ASYMPTOTIC STABILITY OF ORBITAL ATTRACTORS FOR A CLASS OF NON-AUTONOMOUS THERMAL EQUATIONS

by

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Original scientific paper
https://doi.org/10.2298/TSCI2503793D

This paper examines the existence and asymptotic stability of orbital attractors for a class of non-autonomous thermal equations. The study employs the attractor theory in non-autonomous infinite-dimensional dynamical systems, in conjunction with the energy method, compression function method, and Kuratowski non-compactness measure theory. Verification of the existence of the orbital absorption set allows us to conclude that the orbital attractor exists when the nonlinear term is independent of time and dependent on time.

Key words: *orbital, absorption set, conservation of energy*

Introduction

The reaction-diffusion equation [1] is a complex mathematical model that has been applied to a wide range of problems in various fields, including thermal sciences, physics, biology, and chemistry. Its applications extend to the fields of non-Newtonian fluid dynamics, soil mechanics, and heat transfer. Investigations have been conducted to a certain extent from various perspectives, including the uniqueness of the solution to the reaction-diffusion equation, the global attractor, the consistent attractor, and so forth. With regard to the non-autonomous reaction-diffusion equation, the study of orbital attractors has attracted considerable interest among scholars.

Chepyzhov *et al.* [2] studied the existence of tight attractors for non-autonomous reaction-diffusion equations. The existence of pullback consistent attractors when $g(x, t)$ satisfies the translational tightness condition, *i.e.*, $g(x,t)\in L_c^2[R:L^2(\Omega)]$, was further studied in [3] by certain tricks. Caraballo *et al.* [4] studied the existence of pullback consistent attractors when $g(x, t)$ satisfying polynomial growth in $L^2(\Omega)$ with orbital attractors. Khan and Khalid [5] considered the existence of orbital attractors for another class of problems using the Kuratowski non-tightness measure theory proposed by Tao *et al.* [6]. In recent years, there has been a proliferation of articles on the study of orbital attractors for this equation, as evidenced by [7]. A class of systems in nature, such as those pertaining to thermal science and fluid motion, that vary with time are referred to as dynamical systems. Dynamical systems represent one of the most active and rapidly evolving areas of contemporary mathematical research. The French mathematician Poincaré pioneered the theory of stability of differential equations, and the branch of mathematics of dynamical systems originated from the topics they discussed, including the existence and stability of equations, the periodicity of the system, the

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existence of orbital attractors, and other theories. Poincaré's early studies were primarily focused on the theory of dynamical systems of finite dimensions, with fewer studies conducted on infinite-dimensional dynamical systems. Subsequently, Bradshaw and Tsai [8] proposed the concept of fractal sets, while authors in [9-11] conducted a study on some non-linear systems concerning dissipative properties. Their findings included inertial manifolds of the system, the existence of attractors, and other related topics. Additionally, they explored Hausdorff dimensionality and other dimensionality estimation, non-linear Galerkin's method, inertial sets, and other related subjects.

In reaction-diffusion systems, the solution is contingent upon the state at the initial and final times. Consequently, for a given initial value, the solution orbits diverge when the initial moment varies, and the dissipative nature of the system alters with the distinct assumptions of the non-autonomous terms in the equations, which directly impacts the existence of the orbital attractor and its structure [12-14]. Consequently, for reaction-diffusion equations, the primary focus is on the study of consistent attractors and orbital attractors. In this paper, we investigate the existence and stability of orbital attractors for this class of non-autonomous reaction-diffusion equations by applying the Kuratowski non-tightness measure theory and conditions [15].

