THE ENERGY OF COEFFICIENT INEQUALITIES FOR CERTAIN SUBCLASSES OF ANALYTIC FUNCTIONS DEFINED BY DENIZ-OZKAN DIFFERENTIAL OPERATOR

by

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By means of the Deniz-Ozkan differential operator, we introduce and investigate a new subclass of analytic functions. The various results obtained here for this function class include coefficient bounds and Fekete-Szego inequality.

Key words: *analytic functions, differential operator, Fekete-Szego*

Introduction

Let $\mathcal{A}$ denote the family of functions f of the form:

$$f(z)=z+\sum_{n=2}^{\infty}a_n z^n \tag{1}$$

that are analytic in the open unit disk $\mathcal{U}=\{z:|z|<1\}$. A function $f\in\mathcal{A}$ is said to be starlike of order $\alpha(0\le\alpha<1)$ if and only if:

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\}>\alpha,\quad (z\in\mathcal{U})$$

We denote by $\mathcal{S}^*(\alpha)$, the class of all such functions. On the other hand, a function $f\in\mathcal{A}$ is said to be convex of order $\alpha(0\le\alpha<1)$ if and only if:

$$\Re\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\alpha,\quad (z\in\mathcal{U})$$

Let $\mathcal{C}(\alpha)$ denote the class of all those functions which are convex of order α in $\mathcal{U}$.

Note that $\mathcal{S}^*(0)=\mathcal{S}^*$ and $\mathcal{C}(0)=\mathcal{C}$ are, respectively, the classes of starlike and convex functions in $\mathcal{U}$.

Silverman [1] proved the following result. If:

$$g(z)=z-\sum_{n=2}^{\infty}b_n z^n\quad (b_n\ge 0)$$

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is analytic and starlike of order $\alpha(0 \le \alpha < 1)$ which is defined by $\mathcal{T}^*(\alpha)$ in $\mathcal{U}$, then:

$$\sum_{n=2}^{\infty}(n-\alpha)b_n \le 1-\alpha$$

From this result, it can be written:

$$\sum_{n=2}^{\infty} b_n \le \frac{1-\alpha}{2-\alpha}$$

Owa was defined in [2] the following class:

$$\mathcal{S}(\rho,\alpha,\beta)=\left\{ f\in\mathcal{A}:\left|\frac{f(z)-g(z)}{\rho f(z)+g(z)}\right|<\beta,\quad 0\le\rho\le 1,\quad 0<\beta\le 1,\quad g\in\mathcal{T}^*(\alpha)\right\}$$

In his paper, Owa [3] obtained the upper bounds for the coefficients $|a_2|$, $|a_3|$, and $|a_4|$ of the functions belonging to $\mathcal{S}(\rho,\alpha,\beta)$. In addition, he also obtained for the coefficients a_n in this class.

$$|a_n| \le \frac{\beta(3-2\alpha)}{2-\alpha}+\frac{1-\alpha}{n-\alpha}$$

Then, Altintas [4] generalized the result of Owa [3].

Recently, Kamali and Kadioglu [5] have defined the class $\mathcal{S}^*(\rho,\alpha)$:

$$\mathcal{S}^*(\rho,\alpha)=\left\{ f\in\mathcal{A}:\left|\frac{zf'(z)-g(z)}{\rho zf'(z)+g(z)}\right|<1,\quad 0\le\rho\le 1,\quad g\in\mathcal{T}^*(\alpha)\right\}$$

and obtained for the coefficients a_n in this class:

$$|a_n| \le \frac{1}{n}\left[(1+\rho)^{n-1}+(1+\rho)^{n-2}\left(\frac{1-\alpha}{2-\alpha}\right)\right],\quad \text{for}\quad n=2,3,...$$

For a functions f in $\mathcal{A}$, Deniz and Ozkan [9] differential operator D_λ^m as follows.

