FRACTIONAL INEQUALITIES INVOLVING DOUBLE INTEGRALS OF RIEMANN-LIOUVILLE FOR HIGHER-ORDER PARTIAL DIFFERENTIAL FUNCTIONS

by

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The primary aim of this study is to establish new inequalities involving the Riemann-Liouville fractional integrals for different classes of functions in two variables. As a foundational step, we establish two identities concerning the Riemann--Liouville fractional integrals for higher-order partial derivatives of functions. Subsequently, several fractional Ostrowski-type inequalities for bounded functions of two variables are derived. Besides the main results, various special cases derived from the current findings are presented, and the links between these findings and earlier results are explained.

Key words: *bounded functions, fractional integrals, Ostrowski type inequalities*

Introduction

In recent decades, integral inequalities have emerged as powerful tools in various branches of mathematical analysis, with wide-ranging applications in numerical integration, approximation theory, and differential equations. These inequalities not only offer valuable estimates but also provide insight into the behavior of functions under integration. Among the notable contributions in this area, the result established by Ostrowski [1] has garnered considerable attention due to its utility in quantifying the de*via*tion of a function from its integral mean.

Owing to its effectiveness in providing sharp bounds for the deviation of a function from its integral mean, the Ostrowski inequality has become a central topic of investigation in classical and modern analysis. Dragomir and Wang [2, 3] established Ostrowski-type inequalities for functions whose first derivatives belong to various Lebesgue spaces. An Ostrowski-type inequality for functions of two variables, along with its applications in numerical analysis, was also established by Barnett and Dragomir in [4]. In a subsequent contribution, Dragomir *et al.* [5] derived generalized Ostrowski-type inequalities for multivariable functions whose partial derivatives are integrable in the sense of the L_p-norm.

Motivated by the need to extend classical inequalities to broader function classes, some of the earliest and most influential results concerning Ostrowski-type inequalities for functions with bounded higher-order derivatives were established in [6, 7], laying the foundation for subsequent developments in this direction. Subsequently, in [8], some integral inequalities added to the literature by using similar function types. Moreover, a number of Os-

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trowski-type results have been developed in [9, 10] for functions whose derivatives of any order are either L_p-integrable or bounded, thereby extending the classical framework to a broader class of differentiable functions. In addition to deriving inequalities for functions with higher-order derivatives, some authors [11-13] have also explored their natural applications in estimating integral values, often formulating quadrature rules that emerge as a natural consequences of such inequalities.The double integral inequalities established for functions possessing higher-order partial derivatives constitute the foundation of the present study, and the fundamental studies on this topic can be found in references [14-16].

Another significant topic addressed in this article is the theory of Riemann-Liouville fractional integrals. It is essential to review the definitions of Riemann-Liouville fractional integrals.

Definition 1. [17] Assume that $\psi \in L_1[\sigma_1, \sigma_2]$ and $\alpha > 0$. The left-sided Riemann-Liouville fractional integral of order α is defined by:

$$J_{\sigma_1+}^{\alpha}\psi(\zeta) = \frac{1}{\Gamma(\alpha)}\int_{\sigma_1}^{\zeta}(\zeta-\tau)^{\alpha-1}\psi(\tau)\mathrm{d}\tau, \quad \zeta > \sigma_1$$

while the right-sided Riemann-Liouville fractional integral is given by:

$$J_{\sigma_2-}^{\alpha}\psi(\zeta) = \frac{1}{\Gamma(\alpha)}\int_{\zeta}^{\sigma_2}(\tau-\zeta)^{\alpha-1}\psi(\tau)\mathrm{d}\tau, \quad \zeta < \sigma_2$$

where $\Gamma(\alpha)$ denotes the classical Euler gamma function. These operators reduce to the standard integral when $\alpha = 1$, and serve as the foundation for various fractional analogues of classical integral inequalities.

Additionally, the definitions of Riemann-Liouville fractional integrals for functions with two independent variables have also been introduced, providing a natural extension of the classical one-variable case to a bivariate setting.