$$\begin{aligned} & u_t - v\Delta u = f(x,t), \quad (x,t)\in\Omega\times[\tau,\infty) \\ & u(x,t)\big|_{x\in\partial\Omega} = \nabla u(x,t)\big|_{x\in\partial\Omega}, \quad t\ge\tau \\ & u(x,\tau) = u_f, u_t(x,\tau) = u_1, \quad \forall u_\tau \in R \end{aligned} \tag{1}$$

where Ω is a bounded smooth region in R^N, $g(\cdot)\in L^2_{\text{low}}[R,L^2(\Omega)]$, $f\in C^1(R), v,\delta>0$ – a constant, and f satisfies the following conditions:

$$f(s)s\ge 0, \quad f(0)=0, \quad f'(s)\ge\lambda_0$$

$$C_1|u|^p - C_2 \le f(u)u \le C_3|u|^p - C_4$$

$$u=\Delta u=0, \quad (x,t)\in\partial\Omega\times(0,\infty)$$

$$u(x,\tau)=u_\tau(x), \quad t>\tau$$

$$f(u,s)\leqslant C_5(1+|u|^n), \qquad f_\tau(u,\tau)\leqslant C_6(1+|u|\tau+1)$$

$$\int_{-\infty}^{t} e^{\sigma s}|f(x,t)|_H^2\,\mathrm{d}s<\infty, \quad \forall t\in R,\ \sigma \text{ is a constant.}$$

$$F(u,s)\leqslant\lambda f(u,s)+\tau, \quad \forall(u,r)\in R\times R$$

$$F(u,r)=\int_0^u f(\xi,r)\mathrm{d}\xi, \text{ the external force term } f(x,t)\in L^2_{\text{loc}}(R;L^2(\Omega)).$$

where Ω is a bounded set of $R^n (n\le 3)$, and $g(x,t)$ – an external force term dependent on time t and an almost periodic function on t. When $f(x,t)=0$, it is the autonomous Cahn-Hilliard equation. For the non-linear term $f(u)$, there exists $0<\gamma\le\beta\le\infty$, satisfying the following conditions:

$$f'(u)>-k, \quad F(u)>\mu$$

$$|f'(u)| \le k_1(1+|u|^\beta), \quad |f''(u)| \le k_2(1+|u|^\gamma)$$

where $u \in X$, X is some taken Banach (or metric) space. For the unique solution $u(t)$ that exists for this system, it can be expressed:

$$u(t) = S(t)u_0, \quad \forall t \geqslant 0, \quad u(t) = U(t,r)u, \quad t \ge \tau, \quad \tau \in R$$

$s(t): X \to X$, $U(t,\tau): X \to X$ is a family of parametric (non-linear) operators $[U(t,\tau), t \ge \tau, \tau \in R]$ that satisfy the semigroup property:

$$S(t+t) = S(t)S(t), \quad \forall t, \tau \geqslant 0$$

$$U(t,s)N(s,t) = U(t,\tau), \quad \forall t \ge s \ge \tau, \quad \tau \in R$$

$s(0) = 1d$, $U(\tau,\tau) = Id$ (I is the constant operator in X)

The semigroup of solutions, designated $S(t)$, corresponds to the system (1). The phase space is denoted by X, and the trajectory of $u(t)$ in the phase plane is referred to as the solution orbit. The objective of this study is to demonstrate the existence of a solution to the thermal science system and to illustrate the phenomenon of orbital attraction.

When studying the asymptotic behavior of such systems, the main characteristic is that they may not have *forward* dissipation, but when the initial time tends to $-\infty$, and the termination time is fixed at an arbitrarily determined moment, the system has *pull-back* dissipation. These problems are widespread in the equations of thermal science systems. For example, the orbital attractor of a non-autonomous dynamical system and the orbital attractor of the non-autonomous *N-S* equation in *V*-space over a bounded region 2-D. Existence of consistent orbital attractors for the non-autonomous heat equation for non-autonomous infinite-dimensional dynamical systems with non-linear damping terms. Existence of orbital attractors for the non--autonomous reaction-diffusion equation in H_0^1 space. Existence of orbital attractors for the *B-F* equation on the bounded open region Ω in R^3 when the external force term f satisfies:

$$\int_0^t e^{\alpha\tau} \|f(x)\|^2 \, ds < +\infty$$

The methods used include the energy method, the method of compression function, Kuratowski's non-tightness measure theory and so on, which provide us an effective way to study the dynamical behavior of orbital attractors.