Definition 1 Let $f\in\mathcal{A}$. For the parameters $\lambda \ge 0$ and $m\in\mathbb{N}_0$ $\mathbb{N}\cup\{0\}$ define the differential operator D_λ^m on $\mathcal{A}$:

$$\mathrm{D}_\lambda^0 f(z)=f(z)$$

$$\mathrm{D}_\lambda^1 f(z)=\lambda z^3 f'''(z)+(2\lambda+1)z^2 f''(z)+zf'(z)$$

$$\mathrm{D}_\lambda^m f(z)=D[D^{m-1}f(z)]$$

for $z\in\mathcal{U}$.

For a function f in $\mathcal{A}$, from definition of the differential operator D_λ^m, we can easily see that:

$$\mathrm{D}_\lambda^m f(z)=z+\sum_{n=2}^{\infty}B(\lambda,n)^m a_n z^n$$

where $B(\lambda, n)^m = n^{2m}[\lambda(n-1)+1]^m$.

It should be remarked that the operator $\mathrm{D}_\lambda^m f(z)$ is a generalization of the Sălăgean differential operator, see [10, 11].

Making use of the operator $\mathrm{D}_\lambda^m f(z)$, we introduce the following subclass.

Definition 2 By $\mathcal{S}_*^m(\lambda,\rho,\alpha)$, we denote class of functions (1):

$$D_\lambda^m f(z) = z + \sum_{n=2}^{\infty} B(\lambda,n)^m a_n z^n$$

which are analytic in $\mathcal{U}$ and satisfy the condition:

$$\left|\frac{z[\mathrm{D}_\lambda^m f(z)]' - \mathrm{D}_\lambda^m g(z)}{\rho z[\mathrm{D}_\lambda^m f(z)]' + \mathrm{D}_\lambda^m g(z)}\right| < 1 \quad (0 \le \rho \le 1, \quad z \in \mathcal{U})$$

where

$$\mathrm{D}_\lambda^m g(z) = z - \sum_{n=2}^{\infty} B(\lambda,n)^m b_n z^n \ \ (b_n \ge 0) \tag{2}$$

is analytic and starlike of order $\alpha(0 \le \alpha < 1)$.

In this paper, we obtain the upper bounds for the coefficients of the operators belonging to the general class $\mathcal{S}_*^m(\lambda,\rho,\alpha)$.

Lemma 1 Let the function $\mathrm{D}_\lambda^m f(z)$ be defined by (2). The function $\mathrm{D}_\lambda^m f(z)$ is analytic and starlike of order $\alpha(0 \le \alpha < 1)$ in $\mathcal{U}$, if and only if:

$$\sum_{n=2}^{\infty} B(\lambda,n)^m (n-\alpha) b_n \le 1-\alpha$$

From this result, it can be written:

$$\sum_{n=2}^{\infty} B(\lambda,n)^m b_n \le \frac{1-\alpha}{2-\alpha} \tag{3}$$

The result is sharp.

Proof. Assume that the inequality (3) holds and let $|z| = 1$. Then we have:

$$\left|\frac{z[\mathrm{D}_\lambda^m g(z)]'}{\mathrm{D}_\lambda^m g(z)} - 1\right| = \left|\frac{\sum_{n=1}^{\infty}(1-n)B(\lambda,n)^m b_n z^{n-1}}{1-\sum_{n=1}^{\infty} B(\lambda,n)^m b_n z^{n-1}}\right| \le \frac{\sum_{n=1}^{\infty}|n-1| B(\lambda,n)^m |b_n|}{1-\sum_{n=1}^{\infty} B(\lambda,n)^m |b_n|} \le 1-\alpha$$

This shows that the values of $\{z[\mathrm{D}_\lambda^m g(z)]'\}/\mathrm{D}_\lambda^m g(z)$ lies in a circle centered at $w = 1$ whose radius is $1-\alpha$. Hence $\mathrm{D}_\lambda^m g(z)$ is starlike of order α.