Definition 2. [18] Let $\psi \in L_1([\sigma_1, \sigma_2] \times [\rho_1, \rho_2])$. The Riemann-Liouville fractional integrals:

$$J_{\sigma_1+,\rho_1+}^{\alpha,\beta}, \quad J_{\sigma_1+,\rho_2-}^{\alpha,\beta}, \quad J_{\sigma_2-,\rho_1+}^{\alpha,\beta} \quad \text{and} \quad J_{\sigma_2-,\rho_2-}^{\alpha,\beta}$$

are defined by:

$$J_{\sigma_1+,\rho_1+}^{\alpha,\beta}\psi(\zeta,\vartheta) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{\sigma_1}^{\zeta}\int_{\rho_1}^{\vartheta}(\zeta-\tau)^{\alpha-1}(\vartheta-\varsigma)^{\beta-1}\psi(\tau,\varsigma)\mathrm{d}\varsigma\mathrm{d}\tau, \quad \zeta > \sigma_1, \quad \vartheta > \rho_1$$

$$J_{\sigma_1+,d-}^{\alpha,\beta}\psi(\zeta,\vartheta) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{\sigma_1}^{\zeta}\int_{\vartheta}^{\rho_2}(\zeta-\tau)^{\alpha-1}(\varsigma-\vartheta)^{\beta-1}\psi(\tau,\varsigma)\mathrm{d}\varsigma\mathrm{d}\tau, \quad \zeta > \sigma_1, \quad \vartheta < d$$

$$J_{\sigma_2-,\rho_1+}^{\alpha,\beta}\psi(\zeta,\vartheta) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{\zeta}^{\sigma_2}\int_{\rho_1}^{\vartheta}(\tau-\zeta)^{\alpha-1}(\vartheta-\varsigma)^{\beta-1}\psi(\tau,\varsigma)\mathrm{d}\varsigma\mathrm{d}\tau, \quad \zeta|\sigma_2, \ \vartheta|\rho_1$$

and

$$J^{\alpha,\beta}_{\sigma_2-,d-}\psi(\zeta,\vartheta)=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int\limits_{\zeta}^{\sigma_2}\int\limits_{\vartheta}^{\rho_2}(\tau-\zeta)^{\alpha-1}(\varsigma-\vartheta)^{\beta-1}\psi(\tau,\varsigma)\mathrm{d}\varsigma\mathrm{d}\tau,\quad \zeta<\sigma_2,\quad \vartheta<d$$

where $\Gamma(\cdot)$ denotes the classical Euler gamma function.

A substantial body of literature has been dedicated to the study of Riemann-Liouville fractional integrals, reflecting their foundational role in the theory of fractional calculus. Among these, the monographs [17, 19] stand out as fundamental references offering a comprehensive treatment of the subject. Moreover, a significant number of research articles have been devoted to the investigation of various integral inequalities associated with Riemann-Liouville fractional integrals, highlighting their broad applicability and theoretical importance. It is worth emphasizing that one of the pioneering contributions to Hermite-Hadamard type inequalities involving Riemann-Liouville fractional integrals was made by Sarikaya *et al.* [20]. Sarkaya [18] also provided Hermite-Hadamard type inequalities including the fractional integrals defined for functions of two variables, particularly in the framework of co-ordinated convexity. In addition, Dragomir derived Ostrowski-type inequalities for different classes of functions, utilizing identities involving the sum of the right- and left-sided Riemann-Liouville integrals [21-23]. Subsequently, the Montgomery identity involving Riemann-Liouville fractional integrals, along with the associated Ostrowski-type inequalities, was presented by Aglić and Aljinović [24]. In addition to these foundational works, the references [25-30], on Ostrowski-type results involving fractional integrals of functions of one variable are the works that motivated us to write this article. It is also important to highlight the inequalities involving Riemann-Liouville fractional integrals of functions of two variables. For instance. fractional integral inequalities of Ostrowski type for functions with two independent variables are developed by Latif and Hussain [31]. Another significant contribution to the literature is the recent work by Sarkaya [32], who established fractional integral inequalities based on functions of two variables. Erden *et al.* [33, 34] also presented Ostrowski-type results including double fractional integrals for various classes of functions of two variables. In a recent study Erden *et al.* [35] related to the topic of this paper, provided new fractional integral inequalities for different classes of mappings, including functions whose higher-order partial derivatives are of bounded variation.

