SEVERAL FRACTIONAL INTEGRAL FORMULAS AND INTEGRAL TRANSFORMS OF THE HYPERGEOMETRIC SUPERCOSINE FUNCTION

by

Lu-Lu GENG, Wen-Jie LIU*, He-He GENG, and Kang-Kang GENG

School of Mathematics and Physics, Nanjing Institute of Technology,
Nanjing, Jiangsu, China

Original scientific paper
https://doi.org/10.2298/TSCI2503829G

In this paper, we propose fractional integral formulas of the hypergeometric supercosine involving Gauss hypergeometric series, derived from the Riemann--Liouville, Erdelyi-Kober type, and Weyl fractional integral operators. Furthermore, we demonstrate several integral transforms of the hypergeometric supercosine.

Key words: *fractional integral operators, beta transform, laplace transform, Gauss hypergeometric series, special function*

Introduction

Special functions are particular mathematical functions, including gamma [1], beta [2], Gauss hypergeometric [3], Mittag-Leffer [4], Bessel [5], and hypergeometric supergeometric functions [6-8], which have been commonly employed in the domains of mathematical analysis, mathematical physics, and engineering science [9, 10].

In the 1820's, Euler proposed the gamma function, Bessel function, and elliptic integral while studying differential equations. Among these, the gamma function exhibited a particularly strong interpolation property [11]. This distinctive attribute not only reinforces the theoretical underpinnings of mathematics but also has a multitude of applications in the realms of physics and engineering. Towards the end of the 18^{th} century, Legendre was proposed for research in astronomy [12]. In the 1820's, the Legendre function was proposed in conjunction with the development of harmonic analysis [13]. A variety of orthogonal polynomials have been proposed. By the end of the 19^{th} century, the framework of special functions that is currently in use had been established to a significant extent [14]. At the outset of the 20^{th} century, the theory and application of special functions began to permeate the field of physics to a greater extent [15].

By means of the exponential, Kohlrausch-Williams-Watts (KWW), and Mittag-Leffler functions, Gauss hypergeometric series, Kummer confluent hypergeometric series, and Y function, Yang [16] studied the definition and properties of a series of new special functions. It was intimately connected to scaling laws and Caputo fractional derivatives, differential equations, Turan-type inequalities, integral expressions, integral transforms, and differential and integral operators.

* Corresponding author, e-mail: ricardo_liu@163.com

Additionally, the authors investigated the significant conjectures surrounding the subtrigonometric functions based on the exponential function, which have been demonstrated to possess zeros. Moreover, the authors constructed mathematical models for the fractional-order dynamics, generalized heat-conduction equation, and one-dimensional heat equation using new special functions. In addition, Yang [17-19] not only investigated the interrelationships between the Y, Fox H, and Meijer G functions, Wright generalized hypergeometric function, Clausen hypergeometric function, and exponential, Mittag-Leffler, Wiman, Prabhakar, and KWW functions, but also studied the representation of the Lambda function and associated results such as the Riemann, Jensen, and Newman conjectures. By considering the initial and boundary conditions of Dirichlet-type, Newman-type, and Cauchy-type, Yang [17-19] constructed mathematical models for the heat equations with solutions for the family of entire functions. He also constructed a connection among analytical number theory, Fourier transform (the sine integral transform), and partial differential equation (the heat-conduction equation).

Throughout this paper, let $\mathbb{C}$, $\mathbb{R}$, and $\mathbb{N}$ be the sets of the complex, real, and natural numbers, respectively.

Let us recall the fractional integral operators involving Gauss hypergeometric function [20]:

$$[\mathrm{I}_{0,x}^{\kappa,\iota,\mu} f(t)](x) = \frac{x^{-\kappa-\iota}}{\Gamma(\kappa)} \int_0^x (x-t)^{\kappa-1} {}_2\mathrm{F}_1\left[\kappa+\mu, -\mu; \kappa; 1-\frac{t}{x}\right] f(t)\mathrm{d}t \tag{1}$$

and

$$[\mathrm{J}_{x,\infty}^{\kappa,\iota,\mu} f(t)](x) = \frac{1}{\Gamma(\kappa)} \int_x^\infty (t-x)^{\kappa-1} t^{-\kappa-\iota} {}_2\mathrm{F}_1\left[\kappa+\iota, -\mu; \kappa; 1-\frac{t}{x}\right] f(t)\mathrm{d}t \tag{2}$$

where $x>0$, $\kappa, \mu, \iota \in \mathbb{C}$, $\Re(\kappa)>0$, and ${}_2\mathrm{F}_1[.]$ is Gauss hypergeometric function.