Conversely, assume that the function $\mathrm{D}_\lambda^m g(z)$ defined by (2), then:

$$Re\left\{\frac{z[\mathrm{D}_\lambda^m g(z)]'}{\mathrm{D}_\lambda^m g(z)}\right\} = Re\left\{\frac{1-\sum_{n=2}^{\infty} nB(\lambda,n)^m b_n z^{n-1}}{1-\sum_{n=2}^{\infty} B(\lambda,n)^m b_n z^{n-1}}\right\} > \alpha \tag{4}$$

for $z \in \mathcal{U}$. Choose values of z on the real axis so that $\{z[\mathrm{D}_\lambda^m g(z)]'\}/\mathrm{D}_\lambda^m g(z)$ is real. Upon clearing the denominator in (4) and letting $z = 1^-$ through real values, we obtain:

$$1-\sum_{n=2}^{\infty} nB(\lambda,n)^m b_n \geq \alpha\left\{1-\sum_{n=2}^{\infty} B(\lambda,n)^m b_n\right\}$$

which gives (3). Finally the result is sharp with the extremal function $\mathrm{D}_\lambda^m g(z)$ given by:

$$\mathrm{D}_\lambda^m g(z) = z - \frac{1-\alpha}{B(\lambda,n)^m (n-\alpha)} z^n, \quad (n \geq 2)$$

Theorem 1 Let $\mathcal{S}_*^m(\lambda,\rho,\alpha)$, we denote class of functions (1):

$$\mathrm{D}_\lambda^m f(z) = z + \sum_{n=2}^{\infty} B(\lambda,n)^m a_n z^n$$

and satisfy the condition:

$$\left|\frac{z[\mathrm{D}_\lambda^m f(z)]' - \mathrm{D}_\lambda^m g(z)}{\rho z[\mathrm{D}_\lambda^m f(z)]' + \mathrm{D}_\lambda^m g(z)}\right| < 1 \quad \left(0 \leq \rho \leq 1, \quad z \in \mathcal{U}\right) \tag{5}$$

where

$$\mathrm{D}_\lambda^m g(z) = z - \sum_{n=2}^{\infty} B(\lambda,n)^m b_n z^n \quad (b_n \geq 0)$$

is analytic and starlike of order $\alpha(0 \leq \alpha < 1)$, $B(\lambda, n) = n^2[\lambda(n-1)+1]$. Then:

$$|a_n| \leq \frac{1}{nB(\lambda,n)^m}\left[(1+\rho)^{n-1} + (1+\rho)^{n-2}\left(\frac{1-\alpha}{2-\alpha}\right)\right], \quad \text{for} \quad n = 2,3,... \tag{6}$$

Proof. If $f \in \mathcal{S}_*^m\left(\lambda,\rho,\alpha\right)$, then we have:

$$z\left(D_\lambda^m f(z)\right)' - D_\lambda^m g(z) = \left[\rho z\left(D_\lambda^m f(z)\right)' + D_\lambda^m g(z)\right] w(z) \tag{7}$$

by (5), where the function $w(z) = \Sigma_{n=1}^{\infty} c_n z^n$ is analytic in $\mathcal{U}$ and $|w(z)| < 1$. We can write:

$$z\left[1+\sum_{n=2}^{\infty} B(\lambda,n)^m n a_n z^{n-1}\right] - \left[z - \sum_{n=2}^{\infty} B(\lambda,n)^m b_n z^n\right] =$$

$$= \left[\rho z\left(1+\sum_{n=2}^{\infty} B(\lambda,n)^m n a_n z^{n-1}\right) + z - \sum_{n=2}^{\infty} B(\lambda,n)^m b_n z^n\right] w(z)$$

by (7) or:

$$\sum_{n=2}^{\infty} B(\lambda,n)^m \left(n a_n + b_n\right) z^n = \left[\left(1+\rho\right) z + \sum_{n=2}^{\infty} B(\lambda,n)^m \left(\rho n a_n - b_n\right) z^n\right] \sum_{n=1}^{\infty}$$

Equalizing the coefficients of z^n in the last equality, we obtain:

$$B(\lambda,n)^m\left(na_n+b_n\right)=\left(1+\rho\right)c_{n-1}+B(\lambda,2)^m\left(2\rho a_2-b_2\right)c_{n-2}+$$
$$+\ldots+B\left[\lambda,(n-1)\right]^m\left[\rho(n-1)a_{n-1}-b_{n-1}\right]c_1 \tag{8}$$

