ANALYSIS AND CONTROL OF FINANCIAL STABILITY BASED ON NON-LINEAR DIFFERENTIAL DYNAMICAL SYSTEMS

by

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As a complex system with multi-body interactions, the financial market functions in accordance with the non-linear differential dynamical system characteristics and properties. Although the financial market is a complex system with multiple constraints and perturbations, it is subject to a number of universal laws. This paper introduces fractional-order non-linear differential dynamical systems as a means of modeling and analyzing financial stability, as well as exploring the dynamical characteristics and large-time behavior of complex financial systems. Empirical simulation analysis and testing of econometrics serve to verify the scientific validity of this stochastic process.

Key words: *dynamical systems, financial stability, control*

Introduction

Financial markets are highly complex dynamical systems, and there is a long historical tradition of research on financial markets by scientists and technologists from different disciplines, including financial scientists and mathematicians [1-3]. The accumulation of vast quantities of economic data has rendered traditional analytical methods inadequate [4, 5] for the current situation of rapidly expanding data. While standard economic methods are only applicable to equilibrium and systems with only one type of trader, many of the more interesting and important phenomena in financial markets depend on the participation of different types of traders and on what happens when the financial system is far from equilibrium. The necessity for the development of more effective methods and theories to facilitate a more profound understanding of financial markets has become an urgent priority [6-9]. In recent decades, mathematicians have produced a substantial body of important results in a number of fields, including phase transitions, statistical mechanics, non-linear dynamics, and disordered systems. The analytical study of financial markets has increasingly employed concepts such as power law distributions, scalar behavior, correlations, and stochastic processes [10-12]. As traditional physical theories continued to evolve, physicists began to explore the potential of applying concepts and methods from physics to the study of financial market dynamics. This resulted in some notable successes.

The global economic situation is undergoing a profound transformation. Global investment and trade growth rates are low, financial markets and commodity price volatility are high, and emerging and developed economies are gradually becoming geographically separated. Political risk is on the rise, from low to high. The economic environment is unstable, and

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the economic situation is worrisome. Enterprises in the fast-changing economic environment are facing increasing uncertainty, which is leading to a rise in the number of risks and uncertainties they must navigate. In the context of ever-changing, uncertain challenges and opportunities, enterprises must possess the capacity to identify and respond to emerging trends, capitalize on opportunities, and accelerate their development. Accordingly, we propose a non-linear dynamical system based on fractional order and wavelet transform to investigate the relationship between financial flexibility and performance [13-15]. This system will also be used to test and improve the regulating effect of absorptive capacity and to realize stability analysis and control.

A nascent field at the nexus of mathematics and finance is financial mathematics. In the present era, concepts and methodologies derived from statistical physics, theoretical physics, complex systems theory, non-linear science, applied mathematics, and other disciplines are extensively employed in the analysis and investigation of financial markets. Financial physics, also known as financial mathematics, and disciplines such as econometric finance and stochastic process theory are all about financial markets. They are concerned with the analysis, prediction, and control theory of financial markets. The application of financial mathematics enables the examination of financial problems at the micro level, with the utilisation of concepts and models drawn from statistical physics, including those pertaining to multi-body interactions, phase and phase transitions, time-length and short-range correlations, multiple fractals, internal symmetry, random matrix theory and other methodologies [16]. Concurrently, it is possible to analyze the substantial high-frequency data present in the financial market with AI in order to explore the laws of financial dynamics. Furthermore, due to the existence of long-range temporal correlations and significant fluctuations in the financial dynamics system, we also endeavor to investigate phase transition-like phenomena and symmetry with the reorganization group method.

Modeling and analysis of financial dynamics systems

The term *economic and financial cybernetics* was first introduced at the World Congress of Cybernetics, held in Paris in 1952. Subsequently, it began to describe the macroeconomic system with second-order ODE, and discussed the problem of open-loop and closed-loop control of the system, the use of PID control principles to improve the stability of economic policy, and then constantly produced the optimal control problem of the macroeconomy, the problem of establishing a national economic planning system based on the methods of control theory, and the modeling of economic and financial cybernetics based on the dynamics of the system.

