OSCILLATION OF SECOND ORDER NON-LINEAR DIFFERENTIAL EQUATIONS WITH A DAMPING TERM

by

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Original scientific paper
https://doi.org/10.2298/TSCI2503839L

The objective of this paper is to present novel sufficient conditions for the oscillation of all solutions of a class of second-order non-linear differential equations with a damping term. Our oscillation criteria represent an improvement, extension, simplification, and unification of a number of existing ones. The advantages of the obtained results are illustrated by an example.

Key words: *oscillation criteria, differential equation, Riccati transformation*

Introduction

The oscillation of differential equations has a profound physical background, and it is necessary to use differential equations in practical problems such as engineering problems, bacterial cultivation problems, population growth problems, and infectious diseases. The study of oscillation of differential equations has considerable potential for further development and application. The study of second-order differential equations has received considerable attention from researchers. The second-order non-linear differential equations with damping terms studied in this article have a wide range of applications in practical problems such as engineering and fluid dynamics:

$$[r(t)|z'(t)|^{\alpha-1}z'(t)]' + p(t)|z'(t)|^{\alpha-1}z'(t) + q(t)|x[\sigma(t)]|^{\beta-1}x[\sigma(t)] = 0, \quad t \ge t_0 \tag{1}$$

where $z(t) = x(t) + c(t)x[\tau(t)]$, $\alpha > 0$, $\beta > 0$. We assume that the following conditions hold:

$$(h_1) \quad r \in C^1([t_0,\infty), R_+), \quad c \in C^2\{[t_0,\infty), R_+\},$$
$$r'(t) \ge 0, \quad 0 \le c(t) \le 1, \quad c'(t) \ge 0, \quad t \ge t_0$$

$$(h_2) \quad \tau \in C^2\{[t_0,\infty), R\}, \quad \tau(t) \le t, \quad \tau'(t) \ge 0, \quad \lim_{t\to\infty}\tau(t) = \infty, \quad \sigma \in C^1\{[t_0,\infty), R\},$$
$$\sigma'(t) > 0, \quad \sigma(t) \le t, \quad \sigma(t) \le \tau(t), \quad \lim_{t\to\infty}\sigma(t) = \infty, \quad t \ge t_0$$

$$(h_3) \quad p \in C\{[t_0,\infty), [0,\infty)\}, \quad q \in C\{[t_0,\infty), R_+\}$$

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The oscillation of second-order differential equations in the case of $\alpha = \beta$ or $\alpha = 1$, $\beta > 0$, or $\beta \geq \alpha$, or $\alpha = 1$, $0 < \beta < 1$ has been studied in references [1-15]. In this paper, we study eq. (1) in the cases $\beta \geq \alpha$ or $\alpha > \beta$. New oscillation criteria for eq. (1) have been derived. The criteria have been enhanced, expanded, simplified, and unified with a number of existing criteria. The advantages of the results obtained are illustrated by several examples.

Oscillation criteria

In this paper, we study the non-canonical form:

$$\Phi(t) = \int_t^\infty R^{-\frac{1}{\alpha}}(s)\mathrm{d}s < \infty, \quad t \geq t_0 \tag{2}$$

where $R(t) = E(t)r(t)$, , $E(t) = \exp\int_{t_0}^t \frac{p(s)}{r(s)}\mathrm{d}s$.

Lemma 1. If $x(t)$ is an eventually positive solution of eq. (1) and $z'(t) < 0$ for $t \geq t_0$, then:

$$\left\{R(t)[-z'(t)]^\alpha\right\}' - Q(t)z^\beta(t) \geq 0, \quad t \geq t_0 \tag{3}$$

where $Q(t) = E(t)q(t)[1 - c(t)]^\beta$.