Since $\left|c_n\right|\le 1$ for every n, from (8), we have:

$$\left|2a_2+b_2\right|\le\frac{1+\rho}{B(\lambda,2)^m}$$

Now, by using the inequality:

$$\sum_{n=2}^{\infty}B\left(\lambda,n\right)b_n\le\frac{1-\alpha}{2-\alpha}$$

we obtain:

$$2\left|a_2\right|-\left|b_2\right|\le\left|2a_2+b_2\right|\le\frac{1+\rho}{B(\lambda,2)^m}$$

Thus:

$$\left|a_2\right|\le\frac{1}{2B(\lambda,2)^m}\left[\left(1+\rho\right)+\left(\frac{1-\alpha}{2-\alpha}\right)\right] \tag{9}$$

On the other hand, we obtain:

$$3\left|a_3\right|-\left|b_3\right|\le\left|3a_3+b_3\right|\le\frac{1+\rho}{B(\lambda,3)^m}+2\rho\frac{B(\lambda,2)^m}{B(\lambda,3)^m}\left|a_2\right|+\frac{B(\lambda,2)^m}{B(\lambda,3)^m}\left|b_2\right|$$

which implies that by (8):

$$\left|a_3\right|\le\frac{1}{3B(\lambda,3)^m}\left[\left(1+\rho\right)^2+\left(1+\rho\right)\left(\frac{1-\alpha}{2-\alpha}\right)\right] \tag{10}$$

Now, we prove the following inequality:

$$\left|a_n\right|\le\frac{1}{nB(\lambda,n)^m}\left[\left(1+\rho\right)^{n-1}+\left(1+\rho\right)^{n-2}\left(\frac{1-\alpha}{2-\alpha}\right)\right]$$

by using mathematical induction. We have:

$$B\left[\lambda,(n+1)\right]^m\left[(n+1)a_{n+1}+b_{n+1}\right]=\left(1+\rho\right)c_n+B(\lambda,2)^m\left(2\rho a_2-b_2\right)c_{n-1}$$
$$+\ldots+B(\lambda,n)^m\left(\rho na_n-b_n\right)c_1$$

which implies that:

$$B\left[\lambda,(n+1)\right]^m (n+1)|a_{n+1}| \le (1+\rho) + [2\rho B(\lambda,2)^m |a_2| + 3\rho B(\lambda,3)^m |a_3|$$
$$+ ... + n\rho B(\lambda,n)^m |a_n|] + B(\lambda,2)^m |b_2|$$
$$+ ... + B(\lambda,n)^m |b_n| + B(\lambda,(n+1))^m |b_{n+1}|$$
$$\le (1+\rho) + \rho\left[(1+\rho) + \left(\frac{1-\alpha}{2-\alpha}\right) + (1+\rho)^2 + (1+\rho)\left(\frac{1-\alpha}{2-\alpha}\right)\right.$$
$$\left. + ... + (1+\rho)^{n-1} + (1+\rho)^{n-2}\left(\frac{1-\alpha}{2-\alpha}\right)\right] + \left(\frac{1-\alpha}{2-\alpha}\right)$$
$$= (1+\rho)^n + (1+\rho)^{n-1}\left(\frac{1-\alpha}{2-\alpha}\right)$$

Thus:

$$|a_{n+1}| \le \frac{1}{(n+1)B\left[\lambda,(n+1)\right]^m}\left[(1+\rho)^n + (1+\rho)^{n-1}\left(\frac{1-\alpha}{2-\alpha}\right)\right]$$

By *Theorem 1*, it can be written that if $f \in \mathcal{S}_*^m(\lambda,\rho,\alpha)$, then:

$$|a_n| \le \frac{1}{nB(\lambda,n)^m}\left[(1+\rho)^{n-1} + (1+\rho)^{n-2}\left(\frac{1-\alpha}{2-\alpha}\right)\right]$$

with equality only for function of the form:

$$f(z) = z + \frac{1}{nB(\lambda,n)^m}\left[(1+\rho)^{n-1} + (1+\rho)^{n-2}\left(\frac{1-\alpha}{2-\alpha}\right)\right]z^n$$

Corollary 1 If $f \in \mathcal{S}_*^m(\lambda,0,\alpha),$ then:

$$|a_n| \le \frac{1}{nB(\lambda,n)^m}\left(\frac{3-2\alpha}{2-\alpha}\right)$$

Remark 1 Taking m = 0 in *Theorem 1*, we obtain *Theorem 4* given by Kamali and Kadioglu [5].