Characteristics of economic control systems

Modeling economic cybernetics entails ascribing a clear economic meaning to a range of control concepts, including:

– The macroeconomic system is frequently subject to the uncertainties of human and economic phenomena, and it is essentially a stochastic non-linear system. Only non-linear dynamics analysis can more accurately simulate the real situation, ensuring that the simulation results accurately reflect reality.
– The majority of identification and modeling data are obtained through periodic sampling, and the available information is not sufficient, the degree of reliability is low, and serious noise interference should be avoided in most cases.

– The macroeconomic system is a large system, which is generally difficult to decompose accurately. Furthermore, the subsystems often have strong coupling, which causes difficulties in mathematical processing.
– The optimal control of macroeconomics is influenced by the subjective factors of decision makers and system analysts, and different individuals may reach disparate conclusions about the same system.
– The mathematical model of the system often includes hundreds or even more mathematical equations and variables, and its non-linear model is more complex, requiring more sophisticated dynamics analysis.

Modeling, analysis and control of systems

Define the price of a financial product as $P(t)$ at time t, where financial products may include stocks, securities, broad market indices, *etc.* The logarithmic price return for a time interval of Δt is. The logarithmic price return for a time interval of Δt is:

$$R_i(t',\triangle t) = \ln P_i(t' + \triangle t) - \ln P_i(t') \tag{1}$$

The normalized return:

$$r_i(t) = \frac{R_i(t') - \langle R_i(t') \rangle}{\sigma_i}$$

where

$$\sigma_i = \sqrt{\langle R_i^2 \rangle - \langle R_i \rangle^2}$$

is the standard deviation of R_i. The absolute value of return $|r_i(t)|$ is an important form of volatility. The yield-volatility correlation function is defined:

$$L(t') = \frac{[|r(t+t')|^2 - 1]r(t)}{Z}, \text{ where } Z = |r(t+t')|^2 \tag{2}$$

Suppose f, g are functions defined on $[0,+\infty) \times R$. Then:

$$^c\mathrm{D}^p X_t = f(t,\ X_t) + g(t, X_t)\frac{\mathrm{d}C_i}{\mathrm{d}t} \tag{3}$$

When $p \in 0,1$, the solution of the equation X_t is a stochastic process and almost certainly satisfies the following fractional order integral equation:

$$X_t = X_0 + \frac{1}{\Gamma(p)}\int_0^t (t-s)^{p-1} f(s, X_s)\mathrm{d}s + \frac{1}{\Gamma(p)}\int_0^t (t-s)^{p-1} g(s, X_s)\mathrm{d}C_s \tag{4}$$

Let p be a real number satisfying $n - 1 < p \le n$, where n is a positive integer. Let f, g be functions defined on $[0,+\infty) \times \mathbb{R}$. Then the initial value problem for a fractional order differential equation of Riemann-Liouville type of order p:

$$\begin{gathered} \mathrm{D}^p X_t = f(t, X_t) + g(t, X_t)\frac{\mathrm{d}C_t}{\mathrm{d}t}, \quad t \in [0,T] \\ (\mathrm{D}^{p-k} X_t)_{t=0} = b_k, \quad k = 1, 2, \cdots, n \end{gathered} \tag{5}$$

If the uncertain external interference term dC/dt is regarded only as a function of time t, then the fractional-order derivative D^p of the integral equation with respect to t on both sides simultaneously is an indeterminate process and almost necessarily satisfies the following estimate:

$$X_t = \sum_{k=1}^{n} \frac{b_k t^{p-k}}{\Gamma(p-k+1)} + I^p f(t, X_t) + I^p \left[g(t, X_t) \frac{\mathrm{d}C_t}{\mathrm{d}t} \right] \le$$

$$\le \mathrm{D}^p \left[\sum_{k=1}^{n} \frac{b_k t^{p-k}}{\Gamma(p-k+1)} \right] + \mathrm{D}^p I^p f(t, X_t) + \mathrm{D}^p I^p \left[g(t, X_t) \frac{\mathrm{d}C_t}{\mathrm{d}t} \right] \le$$