Proof. Equation (1) is multiplied by $E(t)$, we have:

$$\left[R(t)|z'(t)|^{\alpha-1}z'(t)\right]' + E(t)q(t)|x[\sigma(t)]|^{\beta-1}x[\sigma(t)] = 0, \quad t \geq t_0 \tag{4}$$

from $z(t) = x(t) + c(t)x[\tau(t)]$ and $z'(t) < 0$ for $t \geq t_0$, we get:

$$z'(t) = x'(t) + c'(t)x[\tau(t)] + c(t)x'[\tau(t)]\tau'(t) < 0, \quad t \geq t_0 \tag{5}$$

Since $x(t)$ is a positive solution to eq. (1), we have $x(t) > 0$, $t \geq t_0$. From (h_2), $\tau'(t) \geq 0$, $t \geq t_0$, $\lim_{t\to\infty}\tau(t) = \infty$, there exists a $t_1 \geq t_0$ such that $\tau(t) \geq t_0$ for $t \geq t_1$, we get $x[\tau(t)] > 0$ for $t \geq t_1$. From (h_1), $c(t) \geq 0$, $c'(t) \geq 0$, $t \geq t_0$, we have: $c(t) \geq 0$, $c'(t) \geq 0$, $\tau'(t) \geq 0$, $x(\tau(t)) > 0$, and $t \geq t_1$.

From eq. (5), we can obtain that at least one of $x'(t) \leq 0$ and $x'(\tau(t)) \leq 0$ holds true, there exists a $t_2 \geq t_1$ such that $x'(t) \leq 0$ for $t \geq t_2$ or here exists a $t_3 \geq t_1$ such that $x'(\tau(t)) \leq 0$ for $t \geq t_3$.

If $x'(t) \leq 0$ for $t \geq t_2$, we get $x'(t) \leq 0$ for $t \geq t_2$.

If $x'(\tau(t)) \leq 0$ for $t \geq t_3$, from (h_2), $\tau(t) \leq t$, $\tau'(t) \geq 0$, $t \geq t_0$, $\lim_{t\to\infty}\tau(t) = \infty$, that is $\tau(t) \geq \tau(t_3)$ for $t \geq t_3$. We get $x'(\tau(t)) \leq 0$ for $t \geq t_3$, that is $x'(\tau(t)) \leq 0$ for $\tau(t) \geq \tau(t_3)$. From (h_2), we have $t \geq t_3 \geq \tau(t_3)$, hence $x'(t) \leq 0$ for $t \geq t_3$.

Let $t_4 = \max\{t_2, t_3\}$, we have $x'(t) \leq 0$ for $t \geq t_4$.

From (h_2), there exists a $t_5 \geq t_4$ such that $\tau(t) \geq t_4$ for $t \geq t_5$, we get:

$$x[\tau(t)] \geq x(t), \quad z(t) \leq x[\tau(t)] + c(t)x[\tau(t)]), \quad t \geq t_5$$

$$x[\sigma(t)] \geq x[\tau(t)] \geq \frac{z(t)}{1 + c(t)} \geq [1 - c(t)]z(t), \quad t \geq t_5$$

from eq. (4), we get:

$$\{-R(t)[-z'(t)]^{\alpha}\}' + E(t)q(t)[1-c(t)]^{\beta} z^{\beta}(t) \le 0, \quad t \ge t_5$$

let $Q(t) = E(t)q(t)[1-c(t)]^{\beta}$, then:

$$\{-R(t)[-z'(t)]^{\alpha}\}' - Q(t)z^{\beta}(t) \ge 0, \quad t \ge t_5$$

Lemma 2. If $x(t)$ is an eventually positive solution of eq. (1) and $z'(t) < 0$ for $t \ge t_0$, then:

$$\{R(t)[z'(t)]^{\alpha}\}' + Q_1(t)z^{\beta}[\sigma(t)] \le 0 \tag{6}$$

where $Q_1(t) = E(t)q(t)\{1-c[\sigma(t)]\}^{\beta}$.

