VARIATIONAL PRINCIPLE AND PERIODIC WAVE SOLUTIONALS FOR ELASTIC ROD EQUATION WITH FRACTAL DERIVATIVE

by

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The present research paper will demonstrate the variational principle and periodic wave solutions of the elastic rod equation. First, we will illustrate the generalized variational principle in two examples. Secondly, we consider a fractal non-linear elastic rod equation with an unsmooth boundary. Based on two-scale fractal theory and the semi-inverse method, we successfully establish the fractal variational principle for the non-linear elastic rod equation. This is helpful for studying symmetry, finding conserved quantities, and revealing possible traveling solution structures of the equation. Finally, we investigate periodic wave solutions of the non-linear elastic rod equation.

Key words: *two-scale fractal theory, semi-inverse method, variational principle, fractal derivative*

Introduction

Non-linear PDE are the gold standard for describing complex phenomena in the real world. They are used to model everything from the vibration of an atom to the big bang. People are fascinated by how PDE can be analyzed and solved, and they believe that finding the exact solutions is extremely difficult, if not impossible.

The exact solutions, especially the solitary wave solutions of the PDE, can be found using various powerful methods, including the homotopy perturbation method [1-3], the variational iteration method [4-6], the integral transform method [7-10], Taylor series method [11-13], and the exp-function method [14-16]. Each of these methods has its own set of advantages and disadvantages. The Taylor series method is simple, but its low convergence hinders its wide applications. The exp-function method can lead to exact solutions, but its complex calculation makes it inaccessible to those who are not familiar with some mathematical software.

There is no doubt that variational methods [17-23] offer significant advantages over other approximate analytical methods. They can help us to study the symmetries and reveal the possible conserved quantities for complex models. They play a key role in the numerical and analytical analysis of PDE, and the solutions obtained are the best among all possible trial functions. They can be used for the discussed problems from a global perspective and provide physical insight into the nature of the solutions. This paper demonstrates the efficacy of the two-scale fractal theory [24, 25] and the semi-inverse method [26] in establishing the general-

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ized variational principle. The fractal variational principle is successfully established for the elastic rod equation, and soliton wave solutions are obtained via the Ritz-like method.

Problem statement

Consider the general non-linear wave equation of longitudinal oscillation of non-linear elastic rod with lateral inertia is [27]:

$$\frac{\partial^2 u}{\partial t^2}-c_0^2\left[1+na_n\left(\frac{\partial u}{\partial x}\right)^{n-1}\right]\frac{\partial^2 u}{\partial x^2}-\frac{\nu^2 J_\rho}{s}\frac{\partial^4 u}{\partial t^2\partial x^2}=0 \tag{1}$$

where s, J_ρ, $c_0^2=E/\rho, \nu, E,$ and ρ are the cross-section area of the rod, the polar moment of inertia, the square of the linear elastic longitudinal wave velocity, Poisson ratio, the Young's modulus, and the density of the rod, respectively. While a_n is the material constant, n is a positive integer. For the soft non-linear materials such as rubbers and polymers, $a_n<0.$ For the hard non-linear materials, $a_n>0,$ for example, majority of the metals.

For the sake of convenience, denoting that $a=c_0^2$, $b=na_nc_0^2$ and $c=\nu^2 J_\rho / s$, then we obtain the equation:

$$u_{tt}-au_{xx}-bu_x^{n-1}u_{xx}-cu_{xxtt}=0 \tag{2}$$

where n is a positive integer, the parameters $a,b,c\in R$ and $abc\neq 0$ are arbitrary constant real numbers.

Variational principle: method and examples in fractal space

Variational principle is the theoretical bases for many kinds of variational methods, and the core problem is to seek variational formulations. In this section, fractal variational principles are established for the Sharma-Tasso-Olver equation [28] and the parametric coupled KdV system [29] based on the two-scale fractal theory and semi-inverse method.