$$\le \sum_{k=1}^{n} \frac{b_k t^{p-k}}{\Gamma(p-k+1)} + \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} f(s, X_s) \mathrm{d}s + \mathrm{D}^p X_t \le$$

$$\le \sum_{k=1}^{n} \frac{b_k \mathrm{D}^p (t^{p-k})}{\Gamma(p-k+1)} + f(t, X_t) + g(t, X_t) \frac{\mathrm{d}C_t}{\mathrm{d}t} \le$$

$$\le \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} g(s, X_s) \mathrm{d}C_s \tag{6}$$

For $1 \le m \le n$, the fractional order derivative D^{p-m} on each side of the equation with respect to t yields:

$$\mathrm{D}^{p-m} \left[\sum_{k=1}^{n} \frac{b_k t^{p-k}}{\Gamma(p-k+1)} \right] + \mathrm{D}^{p-m} I^p \left[f(t, X_t) + g(t, X_t) \frac{\mathrm{d}C_t}{\mathrm{d}t} \right] \le$$

$$\le \sum_{k=1}^{m} \frac{b_k t^{m-k}}{\Gamma(m-k+1)} + \frac{1}{\Gamma(m)} \int_0^t (t-s)^{m-1} [f(s, X_s)] \mathrm{d}s + I^m g(t, X_t) \frac{\mathrm{d}C_t}{\mathrm{d}t} \le$$

$$\le \sum_{k=1}^{n} \frac{b_k \mathrm{D}^{p-m} t^{p-k}}{\Gamma(p-k+1)} + I^m f(t, X_t) + g(s, X_s) \frac{\mathrm{d}C_s}{\mathrm{d}s} \le$$

$$\le \mathrm{D}^{p-m} X_t + f(t, X_t) + g(t, X_t) \frac{\mathrm{d}C_t}{\mathrm{d}t} \tag{7}$$

Taking the limit $t \to +\infty$ on both sides of the above equation with respect to t yields:

$$(\mathrm{D}^{p-m} X_t)_{t=0} = b_m, \quad m = 1, 2, \cdots, n \tag{8}$$

It can be verified that the integral equation satisfies exactly the fractional-order equation and the initial-value condition in the initial-value problem, and thus is a solution to the initial-value problem for fractional-order differential equations.

Let p be a real number satisfying $0 \le n - 1 < p \le n$, where n is a positive integer. Suppose f and g are functions defined on $[0,+\infty)\times\mathbb{R}$. When $0 \le m \le n - 2$, the m^{th} order derivative of the integral equation with respect to t on both sides gives:

$$
\begin{aligned}
&\mathrm{D}^m X_t + \sum_{k=m}^{n-1} \frac{x_k t^{k-m}}{\Gamma(k-m+1)} + \mathrm{D}^m I^p \left[f(t,X_t) + g(t,X_t)\frac{\mathrm{dC_t}}{\mathrm{dt}} \right] \le \\
&\le \sum_{k=m}^{n-1} \frac{x_k t^{k-m}}{\Gamma(k-m+1)} + \frac{1}{\Gamma(p-m)} \left[f(t,X_t) + g(t,X_t)\frac{\mathrm{d}C_t}{\mathrm{d}t} \right] \le \\
&\le \sum_{k=m}^{n-1} \frac{x_k t^{k-m}}{\Gamma(k-m+1)} + I^{p-m} \int_0^t (t-s)^{p-m-1} f(s,X_s)\mathrm{d}s \le \\
&\le \frac{1}{\Gamma(p-m)} \int_0^t (t-s)^{p-m-1} g(s,X_s)\mathrm{d}C_s \to \\
&\to \frac{1}{\Gamma(p-m)} + x_m \; (t \to +\infty)
\end{aligned}
\tag{9}
$$

