NON-LINEAR STOCHASTIC RESPONSE AND BIFURCATION ANALYSIS OF A MULTISTABLE RAYLEIGH SYSTEM WITH A FRACTIONAL ELEMENT SUBJECTED TO NOISE EXCITATION

by

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The study examines the stochastic bifurcation phenomenon of a generalized and multistable Rayleigh system subjected to fractional damping driven by Gaussian white noise. First, the harmonic balance technique is employed to minimize the error in terms of mean square, thereby deriving the approximate equal integer-order system from the original system with fractional-order elements. Subsequently, the stationary probability density function of the system is determined using the stochastic averaging method. Subsequently, employing singularity theory, the critical conditions of system parameters for stochastic P-bifurcation of the original system are identified. Finally, a qualitative analysis of the stationary probability density function curves of the system amplitude is conducted in each region delineated by the boundary set curves. The analytical solutions were found to align with the numerical findings obtained from Monte-Carlo simulation, thereby corroborating the theoretical deductions. The methodology and findings presented in this study have the potential to enhance system response control through the design of fractional-order controllers.

Key words: *fractional damping, stochastic P-bifurcation, boundary set curves, stochastic averaging method, Monte-Carlo simulation,*

Introduction

Fractional calculus represents an extension of traditional calculus by incorporating non-integer orders, which allows for the modeling of memory properties in viscoelastic materials compared to conventional integer-order derivatives, for examples, the fractal lubrication problem [1], the fractional Schrodinger-KdV system [2], the fractal Potential-YTSF equation [3], the fractal Bogoyavlenskii equation [4], the fractal oscillator [5], fractal MEMS systems [6-9], and fractal concretes [10]. The fractional derivative (FD) involves convolution, which facilitates the representation of memory and cumulative effects over time. Consequently, the fractional derivative has been demonstrated to be a more effective mathematical instrument for characterizing memory properties [11-14], and has emerged as a valuable mathematical

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tool in a range of research domains, including anomalous diffusion [15], viscoelastic mechanics [16, 17], soft matter physics [18], and advection-reaction-diffusion [19]. It offers a more precise description of diverse reaction processes than integer-order calculus, particularly in the context of engineering, where ambient noise is pervasive. Consequently, the investigation of the dynamic properties and the impacts of fractional-order parameters and noise excitations (NE) on stochastic systems is of significant importance.

Recent research has examined the dynamic behaviors of multi-stable and non-linear systems under various NE, yielding numerous insights. For example, studies of Duffing-Van der Pol (DVdP) oscillators subjected to colored noise, Levy noise, and combined random and harmonic noise [20-24] have attracted significant attention. In their study, Wu and Hao [25] examined a generalized and tri-stable DVdP oscillator under multiplicative colored noise. They analyzed the stationary probability density function (sPDF) of system amplitude (SA) and the impact of noise intensities and other parameters on stochastic P-bifurcation (SPB). Qian and Chen [26] investigated the stochastic response of modified single-DoF vibro-impact oscillators with a recovery factor under broadband NE, utilizing the Markov approximation method to obtain the sPDF of SA and energy envelope. Huang and Jin [27] investigated the dynamical response and sPDF of a strongly non-linear, single-DoF system excited by GW noise, while Sun and Yang [28] examined the stability of fractional-order energy acquisition systems under GW noise. In a study by Li *et al.*, [29] the SPB behavior of a DVdP system with FD was examined under both multiplicative and additive colored NE. Variations in parameters that induce SPB behavior were identified. A number of non-linear vibration systems [30-38], including those with extensive applications in energy harvesting and control, have been the subject of extensive study. These include the six DoF system and the 3-DoF auto-parametric system.

Due to the intricate nature of FD, qualitative analysis of parametric influences on vibration is feasible. However, determining critical parameter conditions is challenging yet crucial for analyzing and designing fractional-order systems. It is of paramount importance to consider the critical parameter conditions when analyzing and designing fractional-order systems. This paper examines the non-linear vibration of fractional-order stochastic systems by evaluating the FD and the impact of noise, utilizing a generalized and multistable Rayleigh system with a fractional element. The singularity method and stochastic averaging method (SAM) were employed to determine the critical parametric conditions for SPB. Subsequently, an analysis of the sPDF of the original system across different regions of the parametric plane was conducted.