This section is devoted to the analysis of recent fractional integral inequalities that can be found using functions with higher order partial derivatives, in the light of the aforementioned works. In the second section, two novel double integral identities involving the definition of Riemann-Liouville fractional integrals of functions with higher-order partial derivatives will be established. In the subsequent section, two fundamental fractional Ostrowski-type inequalities are derived for bounded functions of two variables. In the final section, new double integral inequalities are established for functions whose higher-order partial derivatives are supposed to be in the L_1 space.

Some identities for double integrals

In this section, we present the fundamental identities that are essential for deriving the main results. Since the integrals involving higher-order partial derivatives are challenging to compute and express explicitly, we introduce some notations to present the forthcoming identities more clearly and concisely:

$$J_1(\psi;\zeta,\vartheta) = J^{\alpha,\beta}_{\sigma_1+,\rho_1+}\psi(\zeta,\vartheta) + (-1)^m J^{\alpha,\beta}_{\sigma_1+,\rho_2-}\psi(\zeta,\vartheta) + $$
$$+(-1)^n J^{\alpha,\beta}_{\sigma_2-,\rho_1+}\psi(\zeta,\vartheta) + (-1)^{n+m} J^{\alpha,\beta}_{\sigma_2-,\rho_2-}\psi(\zeta,\vartheta) \tag{1}$$

$$J_2(\psi;\zeta,\vartheta) = (-1)^{n+m} J^{\alpha,\beta}_{\zeta-,\vartheta-}\psi(\sigma_1,\rho_1) + (-1)^n J^{\alpha,\beta}_{\zeta-,\vartheta+}\psi(\sigma_1,\rho_2) + $$
$$+(-1)^m J^{\alpha,\beta}_{\zeta+,\vartheta-}\psi(\sigma_2,\rho_1) + J^{\alpha,\beta}_{\zeta+,\vartheta+}\psi(\sigma_2,\rho_2) \tag{2}$$

$$F_1(\psi;\zeta,\vartheta;n,m) =$$
$$=\left(\frac{(\vartheta-\rho_1)^{m+\beta}+(\rho_2-\vartheta)^{m+\beta}}{\Gamma(m+\beta+1)}\right)\left[J^{\alpha}_{\sigma_1+}\frac{\partial^m\psi(\zeta,\vartheta)}{\partial\vartheta^m}+(-1)^n J^{\alpha}_{\sigma_2-}\frac{\partial^m\psi(\zeta,\vartheta)}{\partial\vartheta^m}\right]+$$
$$+\left(\frac{(\zeta-\sigma_1)^{n+\alpha}+(\sigma_2-\zeta)^{n+\alpha}}{\Gamma(n+\alpha+1)}\right)\left[J^{\beta}_{\rho_1+}\frac{\partial^n\psi(\zeta,\vartheta)}{\partial\zeta^n}+(-1)^m J^{\beta}_{\rho_2-}\frac{\partial^n\psi(\zeta,\vartheta)}{\partial\zeta^n}\right] \tag{3}$$

$$F_2(\psi;\zeta,\vartheta;n,m) =$$
$$=\frac{\left[(\zeta-\sigma_1)^{n+\alpha}+(\sigma_2-\zeta)^{n+\alpha}\right]\left[(\vartheta-\rho_1)^{m+\beta}+(\rho_2-\vartheta)^{m+\beta}\right]}{\Gamma(n+\alpha+1)\Gamma(m+\beta+1)}\frac{\partial^{n+m}\psi(\zeta,\vartheta)}{\partial\zeta^n\partial\vartheta^m} \tag{4}$$

$$F_3(f;x,y;n,m) =$$
$$=\frac{(x-a)^{n+\alpha}+(b-x)^{n+\alpha}}{\Gamma(n+\alpha+1)}\left[(-1)^m J^{\beta}_{y-}\frac{\partial^n f(x,c)}{\partial x^n}+J^{\beta}_{y+}\frac{\partial^n f(x,d)}{\partial x^n}\right]+$$
$$+\frac{(y-c)^{m+\beta}+(d-y)^{m+\beta}}{\Gamma(m+\beta+1)}\left[(-1)^n J^{\alpha}_{x-}\frac{\partial^m f(a,y)}{\partial y^m}+J^{\alpha}_{x+}\frac{\partial^m f(b,y)}{\partial y^m}\right] \tag{5}$$

