POINT SPECTRUM, RESIDUAL SPECTRUM, AND CONTINUOUS SPECTRUM OF UNBOUNDED UPPER TRIANGULAR OPERATOR MATRICES

by

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This paper examines the completion of the point spectrum, residual spectrum, and continuous spectrum of an unbounded operator matrix acting in a Hilbert space. The necessary and sufficient conditions for the point spectrum of an unbounded operator matrix to be equal to the union of the point spectra of its diagonal entries are presented.

Key words: *closed operator matrix, point spectrum, residual spectrum, continuous spectrum*

Introduction

Operator matrices can be applied to boundary control problems, particularly those involving the heat equations with dynamic boundary conditions and boundary control [1, 2]. This paper presents a mathematical study of upper triangular operator matrices, which has many potential applications in numerical simulation for the thermal displacement prediction [3, 4].

Let $\mathcal{H}$ and $\mathcal{K}$ be complex infinite-dimensional separable Hilbert spaces, and let $\mathcal{B}(\mathcal{H}, \mathcal{K})$ (resp., $\mathcal{C}(\mathcal{H}, \mathcal{K})$, $\mathcal{C}^{+}(\mathcal{H}, \mathcal{K})$) be the set of all bounded (resp., closed, closable) operators from $\mathcal{H}$ to $\mathcal{K}$. If $\mathcal{K} = \mathcal{H}$, we write $\mathcal{B}(\mathcal{H})$, $\mathcal{C}(\mathcal{H})$, and $\mathcal{C}^{+}(\mathcal{H})$ as usual. For $\mathcal{T} \in \mathcal{C}(\mathcal{H}, \mathcal{K})$, we use $\mathcal{R}(T)$ and $\mathcal{N}(T)$ to denote the range and kernel of T, respectively, and define $\alpha(T) := \dim \mathcal{N}(T)$ and $\beta(T) := \dim[\mathcal{K}/\mathcal{R}(T)]$. If $\overline{\mathcal{D}(T)} = \mathcal{H}$, we denote $\alpha(T^{*})$ by $d(T) = \dim[\mathcal{R}(T)^{\perp}]$, where T^{*} is the adjoint operator of T and $\mathcal{R}(T)^{\perp}$ is the orthonormal complement of $\mathcal{R}(T)$. Clearly, we get $\beta(T) \geq d(T)$. If, in particular, $\mathcal{R}(T)$ is closed, then $\beta(T) = d(T)$.

For $T \in \mathcal{C}(\mathcal{H})$, the spectrum $\sigma(T)$, the point spectrum $\sigma_p(T)$, the residual spectrum $\sigma_r(T)$, the continuous spectrum $\sigma_c(T)$, the approximate point spectrum $\sigma_{ap}(T)$ and the defect spectrum $\sigma_\delta(T)$ of T are, respectively, defined by:

$$\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not bijective}\}$$

$$\sigma_p(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not injective}\}$$

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$$\sigma_r(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is injective, but } \overline{\mathcal{R}(T - \lambda I)} \neq \mathcal{H}\}$$

$$\sigma_c(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is injective, and } \overline{\mathcal{R}(T - \lambda I)} = \mathcal{H}, \text{ but } \mathcal{R}(T - \lambda I) \neq \mathcal{H}\}$$

$$\sigma_{ap}(T) = \{\lambda \in \mathbb{C} : \text{exist } \{x_n\} \subset \mathcal{D}(T), \text{ such that } (T - \lambda I)x_n \to 0 \text{ as } n \to \infty\}$$

$$\sigma_\delta(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not surjective}\}.$$

The subsets of the point spectrum and the residual spectrum of $T \in \mathcal{C}(\mathcal{H})$ are, respectively, defined by:

$$\sigma_{p,1}(T) = \{\lambda \in \sigma_p(T) : \mathcal{R}(T - \lambda I) = \mathcal{H}\}$$

$$\sigma_{r,1}(T) = \{\lambda \in \sigma_r(T) : \mathcal{R}(T - \lambda I) \text{ is closed}\}.$$