Variational principle for Sharma-Tasso-Olver equation

Consider the Sharma-Tasso-Olver equation [28]:

$$u_t-3uu_{xx}-u_{xxx}-3u_x^2-3u^2u_x=0 \tag{3}$$

In the fractal space, eq. (3) can be modified:

$$\frac{\partial u}{\partial t^\alpha}-3u\frac{\partial^2 u}{\partial x^{2\beta}}-\frac{\partial^3 u}{\partial x^{3\beta}}-3\left(\frac{\partial u}{\partial x^\beta}\right)^2-3u^2\frac{\partial u}{\partial x^\beta}=0 \tag{4}$$

where $\partial u/\partial t^\alpha$, $\partial u/\partial x^\beta$ are He's fractal derivatives defined as [30, 31]:

$$\frac{\partial u}{\partial t^\alpha}(t_0,x)=\Gamma(1+\alpha)\lim_{\substack{t-t_0\to\Delta t\\ \Delta t\neq 0}}\frac{u(t,x)-u(t_0,x)}{(t-t_0)^\alpha} \tag{5}$$

$$\frac{\partial u}{\partial x^\beta}(t,x_0)=\Gamma(1+\beta)\lim_{\substack{x-x_0\to\Delta x\\ \Delta x\neq 0}}\frac{u(t,x)-u(t,x_0)}{(x-x_0)^\beta} \tag{6}$$

The following chain rules hold:

$$\frac{\partial^2}{\partial x^{2\beta}} = \frac{\partial}{\partial x^\beta}\frac{\partial}{\partial x^\beta} \tag{7}$$

$$\frac{\partial^3}{\partial x^{3\beta}} = \frac{\partial}{\partial x^\beta}\frac{\partial}{\partial x^\beta}\frac{\partial}{\partial x^\beta} \tag{8}$$

In the fractal space, all variables depend upon the scales used for observation and the fractal dimensions of the discontinuous boundary. Now, we use the two-scale transforms in the fractal time and spatial, respectively [32, 33]:

$$T = t^\alpha \tag{9}$$

$$X = x^\beta \tag{10}$$

where x, t are for the small scale and X, T for large scale, α, β are the two-scale dimensions [34]. Applying eqs. (9) and (10) to eq. (4), we have:

$$u_T - 3uu_{XX} - u_{XXX} - 3u_X^2 - 3u^2 u_X = 0 \tag{11}$$

We rewrite eq. (11) in conservation forms:

$$u_T + (-3uu_X - u_{XX} - u^3)_X = 0 \tag{12}$$

According to eq. (12), we can introduce a special function defined:

$$\psi_T = -3uu_X - u_{XX} - u^3 \tag{13}$$

$$\psi_X = -u \tag{14}$$

Our aim to structure a variational formulation for eq. (11):

$$J(u,\psi) = \iint L \; \mathrm{d}X \; \mathrm{d}T \tag{15}$$

where L is a trial-Lagrange function.

According to the semi-inverse method, we suppose the trial-Lagrange function with the following form:

$$L = u\psi_T + (-3uu_X - u_{XX} - u^3)\psi_X + F \tag{16}$$

here $F$is an unknown function of u, and/or ψ, and/or their derivatives.

Taking a variation on eq. (16) with respect to ψ, yields:

$$-u_T + (3uu_X + u_{XX} + u^3)_X + \frac{\delta F}{\delta \psi} = 0 \tag{17}$$

where $\delta F/\delta\psi$ is called the variational derivative, which takes the following form:

$$\frac{\delta F}{\delta\psi}=\frac{\partial F}{\partial\psi}-\frac{\partial}{\partial X}\left(\frac{\partial F}{\partial\psi_X}\right)+\frac{\partial^2}{\partial X^2}\left(\frac{\partial F}{\partial\psi_{XX}}\right) \tag{18}$$

Adding both sides of eq. (17) and eq. (12), we have:

$$\frac{\delta F}{\delta\psi}=0 \tag{19}$$

Making a variation on eq. (16) with respect to u, yields:

$$\psi_T-3u_X\psi_X-\psi_{XXX}-3u^2\psi_X+\frac{\delta F}{\delta u}=0 \tag{20}$$

where $\delta F/\delta u$ is the variational derivative, it can be written:

$$\frac{\delta F}{\delta u}=\frac{\partial F}{\partial u}-\frac{\partial}{\partial X}\left(\frac{\partial F}{\partial u_X}\right)+\frac{\partial^2}{\partial X^2}\left(\frac{\partial F}{\partial u_{XX}}\right) \tag{21}$$