Proof. Equation (1) is multiplied by $E(t)$, we have:

$$\{R(t)[z'(t)]^{\alpha}\}' + E(t)q(t)x^{\beta}[\sigma(t)] = 0, \quad t \ge t_0 \tag{7}$$

From $z(t) = x(t) + c(t)x[\tau(t)]$ and (h_1), $0 \le c(t) \le 1, t \ge t_0$, there exists a $t_1 \ge t_0$ such that $\tau(t) \ge t_0$ for $t \ge t_1$, we have $x[\tau(t)] > 0$ for $t \ge t_1$, that is:

$$x[\tau(t)] > 0, \quad x(t) > 0, \quad c(t) \ge 0, \quad t \ge t_1$$

We get $z(t) \ge x(t)$ for $t \ge t_1$. From (h_1), there exists a $t_6 \ge t_1$ such that $\tau(t) \ge t_1$ for $t \ge t_6$, we have $z[\tau(t)] \ge x[\tau(t)]$ for $t \ge t_6$. Since $z'(t) > 0$ for $t \ge t_0$, we get $z[\tau(t)] \le z(t)$ for $t \ge t_6$, then:

$$x(t) = z(t) - c(t)x[\tau(t)] \ge z(t) - c(t)z(t), \quad t \ge t_6$$

from (h_2), there exists a $t_7 \ge t_6$ such that $\sigma(t) \ge t_6$ for $t \ge t_7$, we get:

$$x[\sigma(t)] \ge \{1-c[\sigma(t)]\}z[\sigma(t)], \quad t \ge t_7$$

from eq. (7), we have:

$$\{R(t)[z'(t)]^{\alpha}\}' + E(t)q(t)\{1-c[\sigma(t)]\}^{\beta} z^{\beta}[\sigma(t)] \le 0, \quad t \ge t_7$$

let $Q_1(t) = E(t)q(t)\{1-c[\sigma(t)]\}^{\beta}$, we get:

$$\{R(t)[z'(t)]^{\alpha}\}' + Q_1(t)z^{\beta}[\sigma(t)] \le 0, \quad t \ge t_7$$

Theorem 1. Hypothesis (2) holds, if there exists a function $\rho(t) \in C^1\{[t_0,\infty), R_+\}$ such that:

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left(\rho(s)Q_1(s) - \frac{\lambda^{\lambda}[\rho'(s)]^{\lambda+1}}{(\lambda+1)^{\lambda+1}[\rho(s)G(s)]^{\lambda}} \right) \mathrm{d}s = \infty \tag{8}$$

$$\limsup_{t\to\infty} \int_{T}^{s} \left(\frac{1}{R(s)} \int_{T}^{s} Q(u)\Phi^{\beta}(u)\mathrm{d}u \right)^{\frac{1}{\alpha}} \mathrm{d}s = \infty, \quad T \ge t_0 \tag{9}$$

where $\lambda = \min\{\alpha, \beta\}$, $G(t) = \dfrac{\beta k \sigma'(t)}{R^{\frac{1}{\lambda}}(t)}$. Then eq. (1) is oscillatory.

Proof. We use the method of proof to the contrary, suppose eq. (1) has non-oscillatory solutions $x(t)$, suppose $x(t) > 0$ for $t \ge t_0$, we have:

$$[R(t)|z'(t)|^{\alpha-1} z'(t)]' \le 0, \quad t \ge t_0$$

where $R(t)|z'(t)|^{\alpha-1} z'(t)$ is decreasing function. Therefore, $z'(t)$ is also of one sign. Since this article only considers the non trivial solution of eq. (1), there exists a $t_8 \ge t_0$ such that $z'(t) > 0,\ t \ge t_8$ or there exists a $t_9 \ge t_0$ such that $z'(t) < 0, t \ge t_9$.
(i) If $z'(t) > 0$ for $t \ge t_8$, we define:

$$w(t) = \frac{R(t)[z'(t)]^\alpha}{z^\beta[\sigma(t)]}$$

If $\beta \ge \alpha$, from *Lemma 2*, we have:

$$w'(t) = \frac{\{R(t)[z'(t)]^\alpha\}}{z^\beta[\sigma(t)]} - \frac{\beta R(t)[z'(t)]^\alpha z'[\sigma(t)]\sigma'(t)}{z^{\beta+1}[\sigma(t)]} \le$$
$$\le -Q_1(t) - \frac{\beta\sigma'(t)}{R^{\frac{1}{\alpha}}(t)} \frac{z'[\sigma(t)]}{z'(t)} [z(\sigma(t)]^{\frac{\beta-\alpha}{\alpha}} w^{\frac{\alpha+1}{\alpha}}(t), \quad t \ge t_8$$

