FRACTAL SOLITARY WAVE SOLUTIONS AND VARIATIONAL PRINCIPLE OF THE FRACTAL GENERAL KADOMTSEV-PETVIASHVILI EQUATION

by

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Original scientific paper
https://doi.org/10.2298/TSCI2503775S

This work examines the fractal generalized Kadomtsev-Petviashvili equation, which describes the evolution of non-linear long waves of small amplitude. The fractal traveling wave transformation and the fractal semi-inverse method are employed to derive a fractal variational principle, which was found to be a strong minimum according to the He-Weierstrass function. The solution of the two examples is presented in the form of images. This paper demonstrates that the fractal dimension affects the waveform of the generalized Kadomtsev-Petviashvili equation.

Key words: *He's fractal derivatives, fractal variational principle, semi-inverse method, He-Weierstrass function, fractal solitary wave solutions*

Introduction

The generalized Kadomtsev-Petviashvili equation (GKPE) is a PDE that describes non-linear wave motion. It has been demonstrated that this system is integrable [1, 2]. It is an extension of the 1-D Korteweg-de Vries equation in the two spatial dimensions [3]. This enables the description of 2-D fluctuation phenomena in non-linear media in hydrodynamics and 2-D fluctuation phenomena with weak dispersion in plasma physics. It has been the subject of intense study in a number of disciplines, including ocean physics [4], relativistic fluid dynamics [5], condensed matter physics [6], and wave propagation [7, 8]. Initially proposed to treat slowly changing perturbation waves in dispersive media, KPE has been further studied in subsequent studies. Ma *et al.* [9] developed the eKPE and derived multiple solutions and pump solutions. Duan [10] demonstrated the stability of lateral perturbations of non-linear acoustic solitary waves in dusty plasmas by KPE. Kalamvokas *et al.* [11] employed the inverse spectral transform method to investigate the KPE. In a study conducted by Alves *et al.* [12], the existence of solitary waves in KPE was investigated through the use of variational methods with potential in R^2.

The GKPE is a PDE that describes non-linear waves, which can be expressed as [11, 13, 14]:

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$$\frac{\mathrm{D}\left(\dfrac{\mathrm{D}\psi}{\mathrm{D}t}+6\psi\dfrac{\mathrm{D}\psi}{\mathrm{D}x}+\dfrac{\mathrm{D}^3\psi}{\mathrm{D}x^3}\right)}{\mathrm{D}x}+a\frac{\mathrm{D}^2\psi}{\mathrm{D}y^2}=0 \tag{1}$$

where x and y are spatial dimensions, t – time, and a – a constant.

In order to study GKPE in fractal space, it is necessary to introduce the concept of the two-scale fractal derivative [15]. In recent years, the computation of fractional order derivatives has emerged as a prominent topic of research, with applications in the resolution of intricate mathematical and physical problems. The definition of fractal derivative, as proposed by Professor Ji-Huan He and further developed by his students and his colleagues [16-21], represents an effective tool for modeling complex mathematical or physical models in fractal space. The definition of He's fractal derivative is provided below:

$$\frac{{}^{H}\mathrm{D}\Im}{\mathrm{D}\sigma^{\phi}}(\sigma,t)=\Gamma(1+\phi)\lim_{\substack{\sigma-\sigma_0=\Delta\sigma\\ \Delta\sigma\neq0}}\frac{\Im(\sigma,t)-\Im(\sigma_0,t)}{(\sigma-\sigma_0)^{\phi}} \tag{2}$$

$$\frac{{}^{H}\mathrm{D}\Im}{\mathrm{D}t^{\vartheta}}(\sigma,t)=\Gamma(1+\vartheta)\lim_{\substack{t-t_0=\Delta t\\ \Delta t\neq0}}\frac{\Im(\sigma,t)-\Im(\sigma,t_0)}{(t-t_0)^{\vartheta}} \tag{3}$$