Meanwhile, the Erdelyi-Kober type fractional integral operators are given by [21]:

$$(\mathrm{E}_{0,x}^{\kappa,\mu} f)(x) = \frac{x^{-\kappa-\mu}}{\Gamma(\kappa)} \int_0^x (x-t)^{\kappa-1} f(t)\mathrm{d}t \ [\Re(\kappa)>0] \tag{3}$$

and

$$(\mathrm{K}_{x,\infty}^{\kappa,\mu} f)(x) = \frac{x^{\mu}}{\Gamma(\kappa)} \int_x^\infty (t-x)^{\kappa-1} t^{-\kappa-\mu} f(t)\mathrm{d}t \ [\Re(\kappa)>0] \tag{4}$$

In addition, the Riemann-Liouville fractional integral operator is [22, 23]:

$$(\mathrm{R}_{0,x}^{\kappa} f)(x) = \frac{1}{\Gamma(\kappa)} \int_0^x (x-t)^{\kappa-1} f(t)\mathrm{d}t \tag{5}$$

and Weyl fractional integral operator [15]:

$$(\mathrm{W}_{x,\infty}^{\kappa} f)(x) = \frac{1}{\Gamma(\kappa)} \int_x^\infty (x-t)^{\kappa-1} f(t)\mathrm{d}t \tag{6}$$

Gamma function $\Gamma(\kappa)$ is introduced by [24, 25]:

$$\Gamma(\kappa)=\int_0^\infty e^{-t}t^{\kappa-1}\mathrm{d}t\ [\Re(\kappa)>0] \tag{7}$$

Hypergeometric supercosine *via* Gauss hypergeometric series is proposed by (see [26] p. 23):

$${}_2\mathrm{Supercos}_1(\alpha,\beta;\gamma;x)=\sum_{n=0}^{\infty}\frac{(\alpha)_{2n}(\beta)_{2n}}{(\gamma)_{2n}}\frac{(-1)^n x^{2n}}{(2n)!} \tag{8}$$

where $\alpha,\beta,\gamma,x\in\mathbb{C}$, $n\in\mathbb{N}$, $|x|<1$, $\kappa\in\mathbb{C}$, $\mu\in\mathbb{N}$, and the Pochhammer symbol $(\kappa)_\mu$ [27] is expressed:

$$(\kappa)_\mu=\frac{\Gamma(\kappa+\mu)}{\Gamma(\kappa)}=\begin{cases}1\ (\mu=0),\\ \kappa(\kappa+1)\cdots(\kappa+\mu-1)\ (\mu\in\mathbb{N})\end{cases} \tag{9}$$

The Beta transform is defined by [28]:

$$\mathrm{B}\{f(z):a,b\}=\int_0^1 z^{a-1}(1-z)^{b-1}f(z)\mathrm{d}z \tag{10}$$

The Laplace and generalized integral transform are widely used independently in engineering for linear differential equations including fractional differential equations [29]. The Laplace transform is given by [30]:

$$\mathfrak{L}\{f(t)\}=\int_0^\infty e^{-st}f(t)\mathrm{d}t \tag{11}$$

and the corresponding inverse Laplace transform by:

$$f(t)=\mathfrak{L}^{-1}\{\mathrm{F}(s)\}=\frac{1}{2\pi i}\lim_{T\to\infty}\int_{r-iT}^{r+iT}e^{st}f(s)\mathrm{d}s \tag{12}$$

Preliminaries

In this section, let’s recall some important lemmas

Lemma 1 [31] Suppose $\kappa,\iota,\mu,\theta\in\mathbb{C}$. Then:

$$(\mathrm{I}_{0,x}^{\kappa,\iota,\mu}t^{\theta-1})(x)=x^{\theta-\iota-1}\frac{\Gamma(\theta)\Gamma(\theta+\mu-\iota)}{\Gamma(\theta-\iota)\Gamma(\kappa+\mu+\theta)} \tag{13}$$

and

$$(\mathrm{J}_{x,\infty}^{\kappa,\iota,\mu}t^{\theta-1})(x)=x^{\theta-\iota-1}\frac{\Gamma(\iota-\theta+1)\Gamma(\mu-\iota+1)}{\Gamma(1-\theta)\Gamma(\kappa+\iota+\mu-\theta+1)} \tag{14}$$

are true.