Since $R'(t) \ge 0$ for $t \ge t_0$, , from (h_2), there exists a $t_{10} \ge t_8 \ge t_0$ such that $\sigma(t) \ge t_8$ for $t \ge t_{10}$, we get $R(t) \ge R[\sigma(t)]$ for $t \ge t_{10}$. Since $\{R(t)[z'(t)]^\alpha\}' \le 0$ for $t \ge t_0$, we have:

$$R(t)[z'(t)]^\alpha \le R[\sigma(t)]\{z'[\sigma(t)]\}^\alpha, \quad \frac{z'[\sigma(t)]}{z'(t)} \ge \left\{\frac{R(t)}{R[\sigma(t)]}\right\}^{\frac{1}{\alpha}} \ge 1, \quad t \ge t_{10}$$

Since $z'(t) > 0$ for $t \ge t_8$, we have $\sigma(t) \ge t_8$ and $\sigma(t) \ge \sigma(t_{10}) \ge t_8 \ge t_0$ for $t \ge t_{10}$, hence $z[\sigma(t)] \ge z[\sigma(t_{10})]$ for $t \ge t_{10}$. Let $k_1 = \min(1, \{z[\sigma(t_{10})]^{\beta-\alpha/\alpha}\})$, we get:

$$w'(t) \le -Q_1(t) - \frac{\beta k_1 \sigma'(t)}{R^{\frac{1}{\alpha}}(t)} w^{\frac{\alpha+1}{\alpha}}(t), \quad t \ge t_{10}$$

If $\alpha > \beta$, we have:

$$w'(t) \le -Q_1(t) - \frac{\beta\sigma'(t) z'[\sigma(t)]}{R^{\frac{1}{\beta}}(t)[z'(t)]^{\frac{\alpha}{\beta}}} w^{\frac{\beta+1}{\beta}}(t)$$

from $\{R(t)[z'(t)]^\alpha\}' \le 0,$ we have $z''(t) \le 0$ for $t \ge t_8$, hence $z'[\sigma(t)] \ge z'(t)$ and $z'(t) \le z'(t_{10})$ for $t \ge t_{10}$, therefore we have:

$$\frac{z'[\sigma(t)]}{[z'(t)]^{\frac{\alpha}{\beta}}} \ge \frac{z'(t)}{[z'(t)]^{\frac{\alpha}{\beta}}} \ge \frac{1}{[z'(t_{10})]^{\frac{\alpha}{\beta}-1}}, \quad t \ge t_{10}$$

let $k_2 = \min\{1, [z'(t_{10})]^{-\frac{\alpha}{\beta}+1}\}$, we get:

$$w'(t) \le -Q_1(t) - \frac{\beta k_2 \sigma'(t)}{R^{\frac{1}{\beta}}(t)} w^{\frac{\beta+1}{\beta}}(t), \quad t \ge t_{10}$$

synthesis $\beta \ge \alpha$ and $\alpha > \beta$, let $\lambda = \min\{\alpha, \beta\}$, $k = \min\{k_1, k_2\}$, we have:

$$w'(t) \le -Q_1(t) - \frac{\beta k \sigma'(t)}{R^{\frac{1}{\lambda}}(t)} w^{\frac{\lambda+1}{\lambda}}(t), \quad t \ge t_{10}$$

let $G(t) = \dfrac{\beta k \sigma'(t)}{R^{\frac{1}{\lambda}}(t)}$, we get:

$$w'(t) \le -Q_1(t) - G(t) w^{\frac{\lambda+1}{\lambda}}(t), \quad t \ge t_{10} \tag{10}$$

multiply (10) by $\rho(t)$, integral this inequality in $[T, t]$, and using the inequality [6]:

$$Bu - Au^{\frac{\lambda+1}{\lambda}} \le \frac{\lambda^\lambda}{(\lambda+1)^{\lambda+1}} \frac{B^{\lambda+1}}{A^\lambda}, \quad \lambda > 0, \quad A > 0, \quad B \in R$$

we get:

$$\rho(T) w(T) \ge \int_T^t \left\{ \rho(s) Q_1(s) - \frac{\lambda^\lambda [\rho'(s)]^{\lambda+1}}{(\lambda+1)^{\lambda+1} [\rho(s) G(s)]^\lambda} \right\} \mathrm{d}s, \quad t \ge t_{10}$$

which contradicts the fact that (8).
(ii) If $z'(t) < 0$ for $t \ge t_9$. From *Lemma 1*, we have $\{R(t)[-z'(t)]^\alpha\}' \ge 0$, hence $R(s)[-z'(s)]^\alpha \ge R(t)[-z'(t)]^\alpha$, $s \ge t \ge T > t_9$, that is:

$$-z'(s) \ge R^{-\frac{1}{\alpha}}(s) R^{\frac{1}{\alpha}}(t)[-z'(t)], \quad s \ge t \ge T > t_9$$

integral this inequality in $[t, u]$ for s,we get:

$$z(t) \ge z(u) + R^{\frac{1}{\alpha}}(t)[-z'(t)] \int_t^u R^{-\frac{1}{\alpha}}(s)\, \mathrm{d}s \ge R^{\frac{1}{\alpha}}(T)[-z'(T)] \int_t^u R^{-\frac{1}{\alpha}}(s) \mathrm{d}s$$

let $u \to \infty$, we have:

$$z(t) \ge R^{\frac{1}{\alpha}}(T)[-z'(T)] \Phi(t) = k_3 \Phi(t), \quad t \ge T > t_9$$

where $k_3 = R^{1/\alpha}(T)[-z'(T)]$, from eq. (3), we get:

$$\{R(t)[-z'(t)]^\alpha\}' \ge Q(t) k_3^\beta \Phi^\beta(t), \quad t \ge T > t_9$$

integral this inequality in $[T, t]$, we have:

$$-z'(t) \ge \left[\frac{k_3^\beta}{R(t)} \int_T^t Q(u) \Phi^\beta(u)\, \mathrm{d}u \right]^{\frac{1}{\alpha}}, \quad t \ge T > t_9$$

integral this inequality in $[t_9, t]$, we get:

$$z(t_9) \ge z(t) + k_3^{\frac{\beta}{\alpha}} \int_{t_9}^t \left[\frac{1}{R(s)} \int_T^s Q(u) \Phi^\beta(u)\, \mathrm{d}u \right]^{\frac{1}{\alpha}} \mathrm{d}s, \quad t \ge T > t_9$$

which contradicts the fact that (9), then eq. (1) is oscillatory.

Lemma 3. If $x(t)$ is an eventually positive solution of eq. (1) and $z'(t)<0$ for $t \ge t_0$, then:

(i) $V(t)\Phi^{\mu}(t)$ is bounded.

(ii) $V'(t) \ge Q(t) + m\beta R^{-\frac{1}{\alpha}}(t)V^{\frac{\mu+1}{\mu}}(t)$ (11)

where

$$V(t) = \frac{R(t)[-z'(t)]^{\alpha}}{z^{\beta}(t)}, \quad \Phi(t) = \int_t^{\infty} R^{-\frac{1}{\alpha}}(s)\,\mathrm{d}s, \;\; \mu = \max\{\alpha, \beta\}, m > 0$$