Establishment of the comparable system

The initial conditions of the Caputo FD system, as elucidated in [28], not only impart a distinctive physical meaning to the system but also give rise to an integer-order differential equation. Hence, we used the Caputo FD in this study:

$$
\begin{gathered}
{}_{a}^{C}\mathrm{D}^{p}[x(t)] = \frac{1}{\Gamma(m-p)} \int_{a}^{t} \frac{x^{(m)}(u)}{(t-u)^{1+p-m}}\,\mathrm{d}u, \quad m < p < m+1 \\
{}_{a}^{C}\mathrm{D}^{p}[x(t)] = x^{(m)}(t), \quad p = m
\end{gathered}
\tag{1}
$$

where p represents the order of FD:

$$
{}_{a}^{C}D^{p}[x(t)], \quad m-1 < p \le m, \quad m \in N, \quad t \in [a,b], \quad x^{(m)}(t)
$$

is the m-order derivative of $x(t)$ and $\Gamma(m)$ is the Euler Gamma function.

As for a determinate physical system, the initial motion time of the oscillator is $t = 0$, and typically, the Caputo FD is usually adopted:

$$ {}_0^C\mathrm{D}^p[x(t)] = \frac{1}{\Gamma(m-p)}\int_0^t \frac{x^{(m)}(u)}{(t-u)^{1+p-m}}\,\mathrm{d}u \tag{2}$$

where $m - 1 < \leq m,\ m \in N$.

This research explores the generalized Rayleigh system featuring a fractional damping element perturbed by GW noise:

$$\ddot{x} - (-\varepsilon + \alpha_1\dot{x}^2 - \alpha_2\dot{x}^4)\dot{x} + w^2 x + \beta\, {}_0^C\mathrm{D}^p x = \xi(t) \tag{3}$$

where ε denotes the linear damping coefficient, α_1, α_2 stand for the system non-linear damping coefficients, and w represents the system intrinsic frequency. The:

$$ {}_0^C\mathrm{D}^p[x(t)]$$

denotes the p ($0 \leq p \leq 1$) order Caputo derivative of $x(t)$, and $\xi(t)$ represents the GW noise, satisfying:

$$E[\xi(t)] = 0,\ E[\xi(t)\xi(t-\tau)] = 2D\delta(\tau) \tag{4}$$

where D indicates the intensity of the GW noise $\xi(t)$, and $\delta(\tau)$ is the Dirac function.

Supposing that the FD encompasses both restoring and damping forces [39-42], the equivalent system can be described:

$$\ddot{x}(t) - [-\varepsilon + \alpha_1\dot{x}^2 - \alpha_2\dot{x}^4 + C(p,w)]\dot{x} + [K(p,w) + w^2]x = \xi(t) \tag{5}$$

where $C(p,w)$ and $K(p,w)$ represent the undetermined coefficients of the equivalent restoring and damping forces of :

$$ {}_0^C\mathrm{D}^p[x(t)]$$

The error between systems (3) and (5) is:

$$e = C(p,w)\dot{x} + \beta\, {}_0^C\mathrm{D}^p x - K(p,w)x \tag{6}$$

According to the equivalent theory [43], and minimizing the error (6) in the mean-square-sense, then the indefinite coefficients $C(p,w)$ and $K(p,w)$ are determined by:

$$\frac{\partial E[e^2]}{\partial[C(p,w)]} = 0, \quad \frac{\partial E[e^2]}{\partial[K(p,w)]} = 0 \tag{7}$$

Given that the original system (3) exhibits a stationary solution in the periodic form as described below:

$$x(t) = a(t)\cos\varphi(t) \tag{8}$$

where $\varphi(t) = wt + \theta$, thus:

$$\dot{x}(t) = -wa(t)\sin\varphi(t)\,, \quad \ddot{x}(t) = -w^2 a(t)\cos\varphi(t) \tag{9}$$

By inserting eqs. (6), (8), and (9) into eq. (7), and executing the integral averaging of φ, the ultimate expressions of $C(p,w)$ and $K(p,w)$ can be obtained:

$$C(p,\tau) = -\beta w^{p-1} \sin\left(\frac{p\pi}{2}\right), \quad K(p,\tau) = \beta w^p \cos\left(\frac{p\pi}{2}\right) \tag{10}$$

Thus, the equivalent oscillator corresponding to system (5) is:

$$\ddot{x}(t) - \gamma \dot{x} + w_0^2 x = \xi(t) \tag{11}$$

where

$$\begin{gathered} \gamma = -\varepsilon + \alpha_1 \dot{x}^2 - \alpha_2 \dot{x}^4 - \beta w^{p-1} \sin\left(\frac{p\pi}{2}\right) \\ w_0^2 = w^2 + \beta w^p \cos\left(\frac{p\pi}{2}\right) \end{gathered} \tag{12}$$

The sPDF of the system amplitude

To get the sPDF of the SA, we suppose that the system (11) possesses a solution with periodic form. Subsequently, the transformation described by [44] is introduced:

$$\begin{gathered} X = x(t) = a(t) \cos \varPhi(t) \\ Y = \dot{x} = -a(t) w_0 \sin \varPhi(t) \\ \varPhi(t) = w_0 t + \theta(t) \end{gathered} \tag{13}$$

where w_0 denotes intrinsic frequency of the isovalent system (11), $a(t)$ and $\theta(t)$ denote the magnitude and temporal alignment characteristics of the system's response, in that order, and they are both random processes.

