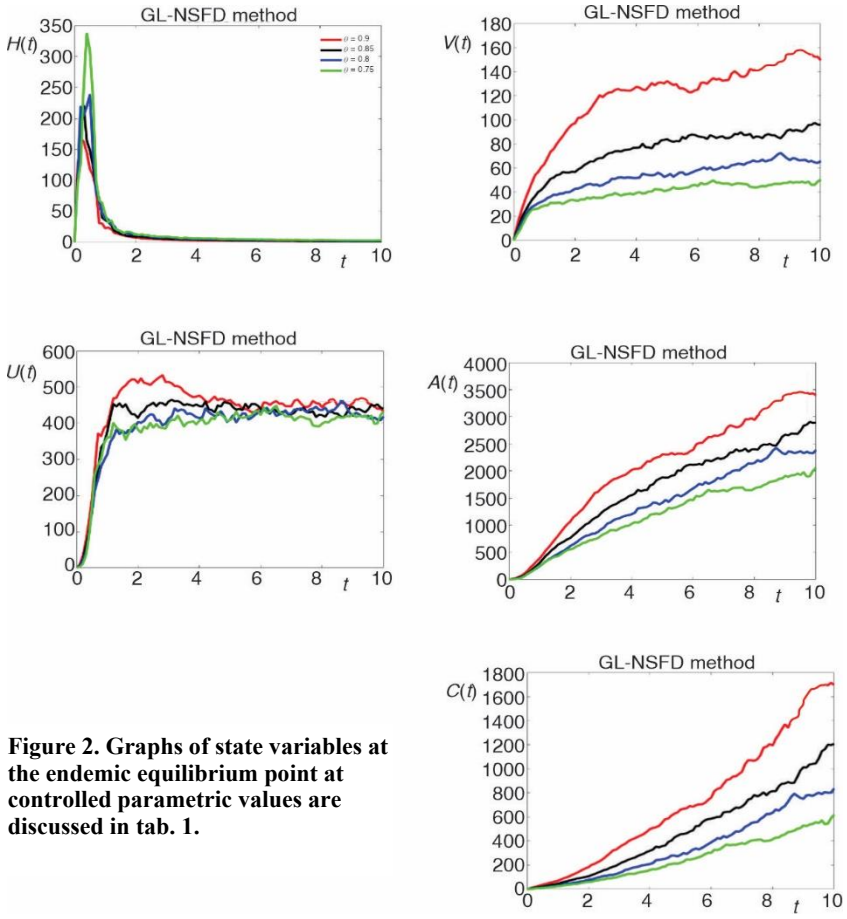

Figure 2. Graphs of state variables at the endemic equilibrium point at controlled parametric values are discussed in tab. 1.

Table 1. Values of parameters and state variables

Parameters	Description
s_{b0}	Initial value of the susceptible baby
s_{d0}	Initial value of susceptible adult
i_0	Initial value of infected
r_0	Initial value of recovered
w_0	Initial value of the water contaminated
μ^θ	Natural birth/death rate
μ	Rate of effective contacts
a_b	Infective contact rate of infected individuals and susceptible babies
a_d	Infective contact rate of infected individuals and susceptible adults
b_b	Infective contact rate of susceptible babies and contaminated water
b_d	Infective contact rate of susceptible adults and contaminated water
δ	Growth rate from susceptible babies to susceptible adults
γ	Recovery rate

Conclusion

Epidemiological modeling serves as a fundamental approach to analyzing the transmission dynamics and developing effective strategies for managing infectious diseases. However, in practical situations, the criteria used to represent the disease might not be precise. A classical integer-order model of chlamydia is transformed into a fractional-order stochastic model for a deeper understanding of the virus's dynamical features. The projected model exhibits positivity and boundedness, and it possesses a unique solution. Additionally, the system reflects the local and global stable behavior at steady-states. Similarly, the numerical algorithm maintains positivity and boundedness, indicating that the numerical scheme maintains the inherent structure and qualitative behavior of the state variables. Finally, a test example is considered to verify the physical properties of the numerical design. The analysis confirms that the proposed scheme yields solutions that remain positive and bounded. Moreover, the solutions converge towards the true steady-states with different rates of convergence. The simulated graph confirms that the rate of convergence is directly proportional to the value of the fractional order parameter θ.

Author declarations

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