When $m = n-1$, making $t \to +\infty$ gives:

$$
X_t^{(n-1)}\Big|_{t=0} = x_{n-1}
$$

which holds almost everywhere. Then we have:

$$
\begin{aligned}
&\mathrm{D}^{n-1} X_t + x_{n-1} + \frac{1}{\Gamma(2p-n+1)} \int_0^t (t-s)^{p-n} f(s,X_s)\mathrm{d}s \le \\
&\le x_{n-1} + \frac{1}{\Gamma(2p-n+1)} \int_0^t (t-s)^{p-n} g(s,X_s)\mathrm{d}C_s \le \\
&\le \frac{1}{\Gamma(2p-n+1)} \int_0^t (t-s)^{p-n} g(s,X_s)\mathrm{d}C_s \le \\
&\le \frac{t^{2p-n+1}}{\Gamma(2p-n+1)} \int_0^1 \frac{f(ut,X_{ut})\mathrm{d}u}{(1-u)^{n-p}} \le \\
&\le t^{p-n+1} \int_0^1 \frac{|g(ut,X_{ut})|\mathrm{d}u}{(1-u)^{n-p}} \le \int_0^t \frac{|g(s,X_s)|\mathrm{d}s}{(t-s)^{n-p}} \to 0, \quad (t \to +\infty)
\end{aligned}
\tag{10}
$$

It follows that the large time behavior of its dynamical system is stochastically stable.

Processing of models

The Fourier transform [17, 18] is a mathematical method that converts information from a time-based to a frequency-based representation. This enables the analysis of global frequency characteristics of information, but not local time characteristics. Although the short-time Fourier transform (Gabor transform) can be employed for time-frequency analysis, the accuracy of this method is constrained by the size of the window. The wavelet transform represents a novel transform analysis method that builds upon the concept of localization inherent to the short-time Fourier transform. It simultaneously addresses the limitations of the aforementioned approach, such as the fixed size of the window, which does not adapt to varying frequencies. In discrete wavelet transform, the information is analyzed at specific scales and positions. This transform has higher computational efficiency than other methods while maintaining the same level of analysis accuracy.

If the signal $f(t)$ is a square-integrable function, *i.e.*:

$$\sum_{-\infty}^{+\infty} |f(t)|^2 < +\infty$$

the wavelet transform of $f(t)$ is:

$$W_{a,b} = \int_{-\infty}^{+\infty} f(t) \frac{1}{\sqrt{|a|}} \psi^* \left(\frac{t-b}{a} \right) \mathrm{d}t \quad \text{(successive type)} \tag{11}$$

$$\psi_{j,k}(t) = 2^{-j/2} \psi(2^{-j} t - k), \quad j,k \in Z \quad \text{(discrete type)} \tag{12}$$

where a, $b \in R$, a is the scale factor, b – the displacement factor, and $a \neq 0$. The $*$ denotes taking the conjugate. Let $a_j = 2^j$, $b_{j,k} = k2^j$, j, $k \in Z$, replace a and b in the wavelet function $\psi_{a,b}(t)$:

$$\psi_{a,b}(t) = \frac{1}{\sqrt{|a|}} \psi \left(\frac{t-b}{a} \right) \tag{13}$$

The original information can be reconstructed using wavelet inverse transformations

$$f(t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W_{a,b} \psi_{a,b}(t) \mathrm{d}a \mathrm{d}b, \quad f(t) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} W_{j,k} \psi_{j,k}(t) \tag{14}$$

For the discrete wavelet transform, the scale function and wavelet function can be expressed, respectively:

$$\phi(2^j t) = \sum_{k=-\infty}^{\infty} h_0(k) \sqrt{2} q(2^{j+1} t - k), \quad \psi(2^j t) = \sum_{k=-\infty}^{\infty} h_1(k) \sqrt{2} \varphi(2^{j+1} t - k) \tag{15}$$

The process of repeating the wavelet decomposition enables the analysis of the original information at multiple resolutions, which can then be used to reconstruct the original signal. The Daubechies wavelet decomposition and reconstruction process allows for the acquisition of both smooth approximations and detail information.