Lemma 2 [31] If $\kappa,\mu,\theta\in\mathbb{C}$, then we have:

$$(\mathrm{E}_{0,x}^{\kappa,\mu}t^{\theta-1})(x)=x^{\theta-1}\frac{\Gamma(\mu+\theta)}{\Gamma(\kappa+\mu+\theta)} \tag{15}$$

and

$$(\mathrm{K}_{x,\infty}^{\kappa,\mu}t^{\theta-1})(x)=x^{\theta-1}\frac{\Gamma(\mu-\theta+1)}{\Gamma(\kappa+\mu-\theta+1)} \tag{16}$$

Lemma 3 [31] Let $\kappa,\theta \in \mathbb{C}$. Then we have:

$$(\mathrm{R}_{0,x}^{\kappa} t^{\theta-1})(x) = x^{\theta+\kappa-1} \frac{\Gamma(\theta)}{\Gamma(\theta+\kappa)} \tag{17}$$

and

$$(\mathrm{W}_{x,\infty}^{\kappa} t^{\theta-1}) = x^{\theta+\kappa-1} \frac{\Gamma(1-\kappa-\theta)}{\Gamma(1-\theta)} \tag{18}$$

Fractional integral operators of hypergeometric supercosine

In this section, we consider the Riemann-Liouville, Erdelyi-Kober, and Weyl fractional integral operators, proposing some fractional integral formulas of hypergeometric supercosine based on Gauss hypergeometric series.

Theorem 1 Suppose $\kappa,\iota,\mu,\theta \in \mathbb{C}$, $\Re(\kappa) > 0$, $\Re(\theta) > \max[0, \Re(\iota-\mu)]$, and $\Re(\gamma) > \Re(\beta) > 0$. Then:

$$\begin{aligned}&[\mathrm{I}_{0,x}^{\kappa,\iota,\mu} t^{\theta-1} {}_2\mathrm{Supercos}_1(\alpha,\beta;\gamma;t)](x) =\\ &= x^{\theta-\iota-1} \frac{\Gamma(\theta)\Gamma(\theta+\mu-\iota)}{\Gamma(\theta-\iota)\Gamma(\kappa+\mu+\theta)} {}_2\mathrm{Supercos}_1(\alpha,\beta;\gamma;x)\\ &* {}_2\mathrm{Supercos}_2(\theta,\theta+\mu-\iota;\theta-\iota,\kappa+\mu+\theta;x)\end{aligned} \tag{19}$$

Proof. According to (8) and then changing the order of integration and summation, we have:

$$[\mathrm{I}_{0,x}^{\kappa,\iota,\mu} t^{\theta-1} {}_2\mathrm{Supercos}_1(\alpha,\beta;\gamma;t)](x) = \sum_{n=0}^{\infty} \left[\frac{(\alpha)_{2n}(\beta)_{2n}}{(\gamma)_{2n}} \frac{(-1)^n}{(2n)!} [\mathrm{I}_{0,x}^{\kappa,\iota,\mu} t^{\theta+2n-1} \right](x) \tag{20}$$

Based on (13), we obtain:

$$\begin{aligned}&\sum_{n=0}^{\infty} \left[\frac{(\alpha)_{2n}(\beta)_{2n}}{(\gamma)_{2n}} \frac{(-1)^n}{(2n)!} [\mathrm{I}_{0,x}^{\kappa,\iota,\mu} t^{\theta+2n-1} \right](x) =\\ &= x^{\theta-\iota-1} \sum_{n=0}^{\infty} \frac{(\alpha)_{2n}(\beta)_{2n}}{(\gamma)_{2n}} \frac{\Gamma(\theta+2n)\Gamma(\theta+\mu-\iota+2n)}{\Gamma(\theta-\iota+2n)\Gamma(\kappa+\mu+\theta+2n)} \frac{(-1)^n x^{2n}}{(2n)!} =\\ &= x^{\theta-\iota-1} \frac{\Gamma(\theta)\Gamma(\theta+\mu-\iota)}{\Gamma(\theta-\iota)\Gamma(\kappa+\mu+\theta)} \sum_{n=0}^{\infty} \frac{(\alpha)_{2n}(\beta)_{2n}}{(\gamma)_{2n}} \frac{(\theta)_{2n}(\theta+\mu-\iota)_{2n}}{(\theta-\iota)_{2n}(\kappa+\mu+\theta)_{2n}} \frac{(-1)^n x^{2n}}{(2n)!}\end{aligned} \tag{21}$$