In the view of eqs. (13) and (14), we have:

$$\frac{\delta F}{\delta u}=-\psi_T+3u_X\psi_X+\psi_{XXX}+3u^2\psi_X=$$

$$=(3uu_X+u_{XX}+u^3)-3uu_X-u_{XX}-3u^3=-2u^3 \tag{22}$$

From eq. (22), F can be identified:

$$F=-\frac{1}{2}u^4 \tag{23}$$

Finally, the following Lagrange function can be obtained:

$$L=u\psi_T+(-3uu_X-u_{XX}-u^3)\psi_X-\frac{1}{2}u^4 \tag{24}$$

Then we get the variational formulation for eq. (11):

$$J(u,\psi)=\iint\left[u\psi_T+(-3uu_X-u_{XX}-u^3)\psi_X-\frac{1}{2}u^4\right]\mathrm{d}X\ \mathrm{d}T \tag{25}$$

Proof. The Euler-Lagrange equations of eq. (25) are:

$$u_T+(-3uu_X-u_{XX}-u^3)_X=0 \tag{26}$$

$$\psi_T-3u_X\psi_X-\psi_{XXX}-3u^2\psi_X-2u^3=0 \tag{27}$$

In view of the constraint condition given by eq. (14), it is easy to prove that eqs. (26) and (27) are equivalent to eqs. (12) and (13), respectively.

In the fractal space (X^β, T^α), the variational formulation can be written:

$$J(u,\psi)=\iint\left[u\frac{\partial\psi}{\partial t^{\alpha}}+\left(-3u\frac{\partial u}{\partial x^{\beta}}-\frac{\partial^2 u}{\partial x^{2\beta}}-u^3\right)\frac{\partial\psi}{\partial x^{\beta}}-\frac{1}{2}u^4\right]\mathrm{d}x^{\beta}\ \mathrm{d}t^{\alpha} \tag{28}$$

which is subject to eq. (4).

Variational principle for the parametric coupled KdV system

Consider the parametric coupled KdV system [29]:

$$u_t+uu_x+u_{xxx}+\lambda vv_x=0 \tag{29}$$

$$v_t+u_x v+v_x u+v_{xxx}=0 \tag{30}$$

where λ is a real parameter and u, v are rapidly decreasing real valued functions depending on the temporal and spatial variables and respectively. When eqs. (29) and (30) with unsmooth boundaries, the fractal derivative will be adopted to describe the model:

$$\frac{\partial u}{\partial t^{\alpha}}+u\frac{\partial u}{\partial x^{\beta}}+\frac{\partial^3 u}{\partial x^{3\beta}}+\lambda v\frac{\partial v}{\partial x^{\beta}}=0 \tag{31}$$

$$\frac{\partial v}{\partial t^{\alpha}}+\frac{\partial u}{\partial x^{\beta}}v+\frac{\partial v}{\partial x^{\beta}}u+\frac{\partial^3 v}{\partial x^{3\beta}}=0 \tag{32}$$

where $\partial u/\partial t^{\alpha}$, $\partial u/\partial x^{\beta}$ are defined as eqs. (5) and (6).

By using the two-scale transforms:

$$T=t^{\alpha} \tag{33}$$

$$X=x^{\beta} \tag{34}$$

Equations (29) and (30) become:

$$u_T+uu_X+u_{XXX}+\lambda vv_X=0 \tag{35}$$

$$v_T+u_X v+v_X u+v_{XXX}=0 \tag{36}$$

We rewrite eqs. (35) and (36) in conservation forms:

$$u_T+\left(\frac{1}{2}u^2+u_{XX}+\frac{1}{2}\lambda v^2\right)_X=0 \tag{37}$$

$$v_T+(uv+v_{XX})_X=0 \tag{38}$$

According to eq. (37), we can introduce a special function ψ defined as:

$$\psi_T=\frac{1}{2}u^2+u_{XX}+\frac{1}{2}\lambda v^2 \tag{39}$$

$$\psi_X=-u \tag{40}$$

similarly, from eq. (38), we can introduce another special function Φ defined as:

$$\Phi_T = uv + v_{XX} \tag{41}$$

$$\Phi_X = -v \tag{42}$$

Our aim is to establish some variational formulation whose stationary conditions satisfy for eqs. (35), (41), and (42), or eqs. (36), (39), and (40). To this end, we will apply the semi-inverse method to construct a trial functional:

$$J(u,v,\psi) = \iint L \, \mathrm{d}X \, \mathrm{d}T \tag{43}$$

where L is a trial-Lagrange function defined:

$$L = v\psi_T + (uv + v_{XX})\psi_X + F \tag{44}$$

Here F is an unknown function of u, v and/or ψ, and/or their derivatives.

Taking a variation on eq. (44) with respect to u and v, yields:

$$v\psi_X + \frac{\delta F}{\delta u} = 0 \tag{45}$$

$$\psi_T + u\psi_X + \psi_{XXX} + \frac{\delta F}{\delta v} = 0 \tag{46}$$

where $\delta F/\delta u$ takes the following form:

$$\frac{\delta F}{\delta u} = \frac{\partial F}{\partial u} - \frac{\partial}{\partial T}\left(\frac{\partial F}{\partial u_T}\right) + \frac{\partial^2}{\partial X^2}\left(\frac{\partial F}{\partial u_{XX}}\right) \tag{47}$$

In view of eqs. (39) and (40), we set:

$$\frac{\delta F}{\delta u} = -v\psi_X = uv \tag{48}$$

$$\frac{\delta F}{\delta v} = -\psi_T - u\psi_X - \psi_{XXX} = -\left(\frac{1}{2}u^2 + u_{XX} + \frac{1}{2}\lambda v^2\right) + u^2 + u_{XX} = \frac{1}{2}u^2 - \frac{1}{2}\lambda v^2 \tag{49}$$

From eqs. (48) and (49), F can be determined:

$$F = \frac{1}{2}u^2 v - \frac{1}{6}\lambda v^3 \tag{50}$$

Finally, we obtain the following Lagrange function:

$$J(u,v,\psi) = \iint\left[v\psi_T + (uv + v_{XX})\psi_X + \frac{1}{2}u^2 v - \frac{1}{6}\lambda v^3\right]\mathrm{d}X \, \mathrm{d}T \tag{51}$$

Proof. The Euler-Lagrange equations of eq. (51) are:

$$-v_T - (uv + v_{XX})_X = 0 \tag{52}$$

$$v\psi_X + uv = 0 \tag{53}$$

$$\psi_T + u\psi_X + \psi_{XXX} + \frac{1}{2}u^2 - \frac{1}{2}\lambda v^2 = 0 \tag{54}$$

Equation (52) is equivalent to eq. (38), eq. (53) is equivalent to eq. (40), in view of the constraint eq. (40), eq. (54) becomes eq. (39).

Solutions of model problem

Variational theory [35-40] is a powerful mathematical tool to finding a suitable solitary waves, however a non-smooth boundary [41] will greatly affect the solitary wave properties, so the smooth space (x, t) should be replace by a fractal space (x^β, t^α), where β and α are fractal dimensions in space and time, respectively. In the fractal space, eq. (2) can be modified:

$$\frac{\partial^2 u}{\partial t^{2\alpha}} - a\frac{\partial^2 u}{\partial x^{2\beta}} - b\frac{\partial u^{n-1}}{\partial x^{\beta}}\frac{\partial^2 u}{\partial x^{2\beta}} - c\frac{\partial^4 u}{\partial t^{2\alpha}\partial x^{2\beta}} = 0 \tag{55}$$

where $\partial u/\partial t^\alpha$, $\partial u/\partial x^\beta$ are defined as eqs. (5) and (6).