Wavelet entropy is derived from the concept of entropy, which is utilized to quantify the uniformity of any kind of energy distribution in space. The greater the uniformity of energy distribution, the greater the entropy. When a system's energy is completely uniformly distributed, the entropy of that system reaches its maximum value. Wavelet entropy is a specific type of entropy value that is calculated by using the information obtained after wavelet decomposition and wavelet reconstruction. Wavelet entropy is a widely utilized analytical tool in the field of financial systems.

Empirical analysis

Numerical simulation steps

The first step is to preprocess the data in a normalized way, using a set of random numbers, A_k, and normalizing them to represent the initial stochastic state of the complex system, a process that corresponds to the values of the initial stochastic distribution of the weights A_{ki} in the theoretical analysis described previously. By writing a computational program based on the neural network toolbox in MATLAB, the system calculates the average Euclidean distance value, d, between P_k and A_k, in which the smallest Euclidean distance value corresponding to A_k is regarded as the distribution of the state that wins this competition, and the distribution of the winning state is adjusted by using the positive feedback rule, and the process is repeated continuously. This process is repeated until all the macroscopic parameter data sets have been input into the network for computation, and the final winning state distribution A_k in the competition and its corresponding a_{ki} can be obtained. The distribution value. Then the distribution value of a_{ki} is fed back into the complex system pattern ξ_k, and the corresponding pattern ξ_k can be derived, which realizes the pattern recognition of the complex system under different inputs, and further combines with the theoretical part of the principle of maximum entropy to obtain the evolution pattern of the complex system. By reflecting the changes in the conditions that occur during the evolution of the complex system to the macroscopic pattern data of the complex system, the results of the pattern of the complex system due to these changes can also be effectively found:

Initialization. Set the initial random state A of the system, and assign any random value between 0 and 1 to a_{ij} (i = 1, 2. N; j = 1, 2. M). Assign an initial value to the positive feedback rate $\eta(0)$ $(0 < \eta(0) < 1)$. Determine the initial value of the neighborhood $N_g(t)$ of $N_g(0)$. Neighborhood $N_g(t)$ is the range of the region containing several tuples centered on the competitive winning pattern determined in fourth step. The value of $N_g(t)$ represents the number of tuples contained in the neighborhood at the t^{th} feedback process. Determine the total number of feedbacks T.

– Select any one of the q input modes P_k to be normalized and input into the system:

$$\overrightarrow{P_k} = \frac{P_k}{|P_k|} = \frac{p_1^k \cdot p_2^k \cdots p_n^k}{[(p_1^k)^2 + (p_2^k)^2 + \cdots (p_n^k)^2]^{\frac{1}{2}}} \tag{16}$$

– Normalize the system state distribution $A_j = (a_{j1}, a_{j2}, \quad a_{jN})$ and compute the Euclidean distance between $\vec{A}_j$ and $\vec{P}_k$:

$$\overrightarrow{A_j} = \frac{A_j}{\|A_j\|} = \frac{(a_{j1}, a_{j2}, \cdots a_{jn})}{[a_{j1}{}^2 + a_{j2}{}^2 + \cdots a_{jn}{}^2]^{\frac{1}{2}}}, \quad \mathrm{d}_j = \sum_{i=1}^{n} \sqrt{p_i} - \bar{a}_{ij}^2)^{1/2}, \quad j = 1 \cdots, M \tag{17}$$

– Find the minimum distance d, which corresponds to the distribution of the competitive winning system A_k : $d_g = \min d_j, j = 1, M$

– Perform system state adjustment. To correct the system mode state between all group elements in the competing neighborhood $N_g(t)$ and the input group elements:

$$a_{ji}(t+1) = a_{ji}(t)[\eta(t) - a_{ji}(t)] \tag{18}$$

where $\eta(t)$ is the positive feedback rate at moment t.

– Continue to select a new input data set normalized input to the system, return to third step, until all the macro-parameter data sets are input.
– Updating the positive feedback rate $\eta(t)$ and its neighborhood $N_g(t)$:

$$\eta(t) = \eta(0) - \frac{t}{T}$$

where $\eta(0)$ is the initial positive feedback rate, t – the number of feedbacks, and T – the total number of feedbacks.