Proof. (i) From *Lemma 1*, we have $\{R(t)[-z'(t)]^{\alpha}\}' \ge 0$ for $t \ge t_0$, there exists a $T \ge t_9$ such that $R(s)[-z'(s)]^{\alpha} \ge R(t)[-z'(t)]^{\alpha}$ for $s \ge t \ge T \ge t_9$, that is:

$$z'(s) \le R^{-\frac{1}{\alpha}}(s)R^{\frac{1}{\alpha}}(t)z'(t), \quad s \ge t \ge T \ge t_9$$

integral this inequality in $[t,l]$ for s, we get:

$$z(t) \ge R^{\frac{1}{\alpha}}(t)[-z'(t)]\int_t^{l} R^{-\frac{1}{\alpha}}(s)\,\mathrm{d}s, \quad t \ge T \ge t_9$$

let $l \to \infty$, we have:

$$z(t) \ge R^{\frac{1}{\alpha}}(t)[-z'(t)]\Phi(t), \;\; t \ge T \ge t_9 \tag{12}$$

If $\alpha > \beta$, from eq. (12), we have:

$$z^{\alpha}(t) \ge R(t)[-z'(t)]^{\alpha}\Phi^{\alpha}(t) = z^{\beta}(t)V(t)\Phi^{\alpha}(t), \quad z^{\alpha-\beta}(t) \ge V(t)\Phi^{\alpha}(t), \quad t \ge T \ge t_9$$

since $z'(t)<0$,we have $z(t) \le z(T)$ for $t > T$, hence:

$$V(t)\Phi^{\alpha}(t) \le z^{\alpha-\beta}(T), \;\; t \ge T \ge t_9$$

$V(t)\Phi^{\alpha}(t)$ is bounded.

If $\beta \ge \alpha$, from (12), we have $z^{\beta}(t) \ge \{R^{\frac{1}{\alpha}}(t)[-z'(t)]\}^{\beta}\Phi^{\beta}(t),$ hence:

$$1 \ge \frac{\{R^{\frac{1}{\alpha}}(t)[-z'(t)]\}^{\beta}}{R(t)[-z'(t)]^{\alpha}}\frac{R(t)[-z'(t)]^{\alpha}}{z^{\beta}(t)}\Phi^{\beta}(t) = R^{\frac{\beta-\alpha}{\alpha}}(t)[-z'(t)]^{\beta-\alpha}V(t)\Phi^{\beta}(t)$$

$$V(t)\Phi^{\beta}(t) \le \frac{1}{[R(t)(-z'(t)^{\alpha}]^{\frac{\beta-\alpha}{\alpha}}}, \quad t \ge T \ge t_9$$

since $\{R(t)[-z'(t)]^{\alpha}\}' \ge 0,$ there exists a $T_1 \ge T$ such that $R(t)[-z'(t)]^{\alpha} \ge R(T_1)[-z'(T_1)]^{\alpha}$ for $t \ge T_1,$ we have:

$$V(t)\Phi^{\beta}(t) \le \frac{1}{[R(T_1)(-z'(T_1)^{\alpha}]^{\frac{\beta-\alpha}{\alpha}}}, \;\; t \ge T_1$$

Thus, $V(t)\Phi^{\beta}(t)$ is bounded. Then $V(t)\Phi^{\mu}(t)$ is bounded, where $\mu=\max\{\alpha,\beta\}$.
(ii) The following proof (11) is correct, if $\alpha>\beta$, from *Lemma 1*, we have:

$$V'(t)\ge Q(t)+\frac{\beta}{R^{\frac{1}{\alpha}}(t)z^{\frac{\alpha-\beta}{\alpha}}(t)}V^{\frac{\alpha+1}{\alpha}}(t)\,,\quad t\ge t_0$$

since $z'(t)<0$, we have $z(t)\le z(T_1)$ for $t>T_1$, let $m_1=\dfrac{1}{z^{\frac{\alpha-\beta}{\alpha}}(T_1)}$, we get:

$$V'(t)\ge Q(t)+\beta m_1 R^{-\frac{1}{\alpha}}(t)V^{\frac{\alpha+1}{\alpha}}(t)\,,\quad t>T_1$$