The previous definition of fractal derivatives also follows the chain rule shown below [22, 23]:

$$\frac{{}^{H}\mathrm{D}T}{\mathrm{D}\sigma^{\phi}\mathrm{D}t^{\vartheta}}=\frac{{}^{H}\mathrm{D}}{\mathrm{D}\sigma^{\phi}}\left(\frac{{}^{H}\mathrm{D}T}{\mathrm{D}t^{\vartheta}}\right) \tag{4}$$

$$\frac{{}^{H}\mathrm{D}T}{\mathrm{D}\sigma^{3\phi}}=\frac{{}^{H}\mathrm{D}}{\mathrm{D}\sigma^{\phi}}\left[\frac{{}^{H}\mathrm{D}}{\mathrm{D}\sigma^{\phi}}\left(\frac{{}^{H}\mathrm{D}T}{\mathrm{D}\sigma^{\phi}}\right)\right] \tag{5}$$

$$\frac{{}^{H}\mathrm{D}T}{\mathrm{D}\sigma^{3\phi}\mathrm{D}t^{\vartheta}}=\frac{{}^{H}\mathrm{D}}{\mathrm{D}\sigma^{\phi}}\left\{\frac{{}^{H}\mathrm{D}}{\mathrm{D}\sigma^{\phi}}\left[\frac{{}^{H}\mathrm{D}}{\mathrm{D}\sigma^{\phi}}\left(\frac{{}^{H}\mathrm{D}T}{\mathrm{D}t^{\vartheta}}\right)\right]\right\} \tag{6}$$

According to the definition of He's fractal derivative with its chain rule, eq. (1) can have the following form:

$$\frac{\mathrm{D}^{\alpha}}{\mathrm{D}x^{\alpha}}\left[\frac{\mathrm{D}^{\gamma}\psi}{\mathrm{D}t^{\gamma}}+6\psi\frac{\mathrm{D}^{\alpha}\psi}{\mathrm{D}x^{\alpha}}+\frac{\mathrm{D}^{3\alpha}\psi}{\mathrm{D}x^{3\alpha}}\right]+a\frac{\mathrm{D}^{2\beta}\psi}{\mathrm{D}y^{2\beta}}=0 \tag{7}$$

where x and y are spatial dimensions, t – the time, and α, β, and γ represent fractional dimensions [24, 25].

Fractal variational principle

In this section, the fractal variational principle (FVP) [26-29] will be utilized to find a variational formulation of eq. (7). We introduce the following transformations [30]:

$$\begin{aligned}&\psi(x^{\alpha},y^{\beta},t^{\gamma})=\Xi(\varepsilon^{\eta})\\&\varepsilon^{\eta}=u^{\eta}x^{\alpha}+v^{\eta}y^{\beta}-w^{\eta}t^{\gamma}\end{aligned}\tag{8}$$

where u, v, and w are non-zero constants and η – the fractional dimension, which will be discussed later.

Through eq. (8), we can transform the complex PDE to get the ODE related to $\Xi(\varepsilon^{\eta})$:

$$-u^{\eta}w^{\eta}\frac{\mathrm{D}^{2\eta}\Xi}{\mathrm{D}\varepsilon^{2\eta}}+6u^{2\eta}\left[\left(\frac{\mathrm{D}^{\eta}\Xi}{\mathrm{D}\varepsilon^{\eta}}\right)^{2}+\Xi\frac{\mathrm{D}^{2\eta}\Xi}{\mathrm{D}\varepsilon^{2\eta}}\right]+u^{4\eta}\frac{\mathrm{D}^{4\eta}\Xi}{\mathrm{D}\varepsilon^{4\eta}}+av^{2\eta}\frac{\mathrm{D}^{2\eta}\Xi}{\mathrm{D}\varepsilon^{2\eta}}=0\tag{9}$$

The integral operation on eq. (9), and ignoring the constant term, we can obtain:

$$(-u^{\eta}w^{\eta}+av^{2\eta})\frac{\mathrm{D}^{\eta}\Xi}{\mathrm{D}\varepsilon^{\eta}}+6u^{2\eta}\left(\Xi\frac{\mathrm{D}^{\eta}\Xi}{\mathrm{D}\varepsilon^{\eta}}\right)+u^{4\eta}\frac{\mathrm{D}^{3\eta}\Xi}{\mathrm{D}\varepsilon^{3\eta}}=0\tag{10}$$

Integrating again for eq. (10), and organizing it, we get:

$$u^{4\eta}\frac{\mathrm{D}^{2\eta}\Xi}{\mathrm{D}\varepsilon^{2\eta}}+3u^{2\eta}\Xi^{2}+(av^{2\eta}-u^{\eta}w^{\eta})\Xi=0\tag{11}$$