By inserting eq. (13) into eq. (11) and employing the deterministic averaging approach, we achieve:

$$\begin{gathered} \frac{\mathrm{d}a}{\mathrm{d}t} = F_{11}(a,\theta) + \mathrm{G}_{11}(a,\theta)\xi(t) \\ \frac{\mathrm{d}\theta}{\mathrm{d}t} = F_{21}(a,\theta) + G_{21}(a,\theta)\xi(t) \end{gathered} \tag{14}$$

in which

$$\begin{gathered} F_{11}(a,\theta) = a \sin^2 \varPhi [-\varepsilon + \alpha_1 a^2 w_0^2 \sin^2 \varPhi - \alpha_2 a^4 w_0^4 \sin^2 \varPhi + \alpha_3 a^6 w_0^6 \sin^2 \varPhi - \\ -\alpha_4 a^8 w_0^8 \sin^8 \varPhi - \beta w^{p-1} \sin\left(\frac{p\pi}{2}\right)\Bigg] \\ F_{21}(a,\theta) = \sin \varPhi \cos \varPhi [-\varepsilon + \alpha_1 a^2 w_0^2 \sin^2 \varPhi - \alpha_2 a^4 w_0^4 \sin^2 \varPhi + \\ +\alpha_3 a^6 w_0^6 \sin^2 \varPhi - \alpha_4 a^8 w_0^8 \sin^8 \varPhi - \beta w^{p-1} \sin\left(\frac{p\pi}{2}\right)\Bigg] \\ G_{11} = -\frac{\sin \varPhi}{w_0} \\ G_{21} = -\frac{\cos \varPhi}{a w_0} \end{gathered} \tag{15}$$

Equation (14) is considered as the Stratonovich-stochastic differential formula [45]. Upon incorporating the respective Wong-Zakai correction term [46], we can derive the relevant Ito stochastic differential formula:

$$\begin{aligned} \mathrm{d}a &= \left[F_{11}(a,\theta) + F_{12}(a,\theta)\right]\mathrm{d}t + \sqrt{2D}\,\mathrm{G}_{11}(a,\theta)\mathrm{d}B(t) \\ \mathrm{d}\theta &= \left[F_{21}(a,\theta) + F_{22}(a,\theta)\right]\mathrm{d}t + \sqrt{2D}G_{21}(a,\theta)\mathrm{d}B(t) \end{aligned} \tag{16}$$

where $B(t)$ is the standard Wiener processes and:

$$\begin{aligned} F_{12}(a,\theta) &= D\frac{\partial G_{11}}{\partial a}G_{11} + D\frac{\partial G_{11}}{\partial \theta}G_{21} \\ F_{22}(a,\theta) &= D\frac{\partial G_{21}}{\partial a}G_{11} + D\frac{\partial G_{21}}{\partial \theta}G_{21} \end{aligned} \tag{17}$$

Based on the SAM [47], and then averaging eq. (16) over Φ, we derive the following averaged Ito differential formula:

$$\begin{aligned} \mathrm{d}a &= m_1(a)\mathrm{d}t + \sigma_1(a)\mathrm{d}B(t) \\ \mathrm{d}\theta &= m_2(a)\mathrm{d}t + \sigma_2(a)\mathrm{d}B(t) \end{aligned} \tag{18}$$

The accurate expression of the averaged diffusion and drift coefficients is determined:

$$\begin{aligned} m_1(a) &= -\frac{1}{2}\left[\beta w^{p-1}\sin\left(\frac{p\pi}{2}\right) + \varepsilon\right]a + \frac{3}{8}\alpha_1 w_0^2 a^3 - \frac{5}{16}\alpha_2 a^5 + \frac{5}{128}\alpha_3 a^7 - \frac{7}{256}\alpha_4 a^9 + \frac{D}{2aw_0^2} \\ \sigma_1^{\,2}(a) &= \frac{D}{w_0^2} \\ m_2(a) &= 0 \\ \sigma_2^{\,2}(a) &= \frac{D}{a^2 w_0^2} \end{aligned} \tag{19}$$

where $w_0^2 = w^2 + \beta w^p \cos(p\pi/2)$.