Background knowledge

Let X be a Banach space, and Q a metric space, often called a symbol (or parameter) space. $\theta = (\theta_t)_{t \in R}$ is a one-parameter homotopy group acting on Q, and has the following properties:

(1) $\theta_0(q) = q, \quad \forall q \in Q$

(2) $\theta_{t+s}(q) = \theta_t(\theta_s), \quad \forall t, s \in R$

(3) The mapping $(t,q) \to \theta_t(q)$ is continuous

The co-circle mapping $\phi: \mathbb{R}^+ \times Q \times X \to X$ induced by θ has the following properties.

(4) $\phi(0,q,x) = x, \quad \forall (q,x) \in Q \times X$

(5) $\phi(s+t,q,x)=\phi[s,\theta_t(q),\phi(t,q,x)], \quad \forall t,s\in\mathbb{R}^+$

(6) The mapping $(t,q,x)\to\phi(t,q,x)$ is continuous

It follows that (θ, ϕ) constitutes a non-autonomous dynamical system on $Q\times X$.

A family of sets:

$$\mathcal{D}=(D_q)_{q\in Q}$$

is orbitally absorbing if for any $q\in Q$, $\mathcal{B}\in\mathcal{B}(X)$, there exists $t_0 = t_0(q, B) \geq 0$ such that $\phi[t,\theta_{-t}(q),B]\subset D_q, \quad \forall t\geq t_0$.

A family of nonempty compact subsets in the space X, $\mathcal{A}=(A_q)_{q\in Q}$, is an orbital attractor of the non-autonomous dynamical system (θ, ϕ), which satisfies:

(7) ϕ – invariance, *i.e.*: $\phi(t,q,A_q)=A_{t(q)}$

(8) Orbital attraction is satisfied for all bounded sets B in X, *i.e.*

$$\lim_{t\to+\infty}\mathrm{d}\{\phi[t,\theta_{-t}(q),B],A_q\}=0$$

To prove the conclusions of this paper, we first introduce the following two lemmas:

Lemma 1. (Young's inequality with ε) Let $a > b$, $b > 0$, $\varepsilon > 0$, $p > 1$, $q > 1$, and:

$$\frac{1}{p}+\frac{1}{q}=1$$

then

$$ab\leq\frac{\varepsilon a^p}{p}+\frac{\varepsilon^{-q/p}b^p}{q}\leq\varepsilon a^p+\varepsilon^{-q/p}b^p$$

In particular, when $p = q = 2$, it becomes:

$$ab\leq\varepsilon a^2+\frac{b^2}{4\varepsilon} \text{ (Cauchy's inequality with } \varepsilon).$$

Lemma 2. (Consistent with Gronwall's inequality) Let x, φ, ψ be three locally cumulative functions on $[t_0,\infty)$, and dx/dt be locally cumulative on $[t_0,\infty)$ as well, and satisfy:

$$\frac{\mathrm{d}x}{\mathrm{d}t}\leq\varphi x+\psi \tag{2}$$

when $t\geq t_0$ then:

$$\int_t^{t+r}\varphi(s)\mathrm{d}s\leq k_1,\quad \int_t^{t+r}v(s)\mathrm{d}s\leq k_2,\quad \int_t^{t+r}x(s)\mathrm{d}s\leq k_3$$

where r, k_1, k_2, k_3, are positive constants, then:

$$x(t+r)\leq\left(\frac{k_3}{r}+k_2\right)\mathrm{e}^{k_1} \tag{3}$$

Main results

Theorem. If $\mathcal{A}=\{A_q\}_{q\in Q}$ satisfies the attraction to all bounded sets B in X in addition to ϕ invariance, *i.e.*:

$$\lim_{t\to+\infty} d[\phi(t,q,B), A_{\theta(q)}]=0 \tag{4}$$

Let the external force term $f(x,t) \in L^2_{1oc}(\Omega;H)$, hold, then the non-autonomous dynamical system corresponding to problem (1) has orbits $D_{\delta,\Omega}$ in Ω_0 that are orbit-absorbing sets, $\hat{D}=\{D_\delta\}_{\delta\in\Omega}\in D$.

At this point $\mathcal{A}=\{A_q\}_{q\propto Q}$ is the orbital attractor of the non-autonomous dynamical system (θ, ϕ).