If $\beta\ge\alpha$, we have:

$$V'(t)\ge Q(t)+\beta R^{-\frac{1}{\beta}}(t)[-z'(t)]^{\frac{\beta-\alpha}{\beta}}V^{\frac{\beta+1}{\beta}}(t)\,,\quad t>T_1$$

from *Lemma 1*, we have $R(t)[-z'(t)]^{\alpha}\ge R(T_1)[-z'(T_1)]^{\alpha}$ for $t>T_1$, we get:

$$-z'(t)\ge\frac{R^{\frac{1}{\alpha}}(t_0)}{R^{\frac{1}{\alpha}}(t)}[-z'(T_1)]\,,\quad t>T_1$$

$$V'(t)\ge Q(t)+\beta m_2 R^{-\frac{1}{\alpha}}(t)V^{\frac{\beta+1}{\beta}}(t)\,,\quad t>T_1$$

where $m_2=R^{\frac{\beta-\alpha}{\alpha\beta}}(T_1)[-z'(T_1)]^{\frac{\beta-\alpha}{\beta}}$.

Synthesis $\alpha>\beta$ and $\beta\ge\alpha$, let $\mu=\max\{\alpha,\beta\}$, $m=\min\{m_1,m_2\}$, we get:

$$V'(t)\ge Q(t)+m\beta R^{-\frac{1}{\alpha}}(t)V^{\frac{\mu+1}{\mu}}(t)\,,\quad t>T_1$$

Theorem 2. Hypothesis (2) and (8) holds, and the following is satisfied:

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[\Phi^{\mu}(s)Q(s)-\frac{k}{\Phi(s)R^{\frac{1}{\alpha}}(s)}\right]\mathrm{d}s=\infty \tag{13}$$

where $\mu=\max\{\alpha,\beta\}$, $k>0$, then eq. (1) is oscillatory.

Proof. We use the method of proof to the contrary, suppose eq. (1) has non-oscillatory solutions $x(t)$, suppose $x(t)>0$ for $t\ge t_0$. We have $[R(t)|z'(t)^{\alpha-1}|z'(t)]'\le 0$ for $t\ge t_0$. Hence $R(t)|z'(t)|^{\alpha-1}z'(t)$ is decreasing function, therefore $z'(t)$ is also of one sign. From the proof process of *Theorem 1*, there exists a $t_8\ge t_0$ such that $z'(t)>0$ for $t\ge t_8$ or there exists a $t_9\ge t_0$ such that $z'(t)<0$ for $t\ge t_9$.
(i) If $z'(t)>0$ for $t\ge t_8$, the first half of *Theorem 1* proves that it contradicts (8).
(ii) If $z'(t)<0$ for $t\ge t_9$, from (11) of *Lemma 3*, we get:

$$Q(t)\le V'(t)-m\beta R^{-\frac{1}{\alpha}}(t)V^{\frac{\mu+1}{\mu}}(t)\,,\quad t>T_1\,.\,T_1\ge T$$

integral this inequality in $[T_2,t]$ for $t>T_2\ge T_1\ge T\ge t_9\ge t_0$, we get:

$$\int_{T_2}^{t}\Phi^{\mu}(s)Q(s)\mathrm{d}s\le\Phi^{\mu}(t)V(t)-\mu\int_{T_2}^{t}\Phi^{\mu-1}(s)\Phi'(s)V(s)\mathrm{d}s-m\beta\int_{T_2}^{t}\Phi^{\mu}(s)R^{-\frac{1}{\alpha}}(s)V^{\frac{\mu+1}{\mu}}(s)\mathrm{d}s$$

from *Lemma 3*, $V(t)\Phi^{\mu}(t)$ is bounded, there exists a $M>0$ such that $V(t)\Phi^{\mu}(t)\le M$, since $\Phi'(t)=-R^{-1/\alpha}(t)$, we get:

$$\int_{T_2}^{t}\Phi^{\mu}(s)Q(s)\mathrm{d}s\le M+\frac{\mu^{2\mu+1}}{(\mu+1)^{\mu+1}m^{\mu}\beta^{\mu}}\int_{T_2}^{t}\frac{1}{R^{\frac{1}{\alpha}}(s)\Phi(s)}\mathrm{d}s$$

let $k=\dfrac{\mu^{2\mu+1}}{(\mu+1)^{\mu+1}m^{\mu}\beta^{\mu}}$, hence:

$$\int_{T_2}^{t}\left[\Phi^{\mu}(s)Q(s)--\frac{k}{R^{\frac{1}{\alpha}}(s)\Phi(s)}\right]\mathrm{d}s\le M$$

which contradicts the fact that (13), then eq. (1) is oscillatory.