In the recent past, numerous mathematicians have studied the completion problems of 2×2 bounded upper triangular operator matrix:

$$M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} : \mathcal{H} \oplus \mathcal{K} \to \mathcal{H} \oplus \mathcal{K}$$

See [5-10]. In particular, the sets:

$$\bigcap_{C\in\mathcal{B}(\mathcal{K},\mathcal{H})} \sigma_p(M_C), \quad \bigcap_{C\in\mathcal{B}(\mathcal{K},\mathcal{H})} \sigma_r(M_C), \text{ and } \bigcap_{C\in\mathcal{B}(\mathcal{K},\mathcal{H})} \sigma_c(M_C)$$

are described in [10]. However, the corresponding properties of unbounded case have not been studied.

In this paper we consider the properties of the point spectrum, the residual spectrum and the continuous spectrum of unbounded operator matrix:

$$T_B = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} : \mathcal{D}(A) \oplus \mathcal{D}(D) \subset \mathcal{H} \oplus \mathcal{K} \to \mathcal{H} \oplus \mathcal{K}$$

acting in Hilbert space $\mathcal{H} \oplus \mathcal{K}$, where the diagonal entries $A \in \mathcal{C}(\mathcal{H})$ and $D \in \mathcal{C}(\mathcal{K})$ with dense domains are given. It is easily seen that T_B is a closed operator for arbitrary closable operator $B : \mathcal{D}(B) \to \mathcal{H}$ with $\mathcal{D}(B) \supset \mathcal{D}(D)$. Firstly, by terms of the set:

$$\bigcap_{B\in\mathcal{C}_D^+(\mathcal{K},\mathcal{H})} \sigma_p(T_B)$$

we characterize the necessary and sufficient conditions, completely described by the diagonal entries of T_B, for the following equation:

$$\sigma_p(T_B) = \sigma_p(A) \cup \sigma_p(D) \tag{1}$$

where

$$\mathcal{C}_D^+(\mathcal{K},\mathcal{H}) = \{B \in \mathcal{C}^+(\mathcal{K},\mathcal{H}) : \mathcal{D}(B) \supset \mathcal{D}(D) \text{ for } D \in \mathcal{C}(\mathcal{K})\}$$

Besides, the sets:

$$\bigcap_{B\in\mathcal{B}(\mathcal{K},\mathcal{H})}\sigma_r(T_B) \quad \text{and} \quad \bigcap_{B\in\mathcal{C}^+_{A,D}(\mathcal{K},\mathcal{H})}\sigma_c(T_B)$$

are obtained, where:

$$\mathcal{C}^+_{A,D}(\mathcal{K},\mathcal{H})=\{B\in\mathcal{C}^+(\mathcal{K},\mathcal{H}):\mathcal{D}(B)\supset\mathcal{D}(D),\mathcal{D}(B^*)\supset\mathcal{D}(A^*)\ \text{for}\, A\in\mathcal{C}(\mathcal{H})\,\text{and}\, D\in\mathcal{C}(\mathcal{K})\}.$$

Moreover, we provide an affirmative answer to the following question.

Is there an operator $B_0\in\mathcal{C}^+_D(\mathcal{K},\mathcal{H})$ such that:

$$\sigma_c(T_{B_0})=\bigcap_{B\in\mathcal{C}^+_{A,D}(\mathcal{K},\mathcal{H})}\sigma_c(T_B)$$

for any given operators $A\in\mathcal{C}(\mathcal{H})$ and $D\in\mathcal{C}(\mathcal{K})$?

In order to prove the main results in the next sections, we first give the following Lemmas.

Lemma 1. [11] Let $A\in\mathcal{B}(\mathcal{H})$ and let $\mathcal{R}(A)$ be not closed. Then there is an infinite dimensional closed subspace $\mathcal{M}$ of $\overline{\mathcal{R}(A)}$ such that $\mathcal{M}\cap\mathcal{R}(A)=\{0\}$.

Lemma 2. Let $A\in\mathcal{C}(\mathcal{H})$ and $D\in\mathcal{C}(\mathcal{K})$ be given operators with dense domains. For some $B\in\mathcal{C}^+_D(\mathcal{K},\mathcal{H})$, T_B is not injective if and only if A or D is not injective.