Therefore, in view of (21), we acquire:

$$\begin{aligned}&[\mathrm{I}_{0,x}^{\kappa,\iota,\mu} t^{\theta-1} {}_2\mathrm{Supercos}_1(\alpha,\beta;\gamma;t)](x) =\\ &= x^{\theta-\iota-1} \frac{\Gamma(\theta)\Gamma(\theta+\mu-\iota)}{\Gamma(\theta-\iota)\Gamma(\kappa+\mu+\theta)} {}_2\mathrm{Supercos}_1(\alpha,\beta;\gamma;x)\\ &* {}_2\mathrm{Supercos}_2(\theta,\theta+\mu-\iota;\theta-\iota,\kappa+\mu+\theta;x)\end{aligned} \tag{22}$$

Theorem 2 Let $\kappa,\iota,\mu,\theta\in\mathbb{C}$, $\Re(\kappa)>0$, and $\Re(c)>\Re(b)>0$. Then:

$$\left[\mathrm{J}_{x,\infty}^{\kappa,\iota,\mu}t^{\theta-1}{}_2\mathrm{Supercos}_1\left(\alpha,\beta;\gamma;\frac{1}{t}\right)\right](x)= = x^{\theta-\iota-1}\frac{\Gamma(\iota-\theta+1)\Gamma(\mu-\theta+1)}{\Gamma(1-\theta)\Gamma(\kappa+\iota+\mu-\theta+1)}{}_2\mathrm{Supercos}_1\left(\alpha,\beta;\gamma;\frac{1}{x}\right) *{}_2\mathrm{Supercos}_2\left(\iota-\theta+1,\mu-\theta+1;1-\theta,\kappa+\iota+\mu-\theta+1;\frac{1}{x}\right) \tag{23}$$

Proof. By (8), we have:

$$\left[\mathrm{J}_{x,\infty}^{\kappa,\iota,\mu}t^{\theta-1}{}_2\mathrm{Supercos}_1\left(\alpha,\beta;\gamma;\frac{1}{t}\right)\right](x)= =\sum_{n=0}^{\infty}\frac{(\alpha)_{2n}(\beta)_{2n}}{(\gamma)_{2n}}\frac{(-1)^n}{(2n)!}\mathrm{J}_{x,\infty}^{\kappa,\iota,\mu}t^{\theta-2n-1}(x) \tag{24}$$

What's more, by means of (14), we get:

$$\sum_{n=0}^{\infty}\left[\frac{(\alpha)_{2n}(\beta)_{2n}}{(\gamma)_{2n}}\frac{(-1)^n}{(2n)!}[\mathrm{J}_{x,\infty}^{\kappa,\iota,\mu}t^{\theta-2n-1}\right](x)=$$

$$=x^{\theta-\iota-1}\sum_{n=0}^{\infty}\frac{(\alpha)_{2n}(\beta)_{2n}}{(\gamma)_{2n}}\frac{\Gamma(\iota-\theta+2n+1)\Gamma(\mu-\theta+2n+1)}{\Gamma(-\theta+2n+1)\Gamma(\kappa+\iota+\mu-\theta+2n+1)}\frac{(-1)^n\left(\frac{1}{x}\right)^{2n}}{(2n)!}= \tag{25}$$

$$=x^{\theta-\iota-1}\frac{\Gamma(\iota-\theta+1)\Gamma(\mu-\theta+1)}{\Gamma(1-\theta)\Gamma(\kappa+\iota+\mu-\theta+1)}\sum_{n=0}^{\infty}\frac{(\alpha)_{2n}(\beta)_{2n}}{(\gamma)_{2n}}\frac{(\iota-\theta+1)_{2n}(\mu-\theta+1)_{2n}}{(1-\theta)_{2n}(\kappa+\iota+\mu-\theta+1)_{2n}}\frac{(-1)^n\left(\frac{1}{x}\right)^{2n}}{(2n)!}$$

In view of the Hadamard product [32], we have:

$$\left[\mathrm{J}_{x,\infty}^{\kappa,\iota,\mu}t^{\theta-1}{}_2\mathrm{Supercos}_1\left(\alpha,\beta;\gamma;\frac{1}{t}\right)\right](x)= = x^{\theta-\iota-1}\frac{\Gamma(\iota-\theta+1)\Gamma(\mu-\theta+1)}{\Gamma(1-\theta)\Gamma(\kappa+\iota+\mu-\theta+1)}{}_2\mathrm{Supercos}_1\left(\alpha,\beta;\gamma;\frac{1}{x}\right) *{}_2\mathrm{Supercos}_2\left(\iota-\theta+1,\mu-\theta+1;1-\theta,\kappa+\iota+\mu-\theta+1;\frac{1}{x}\right) \tag{26}$$