Based on the two-scale transform method, and assume:

$$T = t^\alpha \tag{56}$$

$$X = x^\beta \tag{57}$$

Applying eqs. (56) and (57) to eq. (55), we have:

$$u_{TT} - au_{XX} - bu_X^{n-1}u_{XX} - cu_{XXTT} = 0 \tag{58}$$

Using the traveling wave variable:

$$\xi = X - vT \tag{59}$$

Equation (58) is transformed into the following ODE:

$$(v^2 - a)u'' - b(u')^{n-1}u'' - cv^2u^{(4)} = 0 \tag{60}$$

Taking $u'(\xi) = \varphi(\xi)$ and integrating the obtained equation, we have:

$$(v^2 - a)\varphi - \frac{b}{n}\varphi^n - cv^2\varphi'' = 0 \tag{61}$$

Denoting that:

$$\alpha = \frac{v^2 - a}{cv^2}, \quad \beta = \frac{b}{ncv^2}$$

then we obtain:

$$\alpha\varphi - \beta\varphi^n - \varphi'' = 0 \tag{62}$$

By the semi-inverse method, we can obtain the following variational formulation:

$$J(\varphi)=\int_0^\infty\left(\frac{1}{2}\alpha\varphi^2-\frac{1}{n+1}\beta\varphi^{n+1}-\frac{1}{2}(\varphi')^2\right)\mathrm{d}\xi \tag{63}$$

For the sake of convenience, we only consider $n = 2$, other values of n can be dealt with in a similar way.

Case A. According to [41, 42], we search for a soliton solution in the form:

$$\varphi(\xi)=A\operatorname{sech}(\xi) \tag{64}$$

By substituting eq. (64) into eq. (63), we obtain:

$$J=-\frac{1}{12}A^2(2-6\alpha+A\pi\beta) \tag{65}$$

To find the constant A, we need to solve the following equation:

$$\frac{\partial J}{\partial A}=-\frac{1}{12}A^2\pi\beta-\frac{1}{6}A(2-6\alpha+A\pi\beta)=0 \tag{66}$$

From eq. (66), we obtain:

$$A=\frac{4(-1+3\alpha)}{3\pi\beta} \tag{67}$$

Therefore, the solitary wave solutions to eq. (62) are:

$$\varphi(\xi)=\frac{4(-1+3\alpha)}{3\pi\beta}\operatorname{sech}(\xi) \tag{68}$$

Hence, we obtain:

$$u(\xi)=\int\varphi(\xi)\ \mathrm{d}\xi=\frac{8(-1+3\alpha)}{3\pi\beta}\arctan\left(\tanh\frac{\xi}{2}\right) \tag{69}$$

Case B. According to [41, 42], we search for a soliton solution in the form:

$$u(\xi)=A\operatorname{sech}(\xi)\tanh(\xi) \tag{70}$$

By substituting eq. (70) into eq. (63), we obtain:

$$J=-\frac{1}{90}A^2(21-15\alpha+4A\beta) \tag{71}$$

To find the constant A, we need to solve the following equation:

$$\frac{\partial J}{\partial A}=-\frac{2A^2\beta}{45}-\frac{1}{45}A(21-15\alpha+4A\beta)=0 \tag{72}$$

From eq. (72), we obtain:

$$A=\frac{-7+5\alpha}{2\beta} \tag{73}$$

Therefore, the solitary wave solutions to eq. (62) are:

$$\varphi(\xi)=\frac{-7+5\alpha}{2\beta}\operatorname{sech}(\xi)\tanh(\xi) \tag{74}$$

Hence, we obtain:

$$u(\xi)=\int\varphi(\xi)\ d\xi=\frac{7\ \operatorname{sech}(\xi)}{2\beta}-\frac{5\alpha\ \operatorname{sech}(\xi)}{2\beta} \tag{75}$$

Conclusion

In this paper, the generalized variational principles are illustrated step by step via two kinds of non-linear equations with fractal derivative. Then, the fractal variational principle for the non-linear elastic rod equation is successfully established via the two-scale fractal theory and the semi-inverse method, which can help us to understand the solution structures of the fractal model. Furthermore, solitary wave solutions of the non-linear elastic rod equation are obtained by the Ritz-like method [42]. The results in this thesis will undoubtedly have significant implications for the study of variational theory and periodic wave theory of physical equations, as well as the Hamilton principle and the least action principle. These concepts have a wide range of applications in engineering, for examples, Ma suggested a modification of Hamiltonian-based frequency-amplitude formulation [43], and the minimum condition for a variational principle was given [44] for fractal variational principles [45], applications of the variational principles to complex engineering problems are referred to references [46-48].

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