Proof. Suppose there exists a $B\in\mathcal{C}^+_D(\mathcal{K},\mathcal{H})$ such that T_B is not injective, then there is $0\neq z=(x,y)^T\in\mathcal{D}(T_B)$ such that $Ax+By=0$ and $Dy=0$. If $y\neq 0$, then D is not injective. If $y=0$, then $x\neq 0$, and hence A is not injective.

Conversely, if A is not injective, then T_B is not injective for every $B\in\mathcal{C}^+_D(\mathcal{K},\mathcal{H})$. If D is not injective, then T_0 is not injective, where $B=0$.

Corollary 1. Let $A\in\mathcal{C}(\mathcal{H})$ and $D\in\mathcal{C}(\mathcal{K})$ be given operators with dense domains. Then T_B is injective for every $B\in\mathcal{C}^+_D(\mathcal{K},\mathcal{H})$ if and only if A and D are both injective.

Lemma 3. Let $A\in\mathcal{C}(\mathcal{H})$ and $D\in\mathcal{C}(\mathcal{K})$ be given operators with dense domains. Then $\overline{\mathcal{R}(T_B)}=\mathcal{H}\oplus\mathcal{K}$ for every $B\in\mathcal{C}^+_{A,D}(\mathcal{K},\mathcal{H})$ if and only if $\overline{\mathcal{R}(A)}=\mathcal{H}$ and $\overline{\mathcal{R}(D)}=\mathcal{K}$.

Proof. Since the necessity holds obviously, we only need to prove the sufficiency. Assume that $\overline{\mathcal{R}(A)}=\mathcal{H}$ and $\overline{\mathcal{R}(D)}=\mathcal{K}$, then $\mathcal{N}(A^*)=\{0\}$ and $\mathcal{N}(D^*)=\{0\}$. By *Corollary 1* and *Lemma 4* in [10], we have $\mathcal{N}(T_B{}^*)=\{0\}$ for every $B\in\mathcal{C}^+_{A,D}(\mathcal{K},\mathcal{H})$. Thus $\overline{\mathcal{R}(T_B)}=\mathcal{H}\oplus\mathcal{K}$.

Remark 1. Corollary 1, and *Lemma 3* also hold for bounded operator matrix.

Property of point spectrum

In this section, we describe the set:

$$\bigcap_{B\in\mathcal{C}^+_D(\mathcal{K},\mathcal{H})}\sigma_p(T_B)$$

and characterize the necessary and sufficient conditions for (1), which are completely described by the diagonal operators A and D.

Theorem 1. For given operators $A\in\mathcal{B}(\mathcal{H})$ and $D\in\mathcal{C}(\mathcal{K})$ with dense domain, we have:

$$\bigcap_{B\in\mathcal{C}_D^+(\mathcal{K},\mathcal{H})}\sigma_p(T_B)=\sigma_p(A)\cup\{\lambda\in\mathbb{C}:\alpha(D-\lambda I)>\beta(A-\lambda I)\}$$

Proof. First, we prove that:

$$\bigcap_{B\in\mathcal{C}_D^+(\mathcal{K},\mathcal{H})}\sigma_p(T_B)\supset\sigma_p(A)\cup\{\lambda\in\mathbb{C}:\alpha(D-\lambda I)>\beta(A-\lambda I)\}$$

If $\lambda\in\sigma_p(A)$, then we have $\lambda\in\sigma_p(T_B)$ for every $B\in\mathcal{C}_D^+(\mathcal{K},\mathcal{H})$.

If $\lambda\in\{\lambda\in\mathbb{C}:\alpha(D-\lambda I)>\beta(A-\lambda I)\}\setminus\sigma_p(A)$, for every $B\in\mathcal{C}_D^+(\mathcal{K},\mathcal{H})$, we consider two cases.

Case I: If $\mathcal{N}(B)\cap\mathcal{N}(D-\lambda I)\neq\{0\}$, then there exists some

$$0\neq y_0\in\mathcal{N}(B)\cap\mathcal{N}(D-\lambda I),\text{ and thus }\begin{pmatrix}A-\lambda I & B\\ 0 & D-\lambda I\end{pmatrix}\begin{pmatrix}0\\ y_0\end{pmatrix}=0$$

Hence $\lambda\in\sigma_p(T_B)$.