Let the co-ordinate value of the group element g in the 2-D (or multidimensional) array be x_g, y_g, then the range of the neighborhood is based on the points $[x_g + N_g(t), y_g + N_g(t)]$ and points $[x_g - N_g(t), y_g - N_g(t)]$ are squares with upper right and lower left corners. Its correction is:

$$N_g(t) = \text{int}\, N_g(0) - \frac{t_1}{T}$$

where int[x] is the rounding symbol and $N_g(0)$ is the initial value of $N_s(t)$.

– Make $t = t + 1$, return to second step, until $t = T$.

By empirical mode decomposition (EMD), the original time series $X(t)$ can be written as the sum of all the eigenmode functions (IMF) and residuals $rse(t)$:

$$X(t) = \sum_{k=1}^{n} c_k(t) + rse(t)$$

where n is the total number of decomposed IMF and $c_k(t)$ denotes the k^{th} IMF. Each IMF must satisfy two conditions: the difference between the number of zeros and extremes of the function is less than or equal to 1 throughout the entire time range. At any point in time, the envelopes of the local maxima and local minima, *i.e.*, the upper and lower envelopes, are zero on average. The average value is zero. The IMF decomposed by EMD are quasi-periodic, and the cycle of each IMF represents the cycle of a specific type of event.

In order to obtain the amplitude and phase time series of each IMF, we introduce the Hilbert transform based on the EMD decomposition:

– Compute the conjugate pair for the k^{th} IMF:

$$c_k(t)\,.\,y_k(t) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{c_k(t')}{t - t'} \mathrm{d}t'$$

where c_k is the cauchy principal value.

– The complex conjugate function consisting of $c_k(t)$ and $y_k(t)$ can be expressed:

$$c_k(t) + iy_k(t) = A_k(t)\mathrm{e}^{i\phi_k(t)} \tag{19}$$

– The amplitude and phase can then have the following equation calculated:

$$A_k(t) = [c_k^2(t) + y_k^2(t)]^{\frac{1}{2}}, \quad \phi_k(t) = \arctan\left[\frac{y_k(t)}{c_k(t)}\right] \tag{20}$$

Numerical simulation testing

The stability test of variables primarily employs unit root tests to validate the smoothness of time series data. In the following section, we will simulate and analyze the original data of a financial firm. A stochastic and stable long-term relationship exists between the variables under study. The direct effect test of the variables is applied, followed by regression dynamics analysis to test hypotheses and the relationship of the coefficients of influence between the variables, tab. 1. This is done in order to construct the three types.

Table 1. The ordinary least squares results of regression dynamics analysis

Variant	T_1	T_2	T_3
C	4.866***(49.034)	–1.471***(–7.866)	–1.567***(–8.410)
Financial flexibilities	0.081***(4.808)		0.039***(8.276)
Shareholding concentration		0.018*(2.506)	0.013*(2.063)
Management shareholding		0.115***(10.410)	0.121***(9.997)
Board size		0.009***(5.891)	0.009**(3.814)
R^2	0.229	0.323	0.329
Adjusted R^2	0.228	0.322	0.328
Standard error of regression	7.744	6.469	6.439
Sum squared resid	443043.700	309011.800	306171.800
Log likelihood	–25608.620	–24277.530	–24243.420
F-statistic	219.277***	1172.500***	904.535***
Prob (F-statistic)	0.000	0.000	0.000

*, $p < 0.05$; **, $p < 0.01$; ***, $p < 0.001$

Conclusions

The empirical analysis revealed a significant positive correlation between corporate governance and operating performance. From the perspective of corporate governance, the concentration of the 2nd to 5th largest shareholders is also significantly positively correlated with the company operating performance. Furthermore, the company strong profitability is conducive to the performance of listed companies, which has a positive impact on the company's operating performance. An improvement in a company's financial flexibility will result in enhanced debt capacity and liquidity, thereby enhancing the company's ability to withstand risks. Consequently, enhanced financial flexibility can positively impact operational performance. Furthermore, an increase in cash flexibility and elasticity can diversify the manner in which a business operates, which is conducive to the realization of sustainable growth in business performance.