Now, we give sharp upper bounds for $|a_3 - \eta a_2^2|$ for the class $\mathcal{S}_*^m(\lambda,\mu,\rho,\alpha)$. To prove our main result, we need the following lemmas.

Following first lemma is special case of *Theorem 1* in [12].

Lemma 2 If:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}^*(\alpha)$$

and if v is a complex number, then:

$$\left|a_3 - va_2^2\right| \le (1-\alpha)\max\left\{1, \left|2(1-\alpha)(2v-1)-1\right|\right\}$$

Note that, if we choose:

$$\mathrm{D}_{\lambda,\mu}^m g(z) = z - \sum_{n=2}^{\infty} B(\lambda,n)^m b_n z^n \in \mathcal{S}^*(\alpha)$$

we obtain from *Lemma 2*:

$$\left|b_3 - v\frac{B(\lambda,2)^{2m}}{B(\lambda,3)^m} b_2^2\right| \le \frac{(1-\alpha)}{B(\lambda,3)^m}\max\left\{1, \left|2(1-\alpha)(2v-1)-1\right|\right\} \tag{11}$$

The *Lemma 2* is general case of *Theorem 1* in [4].

Lemma 3 If:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}^*(\alpha)$$

then:

$$|a_n| \le \frac{\prod_{j=0}^{n-1}[j + 2(1-\alpha)]}{(n-1)!}, \quad n \in \mathbb{N}\setminus\{1\} = \{2,3,4,...\}$$

Note that, if we choose:

$$\mathrm{D}_{\lambda}^m g(z) = z - \sum_{n=2}^{\infty} B(\lambda,n)^m b_n z^n \in\in \mathcal{S}^*(\alpha)$$

we obtain from *Lemma 3*:

$$|b_n| \le \frac{\prod_{j=0}^{n-1}[j + 2(1-\alpha)]}{B(\lambda,n)^m (n-1)!}, \quad n \in \mathbb{N}\setminus\{1\} = \{2,3,4,...\}$$

For $n = 2$, we get:

$$|b_2| \le \frac{2(1-\alpha)}{B(\lambda,2)^m} \tag{12}$$

Lemma 4 [2, 13] If $w \in \Omega$, then:

$$\left|w_2 - tw_1^2\right| \le \begin{cases} -t & \text{if} \quad t \le -1 \\ 1 & \text{if} \quad -1 \le t \le 1 \\ t & \text{if} \quad t \ge 1 \end{cases}$$

When $t < -1$ or $t > 1$, the equality holds if and only if $w(z) = z$ or one of its rotations. If $-1 < t < 1$, then equality holds if and only if $w(z) = z^2$ or one of its rotations. Equality holds for $t = -1$ if and only if:

$$w(z)=\frac{z(z+\lambda)}{1+\lambda z}\ (0\le\lambda\le 1)$$

or one of its rotations, while for $t = 1$ the equality holds if and only if:

$$w(z)=-\frac{z(z+\lambda)}{1+\lambda z}\ (0\le\lambda\le 1)$$

or one of its rotations.

Theorem 4 Let $m\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$, $\eta\in\mathbb{R}$, $b_n\ge 0$, and $0\le\mu\le\lambda$. If f of the form (1) is in $\mathcal{S}_*^m(\lambda,\rho,\alpha)$, then:

$$\left|a_3-\eta a_2^2\right|\le$$

$$\le\begin{cases}\dfrac{(1+\rho)}{3B(\lambda,3)^m}\left[\rho-\dfrac{3\eta B(\lambda,3)^m(1+\rho)}{4B(\lambda,2)^{2m}}\right]+\left[\dfrac{2(1+\rho)(1-\alpha)}{3B(\lambda,3)^m}-\dfrac{\eta(1+\rho)(1-\alpha)}{B(\lambda,2)^{2m}}\right]+C(\lambda,\eta,m,\alpha) & \text{if } \eta\le\sigma_1\\ \dfrac{(1+\rho)}{3B(\lambda,3)^m}+\left[\dfrac{2(1+\rho)(1-\alpha)}{3B(\lambda,3)^m}-\dfrac{\eta(1+\rho)(1-\alpha)}{B(\lambda,2)^{2m}}\right]+C(\lambda,\eta,m,\alpha) & \text{if } \sigma_1\le\eta\le\sigma_2\\ \dfrac{(1+\rho)}{3B(\lambda,3)^m}\left[\dfrac{3\eta B(\lambda,3)^m(1+\rho)}{4B(\lambda,2)^{2m}}-\rho\right]+\left[\dfrac{\eta(1+\rho)(1-\alpha)}{B(\lambda,2)^{2m}}-\dfrac{2(1+\rho)(1-\alpha)}{3B(\lambda,3)^m}\right]+C(\lambda,\eta,m,\alpha) & \text{if } \eta\ge\sigma_2\end{cases}$$

where

$$\sigma_1=\frac{4(\rho-1)B(\lambda,2)^{2m}}{3(\rho+1)B(\lambda,3)^m},\quad \sigma_2=\frac{4B(\lambda,2)^{2m}}{3B(\lambda,3)^m}$$

and:

$$C(\lambda,\eta,m,\alpha)=\frac{(1-\alpha)}{3B(\lambda,3)^m}\max\left\{1,\left|2(1-\alpha)\left(\eta\frac{3B(\lambda,3)^m}{2B(\lambda,2)^{2m}}-1\right)-1\right|\right\}$$

Proof. From *Lemma 4*, (12) and choosing:

$$\nu=\eta\frac{3B(\lambda,3)^m}{4B(\lambda,2)^{2m}}$$

in the inequality (11), the proof is completed.

References

[1] Silverman, H., Univalent Functions with Negative Coefficients, *Proc. Amer. Math. Soc.*, *51* (1975), 1, pp. 109-116
[2] Ali, M. R., *et al.*, Coefficient Bounds for p-Valent Functions, *Appl. Math. Comput.*, *187* (2007), 1. pp. 35-46
[3] Owa, S., A Remark on Certain Classes of Analytic Functions, *Math. Japonica, 28* (1983), pp. 15-20
[4] Altintas, O., Coefficients of Some Analytic Functions, *Doga TU Mat. D., 13* (1989), pp. 43-48

[5] Kamali, M., Kadioglu, E., On the Coefficients of Certain Analytic Functions, *Far East J. Math. Sci., 5* (1997), 2, pp. 169-176
[6] Boyd, A. V., Coefficient Estimates for Starlike Functions of Order α, *Proceedings of the American Mathematical Society*, *17* (1996), 5, pp. 1016-1018
[7] Pommerenke, Ch., On Meromorphic Starlike Functions, *Pacific J. Math*., *13* (1963), pp. 221-235
[8] Kaczmarski, J., On the Coefficients of Some Classes of Starlike Functions, *Bulletin DeL academie Polonaise Des Sciences Serie des Sciences Math. Astr. et Phys., 17* (1969), pp. 495-501
[9] Deniz, E., Ozkan, Y., Subclasses of Analytic Functions Defined by a New Differential Operator, *Acta Uni. Apul*, *40* (2014), pp. 85-95
[10] Deniz, E., Orhan, H., The Fekete Szego Problem for a Generalized Subclass of Analytic Functions, *Kyungpook Math. J., 50* (2010), 1, pp. 37-47
[11] Salagean, G. S., Subclasses of Univalent Functions, Complex Analysis – *Proceedings,* 5th Rom.-Finn. Semin., Bucharest 1981, Part 1, Lect. Notes Math. 1013, pp. 362-372, 1983
[12] Keogh, F. R., Merkes, E. P., A Coefficient Inequality for Certain Classes of Analytic Functions, *Proceedings of the American Mathematical Society, 20* (1969), 1, pp. 8-12
[13] Nehari, Z., *Conformal Mapping,* New York, 1952

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