$$S_1(\psi;\zeta,\vartheta;n,m) =$$
$$=\sum_{k=0}^{n-1}\sum_{j=0}^{m-1}\frac{(\zeta-\sigma_1)^{\alpha+k}\left[(\vartheta-\rho_1)^{\beta+j}\frac{\partial^{k+j}\psi(\sigma_1,\rho_1)}{\partial\tau^k\partial\varsigma^j}+(-1)^{m+j}(\rho_2-\vartheta)^{\beta+j}\frac{\partial^{k+j}\psi(\sigma_1,\rho_2)}{\partial\tau^k\partial\varsigma^j}\right]}{\Gamma(\alpha+k+1)\Gamma(\beta+j+1)}+\check{z}$$
$$+\sum_{k=0}^{n-1}\sum_{j=0}^{m-1}\frac{(-1)^{n+k}(\sigma_2-\zeta)^{\alpha+k}\left[\begin{array}{c}(\vartheta-\rho_1)^{\beta+j}\frac{\partial^{k+j}\psi(\sigma_2,\rho_1)}{\partial\tau^k\partial\varsigma^j}+\\+(-1)^{m+j}(\rho_2-\vartheta)^{\beta+j}\frac{\partial^{k+j}\psi(\sigma_2,\rho_2)}{\partial\tau^k\partial\varsigma^j}\end{array}\right]}{\Gamma(\alpha+k+1)\Gamma(\beta+j+1)} \tag{6}$$

$$S_2(\psi;\zeta,\vartheta;n,m)=\sum_{k=0}^{n-1}\frac{(\zeta-\sigma_1)^{\alpha+k}}{\Gamma(\alpha+k+1)}\left[J_{\rho_1+}^{\beta}\frac{\partial^k\psi(\sigma_1,\vartheta)}{\partial\tau^k}+(-1)^m J_{d-}^{\beta}\frac{\partial^k\psi(\sigma_1,\vartheta)}{\partial\tau^k}\right]+$$

$$+\sum_{k=0}^{n-1}\frac{(-1)^{n+k}(\sigma_2-\zeta)^{\alpha+k}}{\Gamma(\alpha+k+1)}\left[J_{\rho_1+}^{\beta}\frac{\partial^k\psi(\sigma_2,\vartheta)}{\partial\tau^k}+(-1)^m J_{d-}^{\beta}\frac{\partial^k\psi(\sigma_2,\vartheta)}{\partial\tau^k}\right]+$$

$$+\sum_{j=0}^{m-1}\frac{(\vartheta-\rho_1)^{\beta+j}}{\Gamma(\beta+j+1)}\left[J_{\sigma_1+}^{\alpha}\frac{\partial^j\psi(\zeta,\rho_1)}{\partial\varsigma^j}+(-1)^n J_{\sigma_2-}^{\alpha}\frac{\partial^j\psi(\zeta,\rho_1)}{\partial\varsigma^j}\right]+$$

$$+\sum_{j=0}^{m-1}\frac{(-1)^{m+j}(d-\vartheta)^{\beta+j}}{\Gamma(\beta+j+1)}\left[J_{\sigma_1+}^{\alpha}\frac{\partial^j\psi(\zeta,d)}{\partial\varsigma^j}+(-1)^n J_{\sigma_2-}^{\alpha}\frac{\partial^j\psi(\zeta,d)}{\partial\varsigma^j}\right] \tag{7}$$

$$S_3(\psi;\zeta,\vartheta;n,m)=$$

$$=\left(\frac{(\vartheta-\rho_1)^{m+\beta}+(d-\vartheta)^{m+\beta}}{\Gamma(m+\beta+1)}\right)\sum_{k=0}^{n-1}\frac{(\zeta-\sigma_1)^{\alpha+k}\frac{\partial^{k+m}\psi(\sigma_1,\vartheta)}{\partial\tau^k\partial\vartheta^m}+(-1)^{n+k}(\sigma_2-\zeta)^{\alpha+k}\frac{\partial^{k+m}\psi(\sigma_2,\vartheta)}{\partial\tau^k\partial\vartheta^m}}{\Gamma(\alpha+k+1)}+$$