Suppose (θ, ϕ) is a non-autonomous dynamical thermal system defined on $Q\times X$, and $D=\{D_q\}_{q\in Q}$ is a family of sets in X, if for any sequence $t_n\to\infty, x_n\in D_{t_n}(q_n)$, the sequence $\phi(t_n,\theta_{-t_n},x_n)$ is column-tight in X, at which point ϕ is $\mathcal{D}$-asymptotically orbital tight.

For this conclusion, we first verify the existence and uniqueness of the solution of this non-autonomous system, then we prove the existence of orbital absorbing sets in $L^2(\Omega)$ for this equation, and finally we prove the existence of orbital attractors for this equation.

Using u as an inner product for both sides of eq. (1), we obtain:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u|^2+v|\Delta u|^2=[\Delta f(u),u]+[f(x,t),u] \tag{5}$$

$$[\Delta f(u),u]\le -\int_\Omega f'(u)|\nabla u|^2\,\mathrm{d}x\le k|\nabla u|^2 \tag{6}$$

By Young's inequality and Holder's inequality we know that:

$$[\Delta f(u),u]\le k|\nabla u|^2\le \frac{v}{2}|\Delta u|^2+\frac{k^2}{2v}|u|^2 \tag{7}$$

Combined with Poincare's inequality, this yields:

$$[f(x,t),u]\le |f||u|\le \frac{1}{2}|f|^2\lambda_0+\frac{1}{2}|u|^2\lambda_1 \tag{8}$$

Therefore:

$$[\Delta f(u),u]+[u(x,t),u]\le \frac{1}{2}v|\Delta u|^2+\frac{C^2}{2}\sigma_0|u|^2+\frac{1}{2}|f|^2\sigma_1+\frac{1}{2}|u|^2\sigma_2 \tag{9}$$

Substituting the previous equation into eq. (4) we have:

$$\frac{\mathrm{d}}{\mathrm{d}t}|u|^2+v|\Delta u|^2\le \frac{1}{2}v|\Delta u|^2+\frac{m^2}{2}|u|^2+\frac{|f|^2}{2}\upsilon+\frac{1}{2}|u|^{2\upsilon} \tag{10}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}|u|^2+v|\Delta u|^2\le \frac{m^2}{2v}|u|^2+\frac{|f|^2}{2\tau_2}+\tau_3|u|^2 \tag{11}$$

Continuing the simplification leads to:

$$\frac{\mathrm{d}}{\mathrm{d}t}|u|^2+\lambda|\Delta u|^2-\frac{1}{\lambda}m^2|u|^2-\vartheta|u|^2\le \rho|f|^2 \tag{12}$$

Choosing the appropriate λ, there exists any constant $\omega_1 > 0$ such that $\lambda = v\omega_1^2 - vk^2 - \theta$.

Combined with Poincare's inequality, eq. (11) reduces to:

$$\frac{\mathrm{d}}{\mathrm{d}t}|u|^2 + \sigma |u|^2 \le \lambda |f|^2 + \varepsilon C_0 \int_\Omega F[u(x,t),t]\,\mathrm{d}x \tag{13}$$

Applying Gronwall's Lemma to the previous equation shows that $D'(t) = \{D(t): t \in R\}$ is the bounded orbital absorption set for the process.

Using $\psi = u_t + \varepsilon u$ with the previous equation as an inner product in H yields:

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\Big[(1-\varepsilon)\|u\|^2 + |\psi|^2 + (|\nabla u|^2 - p)^2\Big] + \varepsilon\|u\|^2 \le \\
&\le \varepsilon\big(|\nabla u|^2 - p\big)^2 - \varepsilon^2\|u\|^2 + p\varepsilon\big(|\nabla u|^2 - p\big) + \varepsilon^2(u,\psi) \le \\
&\le |\psi|^2 + k^2(u^+,\psi) - \varepsilon|\psi|^2 \le [f(t),\psi] - [f(u,t),\psi] \le \\
&\le (u^+,\psi) + \frac{\mathrm{d}}{\mathrm{d}t}k\left|u^+\right|^2 + \varepsilon\left|u^+\right|^2 \le \\
&\le [f(t),\psi] + \frac{1}{2}\lambda|\psi|^2 + \frac{1}{2}\tau|f(t)|^2
\end{aligned} \tag{14}$$