Example. Consider the following differential equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi_{\alpha}[z'(t)]+\frac{2\alpha}{t}\phi_{\alpha}[z'(t)]+t^{\beta}\phi_{\beta}[x(t-2)]=0 \tag{14}$$

Let $z(t)=x(t)+\dfrac{1}{2}x(t-1)$, then:

$$r(t)=1,\ \tau(t)=t-1,\ c(t)=\frac{1}{2},\ p(t)=\frac{2\alpha}{t},\ q(t)=t^{\beta},\ \sigma(t)=t-2$$

Let $t_0=1$, then:

$$E(t)=t^{2\alpha},\ R(t)=t^{2\alpha},\ \int_{t_0}^{\infty}R^{-\frac{1}{\alpha}}(s)\mathrm{d}s=\int_{1}^{\infty}\frac{1}{s^2}\mathrm{d}s=1,\qquad Q(t)=\frac{t^{2\alpha+\beta}}{2^{\beta}}$$

$$Q_1(t)=\frac{t^{2\alpha+\beta}}{2^{\beta}},\ G(t)=\frac{\beta k}{t^{\frac{2\alpha}{\lambda}}},\ \Phi(t)=\frac{1}{t}$$

let $\rho(t)=t$, we have:

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[\rho(s)Q_1(s)-\frac{\lambda^{\lambda}[\rho'(s)]^{\lambda+1}}{(\lambda+1)^{\lambda+1}[\rho(s)G(s)]^{\lambda}}\right]\mathrm{d}s=$$

$$=\limsup_{t\to\infty}\int_{1}^{t}\left[\frac{s^{2\alpha+\beta+1}}{2^{\beta}}-\frac{\lambda^{\lambda}s^{2\alpha-\lambda}}{(\lambda+1)^{\lambda+1}\beta^{\lambda}k^{\lambda}}\right]ds=\infty$$

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[\frac{1}{R(s)}\int_{T}^{s}Q(u)\Phi^{\beta}(u)\,\mathrm{d}u\right]^{\frac{1}{\alpha}}\mathrm{d}s=\limsup_{t\to\infty}\int_{1}^{t}\frac{(s^{2\alpha+1}-T^{2\alpha+1})^{\frac{1}{\alpha}}}{2^{\frac{\beta}{\alpha}}(2\alpha+1)^{\frac{1}{\alpha}}s^2}\mathrm{d}s=\infty$$

Then the conditions of *Theorem 1* are satisfied, then eq. (14) is oscillatory.

Conclusion

This paper examines the oscillation of a class of non-linear differential equations with damping terms. By employing the generalized Riccati transformation technique and certain specialized techniques, a novel oscillation criterion for the differential equation was derived. The results have potential applications to non-linear oscillators [16-18] to find the criterion of the period motion of a non-linear vibration system, for example, MEMS systems [19-22].

Acknowledgment

This work was supported by Chinese Natural Science Foundation Grant No. 62462056，the Yunnan Fundamental Research Projects under Grant 202501AT070393，the Special Basic Cooperative Research Innovation Programs of Qujing Science and Technology Bureau & Qujing Normal University Grant No. KJLH2023YB12, the Basic Research Joint Project of Local Undergraduate Universities in Yunnan Province Grant No. 202401BA070001-135, the Scientific Research Foundation of the Yunnan Provincial Department of Education Grant No. 2023J1010, 2024J0932, the Yunnan Education Science Planning Fund Grant No. BE22034 and the Yunnan College of Business Management Research Fund Grant No. 2025XK13, 2024JK16.

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Paper submitted: October 17, 2023
Paper revised: June 7, 2024
Paper accepted: June 7, 2024