As direct results, we have following:

Corollary 1 Let $\kappa,\mu,\theta\in\mathbb{C}$, $\Re(\kappa)>0$, and $\Re(\gamma)>\Re(\beta)>0$. Then:

$$[\mathrm{E}_{0,x}^{\kappa,\mu}t^{\theta-1}{}_2\mathrm{Supercos}_1(\alpha,\beta;\gamma;t)](x)=x^{\theta-1}\frac{\Gamma(\mu+\theta)}{\Gamma(\kappa+\mu+\theta)}{}_2\mathrm{Supercos}_1(\alpha,\beta;\gamma;x) *{}_1\mathrm{Supercos}_1(\mu+\theta;\kappa+\mu+\theta;x) \tag{27}$$

Corollary 2 Suppose $\kappa,\theta \in \mathbb{C}$, $\Re(\kappa)>0$, and $\Re(\gamma)>\Re(\beta)>0$. Then:

$$[\mathrm{R}_{0,x}^{\kappa} t^{\theta-1}{}_2\mathrm{Supercos}_1(\alpha,\beta;\gamma;t)](x)=\frac{\Gamma(\theta)}{\Gamma(\theta+\kappa)}{}_2\mathrm{Supercos}_1(\alpha,\beta;\gamma;x) *_1\mathrm{Supercos}_1(\theta;\theta+\kappa;x) \tag{28}$$

Corollary 3 If $\kappa,\theta \in \mathbb{C}$, $\Re(\kappa)>0$, and $\Re(\gamma)>\Re(\beta)>0$, then:

$$\left[\mathrm{W}_{x,\infty}^{\kappa} t^{\theta-1}{}_2\mathrm{Supercos}_1\left(\alpha,\beta;\gamma;\frac{1}{t}\right)\right](x)=x^{\theta+\kappa-1}\frac{\Gamma(1-\kappa-\theta)}{\Gamma(1-\theta)}{}_2\mathrm{Supercos}_1\left(\alpha,\beta;\gamma;\frac{1}{x}\right) *_1\mathrm{Supercos}_1\left(1-\kappa-\theta;1-\theta;\frac{1}{x}\right) \tag{29}$$

Some integral transforms of hypergeometric supercosine

In this section, the Beta transform and Laplace transform of the hypergeometric supercosine based on Gauss hypergeometric series are introduced.

Theorem 3 Let $\kappa,\iota,\mu,\theta,l \in \mathbb{C}$, $\Re(\kappa)>0$, $\Re(\theta)>\max[0,\Re(\iota-\mu)]$, and $\Re(\gamma)>\Re(\beta)>0$. Then:

$$\begin{aligned}&\mathrm{B}\{[\mathrm{I}_{0,x}^{\kappa,\iota,\mu} t^{\theta-1}{}_2\mathrm{Supercos}_1(\alpha,\beta;\gamma;tz)](x):l,m\}=\\&=x^{\theta-\iota-1}\mathrm{B}(l,m)\frac{\Gamma(\theta)\Gamma(\theta+\mu-\iota)}{\Gamma(\theta-\iota)\Gamma(\kappa+\mu+\theta)}{}_2\mathrm{Supercos}_1(\alpha,\beta;\gamma;x)\\&*_3\mathrm{Supercos}_3(\mu,\theta+\mu-\iota,l;\theta-\iota,\kappa+\mu+\theta,l+m;x)\end{aligned} \tag{30}$$

Proof. Let:

$$\mathfrak{B}=\mathrm{B}\{[\mathrm{I}_{0,x}^{\kappa,\iota,\mu} t^{\theta-1}{}_2\mathrm{Supercos}_1(\alpha,\beta;\gamma;tz)](x):l,m\} \tag{31}$$

Then we have:

$$\mathfrak{B}=\int_0^1 z^{l-1}(1-z)^{m-1}[\mathrm{I}_{0,x}^{\kappa,\iota,\mu} t^{\theta-1}{}_2\mathrm{Supercos}_1(\alpha,\beta;\gamma;tz)](x)\mathrm{d}z \tag{32}$$