Equation (19) shows that the averaged Ito equation for $a(t)$ is not dependent of $\theta(t)$, so the random process $a(t)$ represents a 1-D diffusion process. Then the corresponding Fokker-Planck-Kolmogorov (FPK) formula of $a(t)$ is:

$$\frac{\partial p(a,t)}{\partial t} = -\frac{\partial}{\partial a}\left[m_1(a)p(a)\right] + \frac{1}{2}\frac{\partial^2}{\partial a^2}\{[\sigma_1^{\,2}(a)]p(a)\} \tag{20}$$

The conditions for boundary fulfill:

$$\begin{aligned} &p(a) = c,\ \ c \in (-\infty, +\infty) \quad \text{as}\ \ a = 0 \\ &p(a) \to 0,\ \ \partial \overline{p} / \partial a \to 0 \quad \text{as}\ \ a \to \infty \end{aligned} \tag{21}$$

According to the boundary conditions (21), the sPDF of SA is obtained:

$$p(a) = \frac{C}{\sigma_1^{\,2}(a)}\exp\left[\int_0^a \frac{2m_1(u)}{\sigma_1^{\,2}(u)}\,\mathrm{d}u\right] \tag{22}$$

where C is the constant after normalization.

By inserting eq. (19) into eq. (22), the detailed equation for the sPDF of SA is attained:

$$p(a)=\frac{Caw_0^2}{D}\exp\left(-\frac{a^2w_0^2\Delta}{7680D}\right)$$

in which:

$$\Delta=3840\left[\varepsilon+\beta w^{p-1}\sin\left(\frac{p\pi}{2}\right)\right]-1440\alpha_1w_0^2a^2+800\alpha_2w_0^4a^4-525\alpha_3w_0^6a^6+378\alpha_4w_0^8a^8 \quad (23)$$

The SPB analysis of the system amplitude

The SPB phenomenon denotes the variation in the quantity of peaks observed in the sPDF curves. In this section, we utilize singularity theory to discuss the parametric impacts on the SPB behaviors of the system, aiming to determine the crucial parametric conditions.

For simplicity $p(a)$ is presented by:

$$p(a)=CR(a,D,\varepsilon,w,p,\alpha_1,\alpha_2)\exp[Q(a,D,\varepsilon,w,p,\alpha_1,\alpha_2)] \quad (24)$$

where

$$R(a,D,\varepsilon,w,p,\alpha_1,\alpha_2)=\frac{aw_0^2}{D}$$

$$Q(a,D,\varepsilon,w,p,\alpha_1,\alpha_2)=-\frac{a^2w_0^2}{7680D}\begin{Bmatrix}3840\left[\varepsilon+\beta w^{p-1}\sin\left(\frac{p\pi}{2}\right)\right]-1440\alpha_1w_0^2a^2+\\+800\alpha_2w_0^4a^4-525\alpha_3w_0^6a^6+378\alpha_4w_0^8a^8\end{Bmatrix} \quad (25)$$

Based on the singularity theory [48], it is necessary for the sPDF of the SA to satisfy the requirements:

$$\frac{\partial p(a)}{\partial a}=0,\quad \frac{\partial^2p(a)}{\partial a^2}=0 \quad (26)$$

By inserting eq. (24) into eq. (26), the following condition can be attained [25, 29]:

$$H=\left\{R'+RQ'=0, R''+2R'Q'+RQ''+RQ'^2=0\right\} \quad (27)$$

where H denotes the crucial condition for the variations of the quantity of peaks in the PDF curve.

Taken the parameters as $\varepsilon=-0.2$, $\alpha_1=2.45$, $\alpha_2=4.6$, w = 1, and $\beta=1$, based on eq. (27), the boundary set for SPB of the system with p and D are acquired, as depicted in fig. 1.

As depicted in fig. 1, the transition set curve's intercepts at D = 0 represent the bifurcation values p_1 = 0.128, p_2 = 0.329, respectively. Under the influence of additive NE, the boundary set curve of the system (3) takes on an approximately triangular shape. Moreover, the unfolding parametric plane is assigned to 2 sub-regions by the boundary set curve. On the basis of the singularity theory, the topological structure of the sPDF curve at various points (p, D) within the same area is qualitatively similar.

Initially, we investigate the sPDF of amplitude $p(a)$ for one point (p, D) in each of the two sub-regions depicted in fig. 1. Subsequently, we contrast the theoretical solution with the numerical result acquired through Monte-Carlo simulation (MCS) of the initial system (3) utilizing the numerical technique for FD [39]. The respective outcomes are displayed in fig. 2.

