ITERATIVE SOLUTION FOR A CLASS OF PARTIAL DIFFERENTIAL EQUATIONS IN FRACTAL SPACES

by

Yi TIAN *

College of Data Science and Application, Inner Mongolia University of Technology, Hohhot, China

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A class of fractal PDE is successfully established by He's fractal derivative in a fractal space, and their variational principles are obtained by the semi-inverse method. The Fourier-Rabbani-He method and the Ritz-like method are used to solve the given fractal equations with initial value conditions. The example is a great demonstration of how the Fourier-Rabbani-He method is a powerful and simple tool that can be used in different ways.

Key words: *Fourier-Rabbani-He Method, semi-inverse transform method, variational principle*

Introduction

A multitude of physical procedures, both natural and artificial, in physics, chemistry, biology, economics, and management can be modeled by PDE. Numerous methods exist for solving non-linear PDE, including the homotopy perturbation method [1-3], variational iteration method [4-6], integral transforms method [7-10], Taylor series method [11-13], and exp-function method [14-16]. In recent times, the integral transforms method has gained considerable popularity as a means of solving differential equations. Among these, the Fourier transformation [17] has demonstrated remarkable versatility, finding applications not only in mathematics but also in other fields such as engineering and physics. Additionally, a range of integral transforms methods, including the Sumudu transform [18], Mohand transform [19], Laplace transform [20], Natural transform [21], and He-transform [22, 23] have been explored, with their properties and applications being extensively investigated by numerous researchers. It is crucial to understand that He-transform [22, 23] is a generalized integral transform that encompasses the Laplace transform, Fourier transform, and other integral transforms as special cases.

In this paper, we apply a hybrid approach called the Fourier-Rabbani-He method (FRHM) [17], which uses the Fourier transform method and HPM to solve a class of PDE in the following form [24]:

$$u_t = g(x,t,u,u_x,u_{xx}), \quad -\infty < x < \infty, \quad t > 0$$

$$u(x,0) = h(x)$$

This hybrid approach offers a clear advantage: it combines two powerful techniques to derive approximate iterative solutions of non-linear problems. This approach provides a so-

* Author's e-mail: ttxsun@163.com

lution in the form of a convergent series with easily computable components, without the need for linearization, perturbation, or restrictive assumptions.

Modified homotopy perturbation method

Consider a non-linear differential equation:

$$\Im(u(x,t)) - f(x,t) = 0, \quad x,t \in \Omega \tag{1}$$

$$\wp\left[u(x,t), \frac{\partial u}{\partial n}\right] = 0, \quad n \in \Gamma \tag{2}$$

where $\Im$ is a general differential operator, $\wp$ – a boundary operator, and $f(x,t)$ – a known function. Operator $\Im$ is decomposed into linear and non-linear operators such as $\bar{L}$ and N, respectively. In the special case, linear operator $\bar{L}$ can be decomposed into $L + R$, where L is the highest order linear differential operator and R – the remainder of that. Thus eq. (1) can be rewritten:

$$\{L[u(x,t)] - f(x,t)\} + \{R[u(x,t)] + N[u(x,t)]\} = 0 \tag{3}$$

We introduce a modified homotopy perturbation (MHP) in the following form [25, 26]:

$$H(v,p) = \{L[v(x,t)] - f(x,t)\} + p\{R[v(x,t)] + N[v(x,t)]\} = 0, \quad p \in [0,1] \tag{4}$$

where v is an approximate solution of eq. (1) and we assume that v is a series in terms of p powers:

$$u(x,t) \simeq v(x,t) = \sum_{i=0}^{\infty} p^i v_i(x,t) \tag{5}$$

The solution of eqs. (1) and (2) are $u(x,t) \simeq \lim_{p \to 1} v(x,t)$.

Fourier-Rabbani-He method

Considering eq. (1) again, if:

$$L[u(x,t)] = \frac{\partial^n}{\partial t^n} u(x,t)$$

according to the FRHM [17], taking Fourier transform of eq. (3), we have:

$$\left\{\frac{\partial^n}{\partial t^n}\mathcal{F}[u(x,t)] - \mathcal{F}[f(x,t)]\right\} + \mathcal{F}\{R[u(x,t)]\} + \mathcal{F}\{N[u(x,t)]\} = 0 \tag{6}$$

Taking inverse Fourier transform of eq. (6), it is concluded that:

$$\left[\frac{\partial^n u(x,t)}{\partial t^n} - f(x,t)\right] + [\mathcal{F}^{-1}(\mathcal{F}\{R[u(x,t)]\}) + \mathcal{F}^{-1}(\mathcal{F}\{N[u(x,t)]\})] = 0 \tag{7}$$