$$+\left[\frac{(\zeta-\sigma_1)^{n+\alpha}+(\sigma_2-\zeta)^{n+\alpha}}{\Gamma(n+\alpha+1)}\right]\sum_{j=0}^{m-1}\frac{(\vartheta-\rho_1)^{\beta+j}\frac{\partial^{n+j}\psi(\zeta,\rho_1)}{\partial\zeta^n\partial\varsigma^j}+(-1)^{m+j}(\rho_2-\vartheta)^{\beta+j}\frac{\partial^{n+j}\psi(\zeta,d)}{\partial\zeta^n\partial\varsigma^j}}{\Gamma(m+\beta+1)} \tag{8}$$

$$S_4(\psi;\zeta,\vartheta;n,m)=$$

$$=\sum_{k=0}^{n-1}\sum_{j=0}^{m-1}\frac{\left[(-1)^{n+k}(\zeta-\sigma_1)^{\alpha+k}+(\sigma_2-\zeta)^{\alpha+k}\right]}{\Gamma(\alpha+k+1)}\frac{\left[(-1)^{m+j}(\vartheta-\rho_1)^{\beta+j}+(\rho_2-\vartheta)^{\beta+j}\right]}{\Gamma(\beta+j+1)}\frac{\partial^{k+j}\psi(\zeta,\vartheta)}{\partial\zeta^k\partial\vartheta^j}-$$

$$-\sum_{k=0}^{n-1}\frac{(-1)^{n+k}(\zeta-\sigma_1)^{\alpha+k}+(\sigma_2-\zeta)^{\alpha+k}}{\Gamma(\alpha+k+1)}\left[(-1)^m J_{\vartheta-}^{\beta}\frac{\partial^k\psi(\zeta,\rho_1)}{\partial\zeta^k}+J_{\vartheta+}^{\beta}\frac{\partial^k\psi(\zeta,\rho_2)}{\partial\zeta^k}\right]-$$

$$-\sum_{j=0}^{m-1}\frac{(-1)^{m+j}(\vartheta-\rho_1)^{\beta+j}+(\rho_2-\vartheta)^{\beta+j}}{\Gamma(\beta+j+1)}\left[(-1)^n J_{\zeta-}^{\alpha}\frac{\partial^j\psi(\sigma_1,\vartheta)}{\partial\vartheta^j}+J_{\zeta+}^{\alpha}\frac{\partial^j\psi(\sigma_2,\vartheta)}{\partial\vartheta^j}\right] \tag{9}$$

$$S_5(\psi;\zeta,\vartheta;n,m)=$$

$$=\frac{(\zeta-\sigma_1)^{n+\alpha}+(\sigma_2-\zeta)^{n+\alpha}}{\Gamma(n+\alpha+1)}\sum_{j=0}^{m-1}\frac{\left[(-1)^{m+j}(\vartheta-\rho_1)^{\beta+j}+(\rho_2-\vartheta)^{\beta+j}\right]}{\Gamma(\beta+j+1)}\frac{\partial^{n+j}\psi(\zeta,\vartheta)}{\partial\zeta^n\partial\vartheta^j}+$$

$$+\frac{(\vartheta-\rho_1)^{m+\beta}+(\rho_2-\vartheta)^{m+\beta}}{\Gamma(m+\beta+1)}\sum_{k=0}^{n-1}\frac{\left[(-1)^{n+k}(\zeta-\sigma_1)^{\alpha+k}+(\sigma_2-\zeta)^{\alpha+k}\right]}{\Gamma(\alpha+k+1)}\frac{\partial^{k+m}\psi(\zeta,\vartheta)}{\partial\zeta^k\partial\vartheta^m} \quad (10)$$

By integration by parts and elementary analysis operations will be used to establish identities involving fractional integrals of functions of two variables. So, identities including Riemann-Liouville fraction integrals for functions of two variables whose higher order partial derivatives are continuous are provided as follows.