After a series of simplifications, we have:

$$\begin{aligned}\mathfrak{B}&=\sum_{n=0}^{\infty}\frac{(\alpha)_{2n}(\beta)_{2n}}{(\gamma)_{2n}}\frac{(-1)^n}{(2n)!}(\mathrm{I}_{0+}^{\kappa,\iota,\mu}t^{2n+\theta-1})(x)\int_0^1 z^{l+2n-1}(1-z)^{m-1}\mathrm{d}z=\\&=x^{\theta-\iota-1}\sum_{n=0}^{\infty}\frac{(\alpha)_{2n}(\beta)_{2n}}{(\gamma)_{2n}}\frac{(-1)^n x^{2n}}{(2n)!}\frac{\Gamma(\theta+2n)\Gamma(\theta+\mu-\iota+2n)}{\Gamma(\theta-\iota+2n)\Gamma(\kappa+\mu+\theta+2n)}\int_0^1 z^{l+2n-1}(1-z)^{m-1}\mathrm{d}z=\\&=x^{\theta-\iota-1}\sum_{n=0}^{\infty}\frac{(\alpha)_{2n}(\beta)_{2n}}{(\gamma)_{2n}}\frac{(-1)^n x^{2n}}{(2n)!}\frac{\Gamma(\theta+2n)\Gamma(\theta+\mu-\iota+2n)}{\Gamma(\theta-\iota+2n)\Gamma(\kappa+\mu+\theta+2n)}\frac{\Gamma(l+2n)\Gamma(m)}{\Gamma(l+m+2n)}=\\&=x^{\theta-\iota-1}\mathrm{B}(l,m)\frac{\Gamma(\theta)\Gamma(\theta+\mu-\iota)}{\Gamma(\theta-\iota)\Gamma(\kappa+\mu+\theta)}\sum_{n=0}^{\infty}\frac{(\alpha)_{2n}(\beta)_{2n}}{(\gamma)_{2n}}\frac{(\theta)_{2n}(\theta+\mu-\iota)_{2n}}{(\theta-\iota)_{2n}(\kappa+\mu+\theta)_{2n}}\frac{(l)_{2n}}{(l+m)_{2n}}\frac{(-1)^n x^{2n}}{(2n)!}\end{aligned}$$

Therefore, we obtain:

$$\begin{aligned}&B\{(\mathrm{I}_{0,x}^{\kappa,\iota,\mu}t^{\theta-1}{}_2\mathrm{Supercos}_1(\alpha,\beta;\gamma;tz))(x):l,m\}=\\&=x^{\theta-\iota-1}\mathrm{B}(l,m)\frac{\Gamma(\theta)\Gamma(\theta+\mu-\iota)}{\Gamma(\theta-\iota)\Gamma(\kappa+\mu+\theta)}{}_2\mathrm{Supercos}_1(\alpha,\beta;\gamma;x)\\&*_3\mathrm{Supercos}_3(\mu,\theta+\mu-\iota,l;\theta-\iota,\kappa+\mu+\theta,l+m;x).\end{aligned}\tag{33}$$

Theorem 4 Let $\kappa,\iota,\mu,\theta,l\in\mathbb{C}$, $\Re(\kappa)>0$, $\Re(\theta)>\max[0,\Re(\iota-\mu)]$, and $\Re(\gamma)>\Re(\beta)>0.$ Then we have:

$$\begin{aligned}&\mathfrak{L}\{z^{l-1}[\mathrm{I}_{0,x}^{\kappa,\iota,\mu}t^{\theta-1}{}_2\mathrm{Supercos}_1(\alpha,\beta;\gamma;tz)](x)\}(s)=\\&=\frac{x^{\theta-\iota-1}}{s^l}\frac{\Gamma(l)\Gamma(\theta)\Gamma(\theta+\mu-\iota)}{\Gamma(\theta-\iota)\Gamma(\kappa+\mu+\theta)}{}_2\mathrm{Supercos}_1(\alpha,\beta;\gamma;\frac{x}{s})\\&*_3\mathrm{Supercos}_2(\theta,\theta+\mu-\iota,l;\theta-\iota,\kappa+\mu+\theta;\frac{x}{s})\end{aligned}\tag{34}$$

Proof. By applying the Laplace transform, we have:

$$\begin{aligned}&\mathfrak{L}\{z^{l-1}[\mathrm{I}_{0,x}^{\kappa,\iota,\mu}t^{\theta-1}{}_2\mathrm{Supercos}_1(\alpha,\beta;\gamma;tz)](x)\}(s)=\\&=\int_0^\infty e^{-sz}z^{l-1}[\mathrm{I}_{0,x}^{\kappa,\iota,\mu}t^{\theta-1}{}_2\mathrm{Supercos}_1(\alpha,\beta;\gamma;tz)(x)]\mathrm{d}z\end{aligned}\tag{35}$$