Case II: If $\mathcal{N}(B)\cap\mathcal{N}(D-\lambda I)=\{0\}$, then:

$$\dim[\mathcal{R}(B|_{\mathcal{N}(D-\lambda I)})]=\dim[\mathcal{N}(D-\lambda I)]=\alpha(D-\lambda I)>\beta(A-\lambda I)$$

It follows from $\beta(A-\lambda I)<\infty$ that $\mathcal{R}(A-\lambda I)$ is closed, and thus:

$$\mathcal{R}(A-\lambda I)\cap\mathcal{R}[B|_{\mathcal{N}(D-\lambda I)}]\neq\{0\}$$

Set $0\neq x\in\mathcal{R}(A-\lambda I)\cap\mathcal{R}(B|_{\mathcal{N}(D-\lambda I)})$, then there exists a $0\neq x\in\mathcal{D}(A-\lambda I)$ and $y\in\mathcal{N}(D-\lambda I)$ such that $(A-\lambda I)x=-By=x$. Set $z=(x,y)^T\neq 0$, then $(T_B-\lambda I)z=0$. That is $\lambda\in\sigma_p(T_B)$.

Considering the above two cases, we obtain that:

$$\lambda\in\bigcap_{B\in\mathcal{C}_D^+(\mathcal{K},\mathcal{H})}\sigma_p(T_B)$$

Next, we deduce the opposite inclusion. Assume that:

$$\lambda\notin\sigma_p(A)\cup\{\lambda\in\mathbb{C}:\alpha(D-\lambda I)>\beta(A-\lambda I)\}$$

then $A-\lambda I$ is injective and $\alpha(D-\lambda I)\leq\beta(A-\lambda I)$.

If $\mathcal{R}(A-\lambda I)$ is closed, then there exists an injective operator:

$$B_0:\mathcal{N}(D-\lambda I)\to\mathcal{R}(A-\lambda I)^{\perp}\quad\text{since}\quad\alpha(D-\lambda I)\leq\beta(A-\lambda I)=d(A-\lambda I)$$

Set:

$$B=\begin{pmatrix}0 & 0\\ B_0 & 0\end{pmatrix}:\begin{pmatrix}\mathcal{N}(D-\lambda I)\\ \mathcal{N}(D-\lambda I)^{\perp}\end{pmatrix}\to\begin{pmatrix}\mathcal{R}(A-\lambda I)\\ \mathcal{R}(A-\lambda I)^{\perp}\end{pmatrix}$$

then it is not hard to verify that $T_B-\lambda I$ is injective.

If $\mathcal{R}(A-\lambda I)$ is not closed, then $\infty=\beta(A-\lambda I)$. Next, we consider two cases:

Case I: If $\alpha(D-\lambda I)\le d(A-\lambda I)$, from the above proof, there exists a bounded operator B such that $T_B-\lambda I$ is injective.

Case II: If $\alpha(D-\lambda I)>d(A-\lambda I)$, then exist two subspaces Δ_1,Δ_2 of $\mathcal{N}(D-\lambda I)$ such that $\mathcal{N}(D-\lambda I)=\Delta_1\oplus\Delta_2$, where $\dim\Delta_2=d(A-\lambda I)$. Set:

$$B=\begin{pmatrix} J & 0 & 0\\ 0 & U & 0\end{pmatrix}:\begin{pmatrix}\Delta_1\\ \Delta_2\\ \mathcal{N}(D-\lambda I)^{\perp}\end{pmatrix}\to\begin{pmatrix}\overline{\mathcal{R}(A-\lambda I)}\\ \mathcal{R}(A-\lambda I)^{\perp}\end{pmatrix}$$

where $U:\Delta_2\to\mathcal{R}(A-\lambda I)^{\perp}$ is a unitary operator, and:

$$J:\Delta_1\to\mathcal{M}\subset\overline{\mathcal{R}(A-\lambda I)}$$

is also a unitary operator by *Lemma 1*. Then we claim that:

$$T_B-\lambda I=\begin{pmatrix} A_1(\lambda) & J & 0 & 0\\ 0 & 0 & U & 0\\ 0 & 0 & 0 & D_1(\lambda)\end{pmatrix}:\begin{pmatrix}\mathcal{D}(A-\lambda I)\\ \Delta_1\\ \Delta_2\\ \mathcal{N}(D-\lambda I)^{\perp}\cap\mathcal{D}(D-\lambda I)\end{pmatrix}\to\begin{pmatrix}\overline{\mathcal{R}(A-\lambda I)}\\ \mathcal{R}(A-\lambda I)^{\perp}\\ \mathcal{K}\end{pmatrix}$$

is injective. Otherwise, there exists:

$$0\ne x=(x_1,x_2,x_3,x_4)^T\in\mathcal{D}(T_B-\lambda I)$$

such that $(T_B-\lambda I)x=0$. It is easy to check that $x_3=x_4=0$. By the injection of $A-\lambda I, J$ and $\mathcal{R}(A-\lambda I)\cap\mathcal{M}=\{0\}$, we also get $x_1=x_2=0$, which gives a contradiction.

Example 1. Let $\mathcal{H}=C[0,1]$, $\mathcal{K}=L_2[0,1]$ and let the entries *A*, *D* of upper triangular operator matrix T_B be defined by $Au(t)=tu(t), u\in\mathcal{H}, t\in[0,1]$ and $Dx=x'', x\in\mathcal{D}(D)$, where:

$$\mathcal{D}(D)=\{x\in\mathcal{K}: x,x'\in AC[0,1], x''\in\mathcal{K}, x(0)=x(1)=0\}$$

By simple calculations, we see that $0\notin\sigma_p(A)$ and $0=\alpha(D)<\beta(A)$. Applying *Theorem 1*, there exists some $B_0\in\mathcal{C}_D^+(\mathcal{K},\mathcal{H})$ such that $0\notin\sigma_p(T_{B_0})$.

On the other hand, we also get $0\notin\sigma_p(T_{B_0})$ for every $B_0\in\mathcal{C}_D^+(\mathcal{K},\mathcal{H})$, since $0\notin\sigma_p(A)\cup\sigma_p(D)$.

In [10], the authors obtained the necessary and sufficient conditions for eq. (1) as:

Theorem 2. [10] Let $A\in\mathcal{C}(\mathcal{H})$ and $D\in\mathcal{C}(\mathcal{K})$ be given operators with dense domains. Then eq. (1) holds for every $B\in\mathcal{C}_D^+(\mathcal{K},\mathcal{H})$ if and only if $\sigma_{p,1}(D)\cap\sigma_{r,1}(A)\subset\sigma_p(T_B)$.

From *Theorem 1* and *2*, immediately, we get the following theorem.

Theorem 3. Let $A\in\mathcal{B}(\mathcal{H})$ and $D\in\mathcal{C}(\mathcal{K})$ with dense domain. Then eq. (1) holds for every $B\in\mathcal{C}_D^+(\mathcal{K},\mathcal{H})$ if and only if $\lambda\in\sigma_{p,1}(D)\cap\sigma_{r,1}(A)$ implies $\alpha(D-\lambda I)>\beta(A-\lambda I)$.

Proof. According to *Theorem 1* and *2*, we see that, for every $B\in\mathcal{C}_D^+(\mathcal{K},\mathcal{H})$, the equation $\sigma_p(T_B)=\sigma_p(A)\cup\sigma_p(D)$ holds if and only if:

$$\sigma_{p,1}(D)\cap\sigma_{r,1}(A)\subset\bigcap_{B\in\mathcal{C}_D^+(\mathcal{K},\mathcal{H})}\sigma_p(T_B).$$

Thus, from *Theorem 2*, eq. (1) holds if and only if:

$$\sigma_{p,1}(D)\cap\sigma_{r,1}(A)\subset\{\lambda\in\mathbb{C}:\alpha(D-\lambda I)>\beta(A-\lambda I)\}$$

Remark 2. *Theorem 2* and *3* are also valid for bounded upper triangular operator matrix, and hence these results are extensions of the *Theorem 7* in [2] and the *Theorem 2.5* in [11], respectively.