Lemma 1. Suppose that ψ: $[\sigma_1, \sigma_2] \times [\rho_1, \rho_2] =:\Delta \subset R^2 \to R$ is an absolutely continuous function such that the partial derivatives $[\partial^{k+j}\psi(u, v)]/(\partial u^k \partial v^j)$ exists and are continuous on Δ for $k = 0, 1, 2,\ldots, n$, $j = 0, 1, 2,\ldots, m$ with $n, m \in N^+$. Then, for any $(\zeta, \vartheta) \in \Delta$ one has:

$$\frac{1}{\Gamma(n+\alpha)\Gamma(m+\beta)}\int_{\sigma_1}^{\sigma_2}\int_{\rho_1}^{\rho_2}Q(\zeta,\tau,\vartheta,\varsigma)\left[\int_{\zeta}^{\tau}\int_{\vartheta}^{\varsigma}\frac{\partial^{n+m+2}\psi(u,v)}{\partial u^{n+1}\partial v^{m+1}}\mathrm{d}v\mathrm{d}u\right]\mathrm{d}\varsigma\mathrm{d}\tau=$$

$$=J_1(\psi;\zeta,\vartheta)+S_1(\psi;\zeta,\vartheta;n,m)-S_2(\psi;\zeta,\vartheta;n,m)+$$

$$+S_3(\psi;\zeta,\vartheta;n,m)-F_1(\psi;\zeta,\vartheta;n,m)+F_2(\psi;\zeta,\vartheta;n,m) \quad (11)$$

where $Q(\zeta, \tau, \vartheta, \varsigma)$ are defined by:

$$Q(\zeta,\tau,\vartheta,\varsigma):=\begin{cases}(\zeta-\tau)^{n+\alpha-1}(\vartheta-\varsigma)^{m+\beta-1}, & \sigma_1\le\tau<\zeta \text{ and } \rho_1\le\varsigma<\vartheta\\(\zeta-\tau)^{n+\alpha-1}(\varsigma-\vartheta)^{m+\beta-1}, & \sigma_1\le\tau<\zeta \text{ and } \vartheta\le\varsigma\le\rho_2\\(\tau-\zeta)^{n+\alpha-1}(\vartheta-\varsigma)^{m+\beta-1}, & \zeta\le\tau\le\sigma_2 \text{ and } \rho_1\le\varsigma<\vartheta\\(\tau-\zeta)^{n+\alpha-1}(\varsigma-\vartheta)^{m+\beta-1}, & \zeta\le\tau\le\sigma_2 \text{ and } \vartheta\le\varsigma\le\rho_2\end{cases} \quad (12)$$

were the expressions $J_1(\psi; \zeta, \vartheta)$, $F_1(\psi; \zeta, \vartheta; n, m)$, $F_2(\psi; \zeta, \vartheta; n, m)$, $S_1(\psi; \zeta, \vartheta; n, m)$, $S_2(\psi; \zeta, \vartheta; n, m)$, and $S_3(\psi; \zeta, \vartheta; n, m)$ are also as given in eqs. (1), (3), (4), (6)-(8), respectively.

Proof. In the main integral stated in the Lemma, the four integrals that arise when the definition of the kernel $Q(\zeta, \tau, \vartheta, \varsigma)$ is written must be calculated. The desired identity (11) can be obtained by employing fundamental properties of integrals and the method of integration by parts.

In what follows, we introduce a new integral identity based on different versions of the Riemann-Liouville fractional integrals in two variables.

Lemma 2. Suppose that ψ: $[\sigma_1, \sigma_2] \times [\rho_1, \rho_2] =:\Delta \subset R^2 \to R$ is an absolutely continuous function such that the partial derivatives $[\partial^{k+j}\psi(u, v)]/(\partial u^k \partial v^j)$ exists and are continuous on Δ for $k = 0, 1, 2,\ldots, n$, $j = 0, 1, 2,\ldots, m$ with $n, m \in N^+$. Then, for any $(\zeta, \vartheta) \in \Delta$ one has:

$$\frac{1}{\Gamma(n+\alpha)\Gamma(m+\beta)}\int_{\sigma_1}^{\sigma_2}\int_{\rho_1}^{\rho_2}\Omega(\tau,\varsigma)\left[\int_{\zeta}^{\tau}\int_{\vartheta}^{\varsigma}\frac{\partial^{n+m+2}\psi(u,v)}{\partial u^{n+1}\partial v^{m+1}}\mathrm{d}v\mathrm{d}u\right]\mathrm{d}\varsigma\mathrm{d}\tau =$$

$$= J_2(\psi;\zeta,\vartheta) + S_4(\psi;\zeta,\vartheta;n,m) + S_5(\psi;\zeta,\vartheta;n,m) -$$