We introduce a MHP:

$$H(v,p)=\left[\frac{\partial^n u(x,t)}{\partial t^n}-f(x,t)\right]+p[\mathcal{F}^{-1}(\mathcal{F}\{R[u(x,t)]\})+\mathcal{F}^{-1}(\mathcal{F}\{N[u(x,t)]\})] \tag{8}$$

where v is an approximation of u and by substituting eq. (5) into eq. (8),we have:

$$H(v,p)=\left[\frac{\partial^n}{\partial t^n}\sum_{i=0}^{\infty}p^i v_i(x,t)-f(x,t)\right]+p\left[\mathcal{F}^{-1}\left(\mathcal{F}\left\{R\left[\sum_{i=0}^{\infty}p^i v_i(x,t)\right]\right\}\right)+\right.$$
$$\left.+\mathcal{F}^{-1}\left(\mathcal{F}\left\{N\left[\sum_{i=0}^{\infty}p^i v_i(x,t)\right]\right\}\right)\right] \tag{9}$$

We apply Adomian decomposition method to convert $N[v(x,t)]$ to sum of some simple Adomian polynomials in this form:

$$N[v(x,t)]=N\left[\sum_{i=0}^{\infty}p^i v_i(x,t)\right]=\sum_{i=0}^{\infty}p^i A_i(x,t) \tag{10}$$

where Adomian polynomials are:

$$A_i(x,t)=\frac{1}{i!}\left\{\frac{d^i}{dp^i}N\left[\sum_{i=0}^{\infty}p^i v_i(x,t)\right]\right\}_{p=0} \tag{11}$$

Putting eq. (10) into eq. (9) and rearranging it in terms of p powers, we can give the following FRH-algorithm:

$$\frac{\partial^n v_0(x,t)}{\partial t^n}=f(x,t)$$

$$\frac{\partial^n v_1(x,t)}{\partial t^n}=-\mathcal{F}^{-1}[\mathcal{F}\{R[v_0(x,t)]\}-A_0(x,t)$$

$$\frac{\partial^n v_i(x,t)}{\partial t^n}=-\mathcal{F}^{-1}(\mathcal{F}\{R[v_{i-1}(x,t)]\}-A_{i-1}(x,t),\quad i=2,3,\cdots$$

Application of Fourier-Rabbani-He method

Consider the following PDE [24]:

$$u_t=uu_x \tag{12}$$

with the initial condition:

$$u(x,0)=0.2x^2 \tag{13}$$

which has the exact solution:

$$u(x,t)=\frac{[1-(0.4)tx]-\sqrt{1-(0.8)tx}}{(0.4)t^2} \tag{14}$$

In a fractal space, eq. (12) can be modified:

$$\frac{\partial u}{\partial t^{\alpha}} = u\frac{\partial u}{\partial x^{\beta}} \tag{15}$$

with the initial condition:

$$u(x^{\beta},0) = 0.2x^{2\beta} \tag{16}$$

where

$$\frac{\partial u}{\partial t^{\alpha}},\ \frac{\partial u}{\partial x^{\beta}}$$

are He's fractal derivatives [27, 28] defined:

$$\frac{\partial u}{\partial t^{\alpha}}(t_0,x) = \Gamma(1+\alpha)\lim_{\substack{t-t_0\to\Delta t\\ \Delta t\neq 0}}\frac{u(t,x)-u(t_0,x)}{(t-t_0)^{\alpha}} \tag{17}$$

$$\frac{\partial u}{\partial x^{\beta}}(t,x_0) = \Gamma(1+\beta)\lim_{\substack{x-x_0\to\Delta x\\ \Delta x\neq 0}}\frac{u(t,x)-u(t,x_0)}{(x-x_0)^{\beta}} \tag{18}$$

Using the two-scale transform method [29, 30] to eq. (15) and assume:

$$T = t^{\alpha} \tag{19}$$

$$X = x^{\beta} \tag{20}$$

where x, t are for the small scale and X, T for large scale, α,β are the two-scale dimensions [31]. Applying eqs. (19) and (20) to eqs. (15) and (16), we have:

$$\frac{\partial u}{\partial T} = u\frac{\partial u}{\partial X} \tag{21}$$

with the initial condition:

$$u(X,0) = 0.2X^{2} \tag{22}$$

We introduce the following operators and function:

$$\begin{gathered} L[u(X,T)] = \frac{\partial u(X,T)}{\partial t},\quad R=0,\quad f(X,T)=0,\\ N[u(X,T)] = -u(X,T)u_X(X,T) \end{gathered} \tag{23}$$

We get the following initial values for $v_i(X,T),\ i=0,1,2,\cdots,$

$$v_0(X,0)=0.2X^2,\quad v_i(X,0)=0,\quad \forall i\geq 1 \tag{24}$$

From FRH-algorithm, we have:

Step 1.

$$\frac{\partial v_0(X,T)}{\partial t} = f(X,T) = 0 \tag{25}$$

From eq. (24), we set $v_0(X,T)=0.2X^2$.
Step 2.

$$\frac{\partial v_1(X,T)}{\partial t}=-A_0(X,T)=v_0(X,T)\frac{\partial v_0(X,T)}{\partial X}=0.08X^3 \tag{26}$$

From eq. (24), we obtain $v_1(X,T)=0.08X^3T$.
Step 3.

$$\begin{aligned}\frac{\partial v_2(X,T)}{\partial t}&=-A_1(X,T)\\&=v_0(X,T)\frac{\partial v_1(X,T)}{\partial X}+v_1(X,T)\frac{\partial v_0(X,T)}{\partial X}=0.08X^4T\end{aligned} \tag{27}$$

From eq. (24), we obtain $v_2(X,T)=0.04X^4T^2$.
The third order iterative solution of eqs. (21) and (22) reads:

$$u(X,T)=0.2X^2+0.08X^3T+0.04X^4T^2 \tag{28}$$

The variational principle is widely used to study non-linear problems [32-39]. In order to establish a variational formulation, we use the following traveling wave variable:

$$\xi=X-cT \tag{29}$$

Equation (21) is transformed into the following ODE:

$$-cu'-uu'=0 \tag{30}$$

by He's semi-inverse method [40], we can obtain the following variational formulation:

$$J(u)=\int_0^\infty\left[\frac{c}{2}u^2+\frac{1}{6}u^3\right]\mathrm{d}\xi \tag{31}$$

Case A. According to [41, 42], we search for a soliton solution in the form:

$$u(\xi)=A\,\mathrm{sech}(\xi) \tag{32}$$

By substituting eq. (32) into eq. (31), we obtain:

$$J=\frac{1}{24}A^2(12c+A\pi) \tag{33}$$

To find the constant A,we need to solve the following equation:

$$\frac{\partial J}{\partial A}=\frac{A^2\pi}{24}+\frac{1}{12}A(12c+A\pi)=0 \tag{34}$$

From eq. (34), we obtain:

$$A=-\frac{8c}{\pi} \tag{35}$$

Therefore, the solitary wave solutions to eq. (21) are:

$$u(X,T) = -\frac{8c}{\pi}\operatorname{sech}(X - cT) \tag{36}$$

From eq. (22), we have:

$$u(X,0) = 0.2X^2 = -\frac{8c\operatorname{sech}(X)}{\pi} \tag{37}$$

Therefore:

$$c = -0.0785398X^2\cosh(X) \tag{38}$$

We have:

$$u(X,T) = 0.2X^2\cosh(X)\operatorname{sech}[X + 0.0785398X^2T\cosh(x)] \tag{39}$$

Case B. According to [41, 42], we search for a soliton solution in the form:

$$u(\xi) = A\operatorname{sech}(\xi)\tanh(\xi) \tag{40}$$

By substituting eq. (40) into eq. (31), we obtain:

$$J = \frac{1}{90}A^2(2A + 15c) \tag{41}$$

We solve the following equation:

$$\frac{\partial J}{\partial A} = \frac{A^2}{45} + \frac{1}{45}A(2A + 15c) = 0 \tag{42}$$

From eq. (42), we obtain:

$$A = -5c \tag{43}$$

Therefore, the solitary wave solutions to eq. (21) is:

$$u(X,T) = -5c\operatorname{sech}(X - cT)\tanh(X - cT) \tag{44}$$

From eq. (22), we have:

$$u(X,0) = 0.2X^2 = -5c\operatorname{sech}(X)\tanh(X) \tag{45}$$

and

$$c = -0.04X^2\cosh(X)\coth(X) \tag{46}$$

We have:

$$\begin{aligned} u(X,T) = {} & 0.2X^2\cosh(X)\coth(X) \\ & \operatorname{sech}[X + 0.04X^2T\cosh(x)\coth(x)] \\ & \tanh[X + 0.04X^2T\cosh(X)\coth(X)] \end{aligned} \tag{47}$$

Conclusion

A class of fractal PDE is successfully established by He's fractal derivative [27, 28] in a fractal space, and their variational principles are obtained by the semi-inverse method [40]. The two-scale transform method [29, 30] and the FRHM [17] are adopted to solve the fractal PDE with initial value conditions. The example is a great demonstration of how the FRHM [17] is an incredibly simple and straightforward tool to solve initial value problems of PDE. The Fourier transform in the present paper can be replaced by Aboodh Transform [43], He Transform [22], Sumudu and Elzaki integral transforms [44], and new modifications occurs.

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