Properties of residual spectrum and continuous spectrum

In this section we concern with the perturbation of the residual spectrum and the continuous spectrum of closed operator matrix.

Theorem 4. For given operators $A\in\mathcal{C}(\mathcal{H})$ with dense domain and $D\in\mathcal{B}(\mathcal{K})$, we have:

$$\bigcap_{B\in\mathcal{B}(\mathcal{K},\mathcal{H})}\sigma_r(T_B)=[\sigma_r(D)\setminus\sigma_p(A)]\cup[\sigma_r(A)\cap\rho(D)]$$

Proof. We first show that the left side contains the right side. Without loss of generality, suppose that $0\in\sigma_r(D)\backslash\sigma_p(A)$ or $0\in\sigma_r(A)\cap\rho(D)$.

If $0\in\sigma_r(D)\backslash\sigma_p(A)$, then A and D are both injective and $\overline{\mathcal{R}(D)}\neq\mathcal{K}$. Thus T_B is also injective from *Corollary 1* and $\overline{\mathcal{R}(T_B)}\subset\mathcal{H}\oplus\overline{\mathcal{R}(D)}\neq\mathcal{H}\oplus\mathcal{K}$ for every $B\in\mathcal{B}(\mathcal{K},\mathcal{H})$.

Hence:

$$0\in\bigcap_{B\in\mathcal{B}(\mathcal{K},\mathcal{H})}\sigma_r(T_B)$$

If $0\in\sigma_r(A)\cap\rho(D)$, then A is injective, $\overline{\mathcal{R}(A)}\neq\mathcal{H}$ and D is bijective. Thus T_B is injective and $\overline{\mathcal{R}(T_B)}\neq\mathcal{H}\oplus\mathcal{K}$ for every $B\in\mathcal{B}(\mathcal{K},\mathcal{H})$ from the following decomposition:

$$T_B=\begin{pmatrix}A & B\\ 0 & D\end{pmatrix}=\begin{pmatrix}I_{\mathcal{H}} & BD^{-1}\\ 0 & I_{\mathcal{K}}\end{pmatrix}\begin{pmatrix}A & 0\\ 0 & D\end{pmatrix}$$

Thus:

$$0\in\bigcap_{B\in\mathcal{B}(\mathcal{K},\mathcal{H})}\sigma_r(T_B).$$

Now we verify the opposite inclusion. Without losing generality, we assume that:

$$0\notin[\sigma_r(D)\setminus\sigma_p(A)]\cup[\sigma_r(A)\cap\rho(D)]$$

Next, we consider the following three cases:

Case I: If $0\in\sigma_p(A)$, then we get:

$$0\notin\bigcap_{B\in\mathcal{B}(\mathcal{K},\mathcal{H})}\sigma_r(T_B)$$

from *Lemma 2*.

Case II: If $0 \notin \sigma_r(A) \cup \sigma_r(D)$, then we also obtain:

$$0 \notin \bigcap_{B\in\mathcal{B}(\mathcal{K},\mathcal{H})} \sigma_r(T_B)$$

by *Corollary 1* and *Lemma 3*.

Case III: If $0 \notin \sigma_r(D) \cup \rho(D)$, then $0 \in \sigma_p(D) \cup \sigma_c(D)$. If $0 \in \sigma_p(D)$, by *Lemma 2*, we also get:

$$0 \notin \bigcap_{B\in\mathcal{B}(\mathcal{K},\mathcal{H})} \sigma_r(T_B)$$

If $0 \in \sigma_c(D)$, *i.e.*, D is injective and $\mathcal{R}(D) \neq \overline{\mathcal{R}(D)} = \mathcal{K}$. We claim that there is B_0 such that $\overline{\mathcal{R}(T_{B_0})} = \mathcal{H} \oplus \mathcal{K}$. In fact, set $B_0 = UD : \mathcal{K} \to \mathcal{H}$, where $U : \mathcal{K} \to \mathcal{H}$ is a unitary operator from $\mathcal{K}$ onto $\mathcal{H}$. Clearly, $B_0 \in \mathcal{B}(\mathcal{K},\mathcal{H})$. For arbitrary $z_1 \in \mathcal{H}$, there is $z_2 \in \mathcal{K}$ such that $Uz_2 = z_1$. It follows from $\overline{\mathcal{R}(D)} = \mathcal{K}$ that there exist $\{y_n\}_{n=1}^{\infty} \subset \mathcal{K}$ such that $Dy_n \to z_2, n \to \infty$, for $z_2 \in \mathcal{K}$. Thus $UDy_n \to Uz_2$ as $n \to \infty$.