Moreover, the empirical analysis indicates that absorptive capacity plays a positive role in the positive correlation between financial flexibility and firm performance. It is inevitable that this will result in an improvement in the solvency of the company and that it will become easier to gain an advantage in high-risk market competition. Which has important reference significance for this research in exploring control strategies for financial stability, and various optimization methods and control theories play a crucial role [19].

References

[1] Samuelson, P. A., Law of Conservation of the Capital-Output Ratio, *Proc. Natl. Acad. Sci. Appl. Math. Sci., 67* (1970), 3, pp. 1477-1479

[2] Wu, Y., *et al.,* A Remark on Samuelson's Variational Principle in Economics, *Applied Mathematics Letters, 84* (2018), Oct., pp. 143-147

[3] Qin, S. T., Ge, Y., A Novel Approach to Markowitz Portfolio Model without Using Lagrange Multipliers, *Int. J. Non-linear Sci. Numer., 11* (2010), Suppl., pp. S331-S334

[4] Anjum, N., *et al.,* Two-Scale Fractal Theory for the Population Dynamics, *Fractals, 29* (2021), 7, 2150182

[5] He, J.-H., *et al.,* Evans Model for Dynamic Economics Revised, *AIMS Mathematics, 6* (2021), 9, pp. 9194-9206

[6] Das, A., Saha, A., Dynamical Survey of the Dual Power Zakharov-Kuznetsov-Burgers Equation with External Periodic Perturbation, *Comput. Math. Appl., 76* (2022), 5, pp. 1174-1193

[7] Matychyn, I., Onyshchenko, V., Optimal Control of Linear Systems with Fractional Derivatives, *Fractional Calculus and Applied Analysis, 21* (2018), 1, pp. 134-150

[8] Bernardo, A. E., Chowdhry, B., Resources, Real Options, and Corporate Strategy, *Journal of Financial Economics, 63* (2002), 2, pp. 211-234

[9] Borman, W. C., Motowidlo, S. J., Task Performance and Contextual Performance: The Meaning for Personnel Selection Research, *Human Performance, 10* (1997), 2, pp. 99-109

[10] Zhu, Y. G., Uncertain Fractional Differential Equations and an Interest Rate Model, *Mathematical Methods in the Applied Science, 38* (2023), 15, pp. 3359-3368

[11] Holderness, C. G., Equity Issuances and Agency Costs: The Telling Story of Shareholder Approval Around the World, *Journal of Financial Economics, 129* (2018), 3, pp. 415-439

[12] Lane, P. J., *et al.,* The Reification of Absorptive Capacity: A Critical Review and Rejuvenation of the Construct, *Academy of Management Review, 31* (2016), 4, pp. 833-863

[13] Marchica, M. T., Mura, R., Financial Flexibility, Investment Ability, and Firm Value: Evidence from Firms with Spare Debt Capacity, *Financial Management, 39* (2020), 4, pp. 1339-1365

[14] Nickell, S. J., Competition and Corporate Performance, *Journal of Political Economy, 104* (2016), 4, pp. 724-746

[15] Machado, J., *Handbook of Fractional Calculus with Applications*, De Gruyter, Berlin, 2019

[16] Kilbas, A., *et al*., *Theory and Application of Fractional Differential Equation*, Elsevier Science B. V., Amsterdam, 2006

[17] He, J.-H., *et al.,* Beyond Laplace and Fourier Transforms: Challenges and Future Prospects, *Thermal Science, 27 (*2023), 6B, 5075-5089

[18] Anjum, N., *et al.,* Free Vibration of a Tapered Beam by the Aboodh Transform-Based Variational Iteration Method, *Journal of Computational Applied Mechanics, 55* (2024), 3, pp. 440-450

[19] Ahmad, N. A. W., *et al.,* Optimizing Royalty Payments for Maximum Economic Benefit: A Case Study Utilizing Modified Shooting and Discretization Methods, *Advances in Differential Equations and Control Processes, 31* (2024), 4, pp. 563-581

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