Also from Young's inequality and Holder's inequality we have:

$$\begin{aligned}
&[f(u,t),\psi] = [f(u,t),u_t + \varepsilon u] + [f(u,t),u_t] = \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega F[u(x,t),t]\mathrm{d}x + \varepsilon\int_\Omega f[u(x,t),t]u(x,t)\mathrm{d}x \le \\
&\le \varepsilon C_0 \int_\Omega F[u(x,t),t]\mathrm{d}x - \int_\Omega F_S[u(x,t),t]\mathrm{d}x \le \\
&\le \varepsilon\int_\Omega f[u(x,t),t]u(x,t)\mathrm{d}x - \varepsilon C_0\int_\Omega F[u(x,t),t]\mathrm{d}x \le \\
&\le -\varepsilon\left(\frac{1-\varepsilon}{4}\|u\|^2 + K_2\right) - \varepsilon\int_\Omega F[u(x,t),t]\mathrm{d}x - C_2|\Omega| \le \\
&\le (1-\varepsilon)\varepsilon|u| - \varepsilon^2\|u\|^2 + |\psi|^2 - \int_\Omega F_S[u(x,t),t]\mathrm{d}x \le \\
&\le \varepsilon\|\psi\|^2 + \varepsilon^2(u,\psi) + \varepsilon\big(|\nabla u|^2 - p\big)^2 + p\varepsilon\big(|\nabla u|^2 - p\big) \le \\
&\le (1-\varepsilon)\varepsilon\|u\|^2 + \big(\lambda_1^2 - \varepsilon\big)|\psi|^2 + 2\int_\Omega F[v(x)]\mathrm{d}x \le
\end{aligned}$$

$$\leq (1-\varepsilon)\varepsilon \| u \|^2 + (\lambda_1^2 - \varepsilon) | \psi |^2 - \frac{\varepsilon^3}{\lambda_1^2} \| u \|^2 | \psi | + |f|^2 - \varepsilon^2 \|u\|^2 \leq$$

$$\leq \varepsilon(| \nabla u |^2 - p)^2 + p\varepsilon\left(| \nabla u |^2 - p\right)^2 + (1-\varepsilon)\varepsilon \| u \|^2 \leq$$

$$\leq (\lambda_1^2 - \varepsilon) | \psi |^2 + \left(\lambda_1^2 - \frac{\varepsilon^2}{\lambda_1^4} - \varepsilon \right) | \psi |^2 + \frac{\varepsilon}{2}\left(| \nabla u |^2 - p\right)^2 - \frac{\varepsilon p^2}{2} \leq$$

$$\leq \frac{3\varepsilon(1-\varepsilon)}{4} \varepsilon \| u \|^2 + \frac{\varepsilon}{2}\left(| \nabla u |^2 - p\right)^2 - \frac{\lambda^2 m}{(1-\varepsilon)} \| u \|^2 \tag{15}$$

From (5)-(8) and (14)-(15), it follows that:

$$\frac{\mathrm{d}}{\mathrm{d}t}[(1-\varepsilon) | f |^2 + |\nabla u|^2 - p]^2 \leq \frac{(1-\varepsilon)}{4} \varepsilon | u |^2 \quad + p\varepsilon\left(|\nabla u|^2 - p\right) + \frac{| f(t) |^2}{2\lambda} \leq$$

$$\leq k^2 (u^+, f) - \varepsilon \| f \|^2 - \varepsilon\left(|\nabla u|^2 - p\right)^2 + \varepsilon\mu \int_\Omega F[u(x)]\mathrm{d}x \leq$$

$$\leq \varepsilon K_2 + C_2 \Omega + \varepsilon^2 (u, f) \tag{16}$$

Let $\delta = \max\left[\varepsilon C_1, \frac{\lambda_1}{2}, \varepsilon, \frac{\epsilon}{4}, \frac{3(1-\varepsilon)}{4} \varepsilon \right],$ then we have:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[(1-\varepsilon)\|u\|^2 + | \psi |^2 + \frac{\varepsilon}{2}\left(| \nabla u |^2 - p\right)^2 + k^2 \left|u^+\right|^2 + 2\int_\Omega F[u(x)]\mathrm{d}x \right] \leq$$