Set $z_n = \begin{pmatrix} 0 \\ y_n \end{pmatrix}$, then: $T_{B_0} z_n = \begin{pmatrix} A & UD \\ 0 & D \end{pmatrix}\begin{pmatrix} 0 \\ y_n \end{pmatrix} \to \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$.

That is $\overline{\mathcal{R}(T_{B_0})} = \mathcal{H} \oplus \mathcal{K}$ by the arbitrary of z_1. Hence:

$$0 \notin \bigcap_{B\in\mathcal{B}(\mathcal{K},\mathcal{H})} \sigma_r(T_B).$$

This completes the proof.

Remark 3. In [8], for bounded operator matrix M_C, the authors described:

$$\bigcap_{C\in\mathcal{B}(\mathcal{K},\mathcal{H})} \sigma_r(M_C) = [\sigma_r(B) \setminus \sigma_p(A)] \cup$$

$$\cup [\{\lambda \in \mathbb{C} : \alpha(D - \lambda I) > 0, \quad \text{and} \quad \mathcal{R}(B - \lambda I) \text{ is closed }\} \setminus (\sigma_p(A) \cup \sigma_p(B))].$$

Note that *Theorem 4* is also valid for bounded operator matrix, and the representation of:

$$\bigcap_{C\in\mathcal{B}(\mathcal{K},\mathcal{H})} \sigma_r(M_C)$$

is simplified in this paper.

Example 2. Let $\mathcal{H} = L_2[0,1]$, and let the entries A, D of upper triangular operator matrix T_B be defined by:

$$A = \begin{pmatrix} 0 & 1 \\ -\dfrac{d^2}{dx^2} & 0 \end{pmatrix}, \quad D = I\big|_{\mathcal{H}}$$

and

$$\mathcal{D}(A) = \{y \in \mathcal{H} : y' \in AC[0,1], y, y'' \in \mathcal{H}, y(0) = y(1) = 0\} \oplus \mathcal{H}$$

By simple calculations, we know that $\pi \in \sigma_p(A) \cap \rho(D)$ and $d(A-\pi I)=0$. Then:

$$\pi \notin [\sigma_r(D) \setminus \sigma_p(A)] \cup [\sigma_r(A) \cap \rho(D)]$$

Applying *Theorem 4*, there exists some $B_0 \in \mathcal{B}(\mathcal{K},\mathcal{H})$ such that $\pi \notin \sigma_r(T_{B_0})$.

On the other hand, we also have $\pi \in \sigma_p(T_B) \setminus \sigma_r(T_B)$ for every $B \in \mathcal{B}(\mathcal{K},\mathcal{H})$, since $\sigma_p(A) \subset \sigma_p(T_B)$.

Now we are ready to give the last result which is an extension of the *Theorem 4* in [10].

Theorem 5. For given densely defined operators $A \in \mathcal{C}(\mathcal{H})$ and $D \in \mathcal{C}(\mathcal{K})$, we have:

$$\bigcap_{B \in \mathcal{C}^+_{A,D}(\mathcal{K},\mathcal{H})} \sigma_c(T_B) = [\sigma_c(A) \cap \sigma_c(D)] \cup [\sigma_c(A) \cap \rho(D)] \cup [\rho(A) \cap \sigma_c(D)]$$

Proof. We first deduce that:

$$\bigcap_{B \in \mathcal{C}^+_{A,D}(\mathcal{K},\mathcal{H})} \sigma_c(T_B) \supset [\sigma_c(A) \cap \sigma_c(D)] \cup [\sigma_c(A) \cap \rho(D)] \cup [\rho(A) \cap \sigma_c(D)]$$