$$-F_3(\psi;\zeta,\vartheta;n,m) + F_2(\psi;\zeta,\vartheta;n,m) \tag{13}$$

where $\Omega(\tau, \varsigma)$ is defined by:

$$\Omega(\tau,\varsigma) := \begin{cases} (\tau-\sigma_1)^{n+\alpha-1}(\varsigma-\rho_1)^{m+\beta-1}, & \sigma_1 \le \tau < \zeta \quad \text{and} \quad \rho_1 \le \varsigma < \vartheta \\ (\tau-\sigma_1)^{n+\alpha-1}(\rho_2-\varsigma)^{m+\beta-1}, & \sigma_1 \le \tau < \zeta \quad \text{and} \quad \vartheta \le \varsigma \le \rho_2 \\ (\sigma_2-\tau)^{n+\alpha-1}(\varsigma-\rho_1)^{m+\beta-1}, & \zeta \le \tau \le \sigma_2 \quad \text{and} \quad \rho_1 \le \varsigma < \vartheta \\ (\sigma_2-\tau)^{n+\alpha-1}(\rho_2-\varsigma)^{m+\beta-1}, & \zeta \le \tau \le \sigma_2 \quad \text{and} \quad \vartheta \le \varsigma \le \rho_2 \end{cases}$$

and the expressions were the expressions $J_1(\psi;\ \zeta,\ \vartheta)$, $F_2(\psi;\ \zeta,\ \vartheta;\ n,\ m)$, $F_3(\psi;\ \zeta,\ \vartheta;\ n,\ m)$, $S_4(\psi;\ \zeta,\ \vartheta;\ n,\ m)$, and $S_5(\psi;\ \zeta,\ \vartheta;\ n,\ m)$ are as defined in eqs. (2), (4), (5), (9), and (10), respectively.

Proof. If the procedures used in the proof of the *Lemma 2* are applied by following the same sequence and considering the kernel $\Omega(\tau, \varsigma)$ instead of the kernel $Q(\zeta, \tau, \vartheta, \varsigma)$, the desired result (14) is obtained.

Double integral inequalities for bounded functions

This section is devoted to the development of recent double integral inequalitie involving the Riemann-Liouville fractional integral definitions for two-variable functions whose partial derivatives of any order are bounded.

Theorem 2. Suppose that all the assumptions of *Lemma 1* hold. If the partial derivative of order $n + m + 2$ of ψ is bounded, *i.e.*:

$$\left\|\psi^{(n+m+2)}\right\|_\infty = \sup_{(u,v)\in(\sigma_1,\sigma_2)\times(\rho_1,\rho_2)}\left|\frac{\partial^{n+m+2}\psi(u,v)}{\partial u^{n+1}\partial v^{m+1}}\right| < \infty$$

then, for any $(\zeta, \vartheta) \in \Delta$, one has the inequalities:

$$\Big|J_1(\psi;\zeta,\vartheta) + \varsigma_1(\psi;\zeta,\vartheta;n,m) - \varsigma_2(\psi;\zeta,\vartheta;n,m) +$$

$$+\varsigma_3(\psi;\zeta,\vartheta;n,m) - \psi_1(\psi;\zeta,\vartheta;n,m) + \psi_2(\psi;\zeta,\vartheta;n,m)\Big| \le$$

$$\leq \frac{(n+\alpha)(m+\beta)}{\Gamma(n+\alpha+2)\Gamma(m+\beta+2)} Z_1(\zeta,\vartheta;n,m) \leq$$

$$\leq \frac{\left[(\zeta-\sigma_1)^{n+\alpha+1}+(\sigma_2-\zeta)^{n+\alpha+1}\right]\left[(\vartheta-\rho_1)^{m+\beta+1}+(\rho_2-\vartheta)^{m+\beta+1}\right]}{(n+\alpha+1)(m+\beta+1)\Gamma(n+\alpha)\Gamma(m+\beta)}\left\|\psi^{(n+m+2)}\right\|_\infty \tag{14}$$

where $Z_1(\zeta, \vartheta; n, m)$ is defined by:

$$Z_1(\zeta,\vartheta;n,m)=$$

$$=(\zeta-\sigma_1)^{n+\alpha+1}\left\{(\vartheta-\rho_1)^{m+\beta+1}\left\|\psi^{(n+m+2)}\right\|_{[\sigma_1,\zeta]\times[\rho_1,\vartheta],\infty}+(\rho_2-\vartheta)^{m+\beta+1}\left\|\psi^{(n+m+2)}\right\|_{[\sigma_1,\zeta]\times[\vartheta,\rho_2],\infty}\right\}+$$

$$+(\sigma_2-\zeta)^{n+\alpha+1}\left\{(\vartheta-\rho_1)^{m+\beta+1}\left\|\psi^{(n+m+2)}\right\|_{[\zeta,\sigma_2]\times[\rho_1,\vartheta],\infty}+\right.$$

$$\left.+(\rho_2-\vartheta)^{m+\beta+1}\left\|\psi^{(n+m+2)}\right\|_{[\zeta,\sigma_2]\times[\vartheta,\rho_2],\infty}\right\} \tag{15}$$

where the expressions $J_1(\psi; \zeta, \vartheta)$, $F_1(\psi; \zeta, \vartheta; n, m)$, $F_2(\psi; \zeta, \vartheta; n, m)$, $S_1(\psi; \zeta, \vartheta; n, m)$, $S_2(\psi; \zeta, \vartheta; n, m)$, and $S_3(\psi; \zeta, \vartheta; n, m)$ are also as given in eqs. (1), (3), (4), (6)-(8), respectively.

Proof. If the definition of the kernel $Q(\zeta, \tau, \vartheta, \varsigma)$ is used after taking the absolute value of both sides of the identity (11), the required inequality (14) is obtained.

Remark 1. With $n = m = 0$ and the same assumption of *Teorem 2*, the sum symbols disappear and 0 is substituted instead of n and m in the remaining expressions, one possesses the inequality which was proved Erden *et al.* [34].

In the following, we examine the results derived from the application of the second identity (13).

Theorem 3. Suppose that all the assumptions of *Lemma 2* hold. If the partial derivative of order $n + m + 2$ of ψ is bounded, *i.e.*:

$$\left\|\psi^{(n+m+2)}\right\|_\infty = \sup_{(u,v)\in(\sigma_1,\sigma_2)\times(\rho_1,\rho_2)}\left|\frac{\partial^{n+m+2}\psi(u,v)}{\partial u^{n+1}\partial v^{m+1}}\right| < \infty$$

then, for any $(\zeta, \vartheta) \in \Delta$, one has the inequalities:

$$\left|J_2(\psi;\zeta,\vartheta)+\varsigma_4(\psi;\zeta,\vartheta;n,m)+\varsigma_5(\psi;\zeta,\vartheta;n,m)-\right.$$

$$\left.-\psi_3(\psi;\zeta,\vartheta;n,m)+\psi_2(\psi;\zeta,\vartheta;n,m)\right| \leq \frac{1}{\Gamma(n+\alpha+2)\Gamma(m+\beta+2)} Z_1(\zeta,\vartheta;n,m) \leq$$

$$\leq \frac{\left[(\zeta-\sigma_1)^{n+\alpha+1}+(\sigma_2-\zeta)^{n+\alpha+1}\right]\left[(\vartheta-\rho_1)^{m+\beta+1}+(\rho_2-\vartheta)^{m+\beta+1}\right]}{\Gamma(n+\alpha+2)\Gamma(m+\beta+2)}\left\|\psi^{(n+m+2)}\right\|_\infty \tag{16}$$

where $Z_1(\psi; \zeta, \vartheta; n, m)$ is defined as in eq. (15). Here, the expressions $J_2(\psi; \zeta, \vartheta)$, $F_2(\psi; \zeta, \vartheta; n, m)$, $F_3(\psi; \zeta, \vartheta; n, m)$, $S_4(\psi; \zeta, \vartheta; n, m)$, and $S_5(\psi; \zeta, \vartheta; n, m)$ are also as defined in eqs. (2), (4), (5), (9), and (10), respectively.

Proof. If the process followed in the proof of *Theorem 2* are applied in the same order by considering the definition of the kernel $\Omega(\tau, \varsigma)$ after taking the absolute value of both sides of eq. (13), the required result (16) is attained.

Remark 2. If $n = m = 0$ is chosen specifically under the conditions of the *Theorem 3*, since the sum symbols vanished, one has the inequality which was proved Erden *et al.* [34].

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