$$\leq \delta\left[(1-\varepsilon)\|u\|^2 + \frac{\varepsilon}{2}\left(| \nabla u |^2 - p\right)^2 + 2\int_\Omega F[u(x)]\mathrm{d}x \right] + \lambda_1 | f(t) |^2 \leq$$

$$\leq 2\varepsilon K_2 + | \psi |^2 + k^2 | u^+ |^2 \leq C_2 | \Omega | + \varepsilon^2 p^2 \tag{17}$$

For the inner product of eq. (1) with $-\Delta u_2$, then:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u_2\|^2 + v\|\nabla\Delta u_2\|^2 = [\Delta f(u), -\Delta u_2] + [f(x,t), -\Delta u_2] = [\Delta f(u), -\Delta u_2] + [\nabla f(u), \nabla\Delta u_2] \leq$$

$$\leq \int_\Omega f'(u)\nabla u_2 \nabla\Delta u_2 \mathrm{d}x \leq \gamma \int_\Omega \nabla u_2 \nabla\Delta u_2 \mathrm{d}x + +2v\sigma_1^2 \left|\nabla u_2\right|^2 \leq \frac{\gamma}{2}\left|\nabla\Delta u_2\right|^2 \tag{18}$$

$$\left\|[f(x,t), -\Delta u_2]\right\| = \frac{1}{2}\varrho\left|\Delta u_2\right|^2 + \frac{1}{2}\delta | f(x,t) |^2 \leq \varrho\left|\nabla\Delta u_2\right|^2 + [f(x,t), -\Delta u_2] \tag{19}$$

$$(\Delta f(u), -\Delta u_2) + \frac{\tau}{2} v\left|\nabla u_2\right|^2 + \tau | f(x,t) |^2 \leq \frac{\tau}{2}\left|\nabla\Delta u_2\right|^2 + \frac{1}{2\varrho} | f(x,t) |^2 \tag{20}$$

Substituting the previous equation into eq. (17), we have:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u_2\|^2+v|\nabla\Delta u_2|^2\le\frac{\rho}{2v}|\nabla u_2|^2+\frac{\rho_1}{2}|\nabla\Delta u_2|^2+\frac{1}{2\rho}|f(x,t)|^2 \tag{21}$$

That is:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u_2\|^2+v|\nabla\Delta u_2|^2\le\frac{\mu}{v}|\nabla u_2|^2+\frac{1}{\tau}|f(x,t)|^2 \tag{22}$$

Organizing the previous equation gives:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u_2\|^2+c\|u_2\|^2\le\kappa|f(x,t)|^2+\frac{v}{2}|\nabla\Delta u_2|^2 \tag{23}$$

There exists a positive constant δ, and $c=v\delta^2-c_1\delta-c_1^2/v,$, applying Poincare's Inequality, we get:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u_2\|^2+\lambda_k\|u_2\|^2\le c_2|f(x,t)|^2+\frac{c_2^2}{2}|\nabla u_2|^2 \tag{24}$$

For eq. (22), it follows from Gronwall's lemma that:

$$\|u_2(t)\|^2\le\|u_2(\tau)\|^2\mathrm{e}^{-\lambda_{m+1}(t-t)}+\frac{1}{c_2}\int_\tau^t\mathrm{e}^{-\lambda_{k+1}(t-s)}|f(s)|^2\,\mathrm{d}s+\frac{c_3}{2}|\nabla\Delta u_2|^2 \tag{25}$$

The two terms on the right-hand side of eq. (25) are estimated below, where we borrow a priori estimation methods.