Without loss of generality, assume that:

$$0 \in (\sigma_c(A) \cap \sigma_c(D)) \cup (\sigma_c(A) \cap \rho(D)) \cup (\rho(A) \cap \sigma_c(D))$$

If $0 \in \sigma_c(A) \cap \sigma_c(D)$, then T_B is injective and $\overline{\mathcal{R}(T_B)} = \mathcal{H} \oplus \mathcal{K}$ for every $B \in \mathcal{C}^+_{A,D}(\mathcal{K},\mathcal{H})$ from *Corollary 1* and *Lemma 3*. We claim that $\mathcal{R}(T_B) \neq \mathcal{H} \oplus \mathcal{K}$ for every $B \in \mathcal{C}^+_{A,D}(\mathcal{K},\mathcal{H})$. Indeed, if not, then T_B is invertible. Thus $0 \notin \sigma_{ap}(A) \cup \sigma_\delta(D)$, that is, *A* is left invertible and *D* is right invertible. These imply that both $\mathcal{R}(A)$ and $\mathcal{R}(D)$ are closed. Hence *A* and *D* are invertible, which gives a contradiction. Therefore:

$$0 \in \bigcap_{B \in \mathcal{C}^+_{A,D}(\mathcal{K},\mathcal{H})} \sigma_c(T_B).$$

If either $0 \in \sigma_c(A) \cap \rho(D)$ or $0 \in \rho(A) \cap \sigma_c(D)$, we can also prove that:

$$0 \in \bigcap_{B \in \mathcal{C}^+_{A,D}(\mathcal{K},\mathcal{H})} \sigma_c(T_B)$$

in the same way.

Now we verify the converse inclusion. Without losing generality, suppose that:

$$0 \in \bigcap_{B \in \mathcal{C}^+_{A,D}(\mathcal{K},\mathcal{H})} \sigma_c(T_B)$$

holds. That is, T_B is injective and $\mathcal{R}(T_B) \neq \overline{\mathcal{R}(T_B)} = \mathcal{H} \oplus \mathcal{K}$ for every:

$$0 \in \bigcap_{B \in \mathcal{C}^+_{A,D}(\mathcal{K},\mathcal{H})} \sigma_c(T_B).$$

Applying *Corollary 1* and *Lemma 3*, we see that both A and D are injective, $\overline{\mathcal{R}(A)} = \mathcal{H}$ and $\overline{\mathcal{R}(D)} = \mathcal{K}$. Next, we claim that either $\mathcal{R}(A) \neq \mathcal{H}$ or $\mathcal{R}(D) \neq \mathcal{K}$. In fact, if not, then both A and D are surjective. Thus A and D are invertible. Then we get T_B is invertible, which leads a contradiction. Hence, we have that either $\mathcal{R}(A) \neq \mathcal{H}$ or $\mathcal{R}(D) \neq \mathcal{K}$. Therefore:

$$0 \in [\sigma_c(A) \cap \sigma_c(D)] \cup [\sigma_c(A) \cap \rho(D)] \cup [\rho(A) \cap \sigma_c(D)].$$

Corollary 2. For given densely defined operators $A \in \mathcal{C}(\mathcal{H})$ and $D \in \mathcal{C}(\mathcal{K})$, let $B_0 = 0$, then:

$$\bigcap_{B \in \mathcal{C}^{+}_{A,D}(\mathcal{K},\mathcal{H})} \sigma_c(T_B) = \bigcap_{B \in \mathcal{B}(\mathcal{K},\mathcal{H})} \sigma_c(T_B) = \sigma_c(T_{B_0})$$

This *Proof* is similar to the proof of *Theorem 5*, so we omit it.

Remark 4. Corollary 2 provides a certain answer to the question mentioned in the first section.

Conclusions

This paper considers several properties of the point spectrum, residual spectrum, and continuous spectrum of an unbounded operator matrix acting in a Hilbert space. Specifically, this paper examines the completion of the point spectrum, residual spectrum, and continuous spectrum of unbounded operator matrices. Subsequently, the necessary and sufficient conditions under which the point spectrum of the unbounded operator matrix is equal to the union of the point spectrum of its diagonal entries are presented.

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