As depicted in fig. 2, the parametric area (p, D), where the sPDF curves exhibiting multi-modal are enclosed by the nearly triangular area in fig. 1, and the area 1 can form a bi-modal area of the sPDF curve for SA.

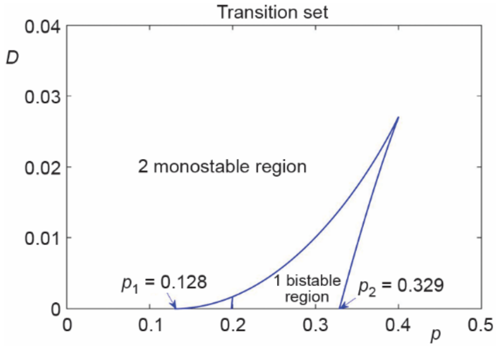

Figure 1. Boundary set under additive NE with *p* and *D* as the unfolding parameters

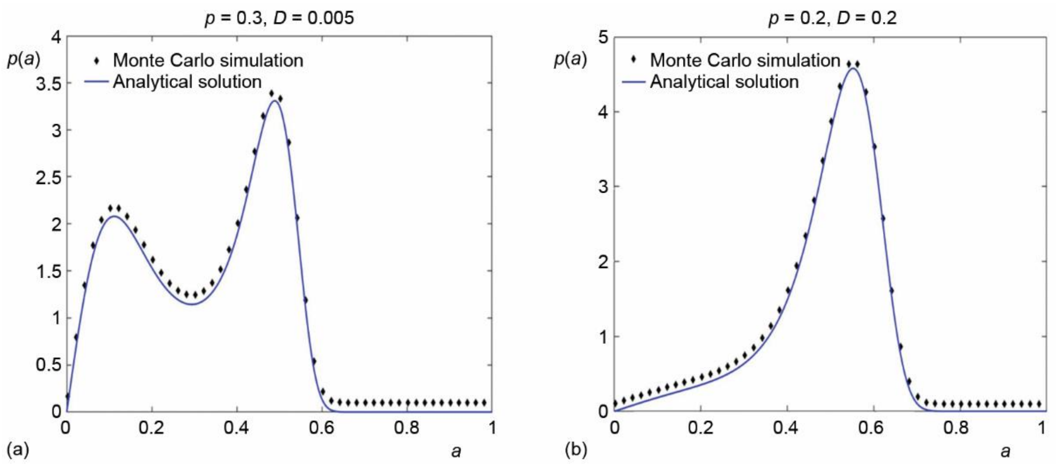

Figure 2. The PDF of amplitude *p*(*a*) in various sub-areas of fig. 1 with *p* and *D* as the unfolding parameters; (a) (*p, D*) in Region 1 of fig. 1 and (b) (*p, D*) in Region 2 of fig. 1

After considering (p, D) as $p = 0.3$, $D = 0.005$ in Region 1, the PDF $p(a)$ of the system has two peaks, and a stable limit cycle emerges with a corresponding amplitude a distant from the original position. Notably, the probability around the origin is non-zero, indicating the coexistence of the equilibrium and limit cycle within the system concurrently, as displayed in fig. 2(a). Conversely, when the $p = 0.2$, $D = 0.002$ in Region 2, the peak of the PDF $p(a)$ is distant from the origin, and a stable limit cycle persists within the system, as illustrated in fig. 2(b). These findings indicate that the sPDF curve of SA can arise in various types depending on the values of noise intensity and FD order. This implies that the sPDF $p(a)$ could be modulated by p and D, respectively. Additionally, a comparison between the numerical data derived from MCS and the analytical solutions derived through stochastic averaging technique demonstrates good alignment, affirming the validity of the conceptual analysis process.

Conclusion

In this study, we examine the SPB behavior of a fractional-order and bistable Rayleigh system under the influence of an additive white noise process. In accordance with the principle of equivalence, we transform the original fractional-order system into an equivalent integer-order system of comparable significance. By employing the SAM, we derive the sPDF of the SA. Moreover, employing singularity theory, we establish crucial parameter conditions for the system's SPB, offering valuable theoretical insights for system design. The congruence between the numerical findings obtained *via* MCS and the analytical solutions provides compelling evidence for the validity of our theoretical analysis. Our findings indicate that both the FD order and noise intensity can induce the SPB phenomenon, resulting in a transition from a single to a dual-peak distribution in the sPDF curve of the system, contingent upon suitable unfolding parameters. The present work can be extended to a generalized Rayleigh system with quadratic non-linearity or singularity as that discussed in [49, 50].

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