For the first term, when $k+1$ is sufficiently large, for any $\varepsilon>0$, there are:

$$\|u_2(\tau)\|^2\mathrm{e}^{-\lambda_{m+1}(t-\tau)}\le\frac{\varepsilon}{2} \tag{26}$$

For item 2, since:

$$\int_\tau^t\mathrm{e}^{-\lambda_k^{(t-\tau)}}|f(s)|^2\,\mathrm{d}s\le$$

$$\le\int_{t-\sigma_0}^{t}\mathrm{e}^{-\lambda_{m=1}(t-\tau)}|f(s)|^2\,\mathrm{d}s+\int_{t-\sigma_1}^{t-\tau}\mathrm{e}^{-\lambda_k(t-\tau)}|f(s)|^2\,\mathrm{d}s\le$$

$$\le\int_{t-\tau}^{t}\mathrm{e}^{-\lambda_k(t-\tau)}|f(s)|^2\,\mathrm{d}s+\cdots+\lambda_1|\nabla\Delta u_2|^2\le$$

$$\le\int_{t-\sigma}^{t}\mathrm{e}^{-\lambda_{k+1}(t-\tau)}|f(s)|^2\,\mathrm{d}s\times\sup_{s\in R}\int_{s-1}^{s}|f(s)|^2\,\mathrm{d}s\le$$

$$\le\mathrm{e}^{-\lambda_k\sigma}(1+\mathrm{e}^{-\lambda_k}+\mathrm{e}^{-2\lambda_{k+1}}+\cdots)\le$$

$$\leq \int_{t-\sigma}^{t} \mathrm{e}^{-\lambda(t-\tau)} \,|\, f(s)\,|^2 \,\mathrm{d}s + \mathrm{e}^{-\lambda_k \eta}(1-\mathrm{e}^{-\lambda_{m+1}})\sup_{s\in R}\int_{\tau}^{t} |\, f(s)\,|^2 \,\mathrm{d}s \tag{27}$$

For the previous equation, when τ is sufficiently small and λ_k is sufficiently large, there holds:

$$\int_{\tau}^{t} \mathrm{e}^{-\lambda_{m+1}^{(t-\tau)}} \,|\, f(s)\,|^2 \,\mathrm{d}s \leq \frac{\tau}{2}\varepsilon \tag{28}$$

From eqs. (26)-(28), one can get:

$$\|u_2(t)\|^2 \leq \mathrm{e}^{-\lambda_k \eta}(1-\mathrm{e}^{-\lambda_{m+1}})\sup_{s\in R}\int_{\tau}^{t} |\, f(s)\,|^2 \,\mathrm{d}s \leq \frac{\tau}{4}\varepsilon \tag{29}$$

It follows that the non-autonomous dynamical system in Ω_0 has an orbital attractor and is stable.

Conclusions

The non-self-contained heat equation, as previously described, was initially proposed in the context of thermodynamics, specifically in the study of two substances. Since then, similar equations have emerged in an increasing number of mathematical models. The study of dynamical systems is derived from the understanding of the natural world and is summarized by two major problems: non-linear dynamics in celestial mechanics and turbulence in fluid mechanics. In the case of non-autonomous systems, the problem of the existence and stability of the orbital attractor is solved by the aforementioned analysis. The more established theories employed in this study include the existence theory of global attractors, dimension theory, inertial manifolds, and exponential attractors. It has been demonstrated that there exists a bounded, open region Ω on R^2, when the external force term f satisfies:

$$\int_{-\infty}^{t} \mathrm{e}^{\sigma s} \|\, f(s)\|^2 \,\mathrm{d}s < \infty$$

This shows that for a non-autonomous system defined on $Q\times X$, if $D=\{D_q\}_{q\in Q}$ is a family of sets in X, for any sequence $t_n \to \infty$, $x_n \in D_{-t_n}(q_n)$, then $u(t_n, \theta_{-t_n}, x_n)$ is column-tight in X, while u is orbitally asymptotically tight.

Acknowledgment

This work was supported by the Key Scientific Research Project of Colleges and Universities in Henan Province (No.22B110019), Special Project of Applied Basic Research and Applied Research of Zhengzhou Shengda University (SD202260), Soft Science Project of Henan Provincial Department of Science and Technology (No. 232400411122).

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Paper submitted: August 4, 2023
Paper revised: June 1, 2024
Paper accepted: June 1, 2024