After a series of simplification, we gain:

$$\begin{aligned}&\int_0^\infty e^{-sz}z^{l-1}[\mathrm{I}_{0,x}^{\kappa,\iota,\mu}t^{\theta-1}{}_2\mathrm{Supercos}_1(\alpha,\beta;\gamma;tz)(x)]\mathrm{d}z=\\&=x^{\theta-\iota-1}\sum_{n=0}^\infty\frac{(\alpha)_{2n}(\beta)_{2n}}{(\gamma)_{2n}}\frac{(-1)^n x^{2n}}{(2n)!}\frac{\Gamma(\theta+2n)\Gamma(\theta+\mu-\iota+2n)}{\Gamma(\theta-\iota+2n)\Gamma(\kappa+\mu+\theta+2n)}\frac{\Gamma(l+2n)}{s^{l+2n}}=\\&=\frac{x^{\theta-\iota-1}}{s^l}\frac{\Gamma(l)\Gamma(\theta)\Gamma(\theta+\mu-\iota)}{\Gamma(\theta-\iota)\Gamma(\kappa+\mu+\theta)}\sum_{n=0}^\infty\frac{(\alpha)_{2n}(\beta)_{2n}}{(\gamma)_{2n}}\frac{(-1)^n x^{2n}}{(2n)!}\frac{(\theta)_{2n}(\theta+\mu-\iota)_{2n}}{(\theta-\iota)_{2n}(\kappa+\mu+\theta)_{2n}}\frac{(l)_{2n}}{s^{2n}}\end{aligned}\tag{36}$$

Thus, we get the following result:

$$\begin{aligned}&\mathfrak{L}\{z^{l-1}[\mathrm{I}_{0,x}^{\kappa,\iota,\mu}t^{\theta-1}{}_2\mathrm{Supercos}_1(\alpha,\beta;\gamma;tz)](x)\}(s)=\\&=\frac{x^{\theta-\iota-1}}{s^l}\frac{\Gamma(l)\Gamma(\theta)\Gamma(\theta+\mu-\iota)}{\Gamma(\theta-\iota)\Gamma(\kappa+\mu+\theta)}{}_2\mathrm{Supercos}_1(\alpha,\beta;\gamma;\frac{x}{s})\\&*_3\mathrm{Supercos}_2(\theta,\theta+\mu-\iota,l;\theta-\iota,\kappa+\mu+\theta;\frac{x}{s}).\end{aligned}\tag{37}$$

Conclusion

This paper presents fractional integral formulas of the hypergeometric supercosine involving Gauss hypergeometric series, and several integral transforms of the hypergeometric supercosine are recommended. Further applications of the integral transforms to solving differential equations are discussed in [33].

Acknowledgement

This work is supported by Basic Science (Natural Science) Research Project of Jiangsu Provincial Colleges and Universities (No. 24KJD110003).

References

[1] Artin, E., *The Gamma Function*, Courier Dover Publications, Minerla, N. Y., USA, 2015
[2] Chaudhry, M. A., *et al.*, Extension of Euler's Beta Function, *Journal of Computational of Computational and Applied Mathematics*, *78* (1997), 1, pp. 19-32
[3] Beukers, F., Gauss' Hypergeometric Function, in: *Arithmetic and Geometry Around Hypergeometric Functions,* Birkhauser, Basel, 2007, pp. 23-42
[4] Popov, A. Y., The Spectral Values of a Boundary Value Problem and the Zeros of Mittag-Leffer Functions, *Differential Equations*, *38* (2002), 5, pp. 642-653
[5] McKay, A. T., A Bessel Function Distribution, *Biometrika*, *24* (1932), May, pp. 39-44
[6] Yang, X. J., *An Introduction to Hypergeometric, Supertrigonometric, and Superhyperbolic Functions*, Academic Press, New York, USA, 2021
[7] Geng, L. L., *et al*., Turan-Type Inequalities for the Supertrigonometric Functions, *Mathematical Methods in the Applied Sciences*, *45* (2022), 7, pp. 3514-3519
[8] Geng, L. L., Yang, X. J., Turan-Type Inequalities for Two Hypergeometric Supertrigonometric Functions Associated with Kummer Confluent Hypergeometric Series of First Type, *International Journal of Geometric Methods in Modern Physics*, *19* (2022), 12, 2250191
[9] Andrews, G. E., *et al.*, *Special Functions*, Cambridge University Press, Cambridge, Mass., USA, 1999
[10] He, J.-H., El-Dib, Y. O., A Tutorial Introduction to the Two-Scale Fractal Calculus and its Application to the Fractal Zhiber-Shabat Oscillator, *Fractals*, *29* (2021), 8, pp. 1-9
[11] Euler, L., Letter to Goldbach, *Correspondence Math. et Phys*., (1843), 1, pp. 549-552
[12] Legendre, A. M., Exercices de Calcul Integral, sur divers Ordres de Transcendantes, et sur les Quadratures, Paris, Mme Ve Courcier, *Imprimeur-Libraire pour les Mathematiques*, 1811
[13] Rosler, M., Dunkl operators: Theory and Applications, Orthogonal Polynomials and Special Functions, *Lecture Notes in Mathematics*, *1817* (2003), 1817, pp. 93-135
[14] Gray, A., Mathews, G. B., *A Treatise on Bessel Functions and Their Applications to Physic*, Macmillan, London, 1895
[15] Nikiforov, A. F., Uvarov, V. B., *Special Functions of Mathematical Physics*, Birkhaauser, Basel, 1988
[16] Yang, X. J., A New Integral Transform Operator for Solving the Heat-Diffusion Problem, *Applied Mathematics Letters*, *64* (2017), Feb., pp. 193-197
[17] Yang, X. J., *et al*., New Special Functions Applied to Represent the Weierstrass-Mandelbrot Function, *Fractals, 32* (2023), 4, 2340113
[18] Yang X. J., The Y Function Applied in the Study of an Anomalous Diffusion, *Thermal Science*, *25* (2021), 6B, pp. 4465-4475
[19] Yang, X. J., *et al*., A New Odd Entire Function of Order one Arising in the Heat problem, *Thermal Science*, *27* (2023), 1B, pp. 505-512
[20] Kober, H., On Fractional Integrals and Derivatives, *The Quarterly Journal of Mathematics*, *11* (1940), 1, pp. 193-211
[21] Sahin, R., Yagcl, O., Fractional Calculus of the Extended Hypergeometric Function, *Applied Mathematics and Non-Linear Sciences*, *5* (2020), 1, pp. 369-384
[22] Nonnenmacher, T. F., Metzler, R., On the Riemann-Liouville Fractional Calculus and Some Recent Applications, *Fractals*, *3* (1995), 3, pp. 557-566
[23] Geng, L. L., Yang, X. J., Novel Chebyshev-Type Inequalities for the General Fractional-Order Integrals with the Rabotnov Fractional Exponential Kernel, *Fractals*, *31* (2023), 9, 2350126
[24] Saichev, A. I., Zaslavsky, G. M., Fractional Kinetic Equations: Solutions and Applications, *Chaos: An Interdisciplinary Journal of Nonlinear Science, 7* (1997), 4, pp. 753-764
[25] Euler, L., *Correspondance Mathematique. et physique.de quelques* $\mathrm{C}\ \forall a,b,c\ (a=b)\wedge(a=c)\Rightarrow a=c$ *lebres Geometres Du 18eme siecle*, Publiee Par Fuss, 1. Letter to Goldbach, 1729
[26] Legendre, A. M., *Exercices de Calcul Integral,* Courcier, Paris, 1814

[27] Yang, X. J., *Theory and Applications of Special Functions for Scientists and Engineers*, Springer, Singapore, 2021
[28] Pochhammer, L., Ueber Hypergeometrischen Funktionen Hoheren Ordungen, *Journal fr die Reine und Angewandte Mathematik*, *71* (1870), 216
[29] He, J.-H., *et al*., Beyong Laplace and Fourier Transforms Challenges and Future Prospects, *Thermal Science*, *27* (2023), 6B, pp. 5075-5089
[30] Laplace, P. S., Théorie Analytique Des Probabilités, *Courcier*, 1820, (First appeared in 1812)
[31] Geng, L. L., *et al*., New Fractional Integral Formulas and Kinetic Model Associated with the Hypergeometric Superhyperbolic Sine Function, *Mathematical Methods in the Applied Sciences*, *46* (2023), 2, pp. 1809-1820
[32] Chand, Prajapati J. C., *et al.*, Fractional Calculus and Applications of Family of Extended Generalized Gauss Hypergeometric Functions, *Discrete and Continuous Dynamical Systems-S*, *13* (2020), 3, pp. 539-560
[33] Anjum, N., *et al.*, Free Vibration of a Tapered Beam by the Aboodh Transform-based Variational Iteration Method, *Journal of Computational Applied Mechanics*, *55* (2024), 3, pp. 440-450

Paper submitted: May 16, 2024
Paper revised: July 6, 2024
Paper accepted: July 6, 2024