By using the semi-inverse method [32-40], eq. (11) can be transformed to obtain the fractal variational formula:

$$L(\Xi)=\left[-\frac{1}{2}u^{4\eta}\left(\frac{\mathrm{D}^{\eta}\Xi}{\mathrm{D}\varepsilon^{\eta}}\right)^{2}+u^{2\eta}\Xi^{3}+\frac{1}{2}av^{2\eta}\Xi^{2}-\frac{1}{2}u^{\eta}w^{\eta}\Xi^{2}\right]d\varepsilon^{\eta}\tag{12}$$

From eq. (12), we can obtain the He-Weierstrass function [41]:

$$\begin{aligned}&E\left(\varepsilon^{\eta},\Xi,\frac{\mathrm{D}^{\eta}\Xi}{\mathrm{D}\varepsilon^{\eta}},\omega\right)=\frac{1}{2}\omega^{2}-\left[\frac{\Xi^{3}}{u^{2\eta}}-\left(\frac{w^{\eta}}{2u^{3\eta}}-\frac{av^{2\eta}}{2u^{4\eta}}\right)\Xi^{2}\right]-\\&-\left\{\frac{1}{2}\left(\frac{D^{\eta}\Xi}{D\varepsilon^{\eta}}\right)^{2}-\left[\frac{\Xi}{u^{2\eta}}-\left(\frac{w^{\eta}}{2u^{3\eta}}-\frac{av^{2\eta}}{2u^{4\eta}}\right)\right]\Xi^{2}\right\}+\frac{\mathrm{D}^{\eta}\Xi}{\mathrm{D}\varepsilon^{\eta}}\left(\frac{\mathrm{D}^{\eta}\Xi}{\mathrm{D}\varepsilon^{\eta}}-\omega\right)=\\&=\frac{1}{2}\omega^{2}-\frac{1}{2}\left(\frac{\mathrm{D}^{\eta}\Xi}{\mathrm{D}\varepsilon^{\eta}}\right)^{2}-\left(\omega-\frac{\mathrm{D}^{\eta}\Xi}{\mathrm{D}\varepsilon^{\eta}}\right)\frac{\mathrm{D}^{\eta}\Xi}{\mathrm{D}\varepsilon^{\eta}}\end{aligned}\tag{13}$$

where variable:

$$\omega=\frac{\mathrm{D}^{\eta}\Xi}{\mathrm{D}\varepsilon^{\eta}}$$

From eq. (13), it is clear that:

$$E(\varepsilon^\eta, \Xi, \Xi^\eta, \omega) = 0$$
$$\frac{\partial^2 E}{\partial \omega^2} > 0 \tag{14}$$

Equation (14) indicates that eq. (13) is a minimal FVP.

Solitary wave solutions

The purpose of this part is to find the solitary wave solution of eq. (12) by the obtained FVP. We consider the following form [42-44]:

$$\Xi(\varepsilon^\eta) = \kappa^\eta \operatorname{sech}^2(q^\eta \varepsilon^\eta) \tag{15}$$

where $\kappa \neq 0,\ q \neq 0,$ and the values of κ and q will change as η changes.

Combining eq. (12) with eq. (15), we will get the following expression:

$$L(\kappa^\eta, q^\eta) = \int_0^\infty \Big[-2u^{4\eta}\kappa^{2\eta} q 2\eta \operatorname{sech}^4(q^\eta \varepsilon^\eta)\tanh^2(q^\eta \varepsilon^\eta) + u^{2\eta}\kappa^{3\eta}\operatorname{sech}^6(q^\eta \varepsilon^\eta) +$$
$$+\frac{(av^{2\eta} - u^\eta w^\eta)\kappa^{2\eta}}{2}\operatorname{sech}^4(q^\eta \varepsilon^\eta)\Big]\mathrm{d}\varepsilon^\eta =$$
$$= \int_0^\infty \left[-2u^{4\eta}\kappa^{2\eta} q^\eta \operatorname{sech}^4(\varphi)\tanh^2(\varphi) + \frac{u^{2\eta}\kappa^{3\eta}}{q^\eta}\operatorname{sech}^6(\varphi) + \frac{(av^{2\eta} - u^\eta w^\eta)\kappa^{2\eta}}{2q^\eta}\operatorname{sech}^4(\varphi)\right]\mathrm{d}\varphi =$$
$$= \frac{-4u^{4\eta}\kappa^{2\eta} q^\eta + 8u^{2\eta}\kappa^{3\eta} + 5(av^{2\eta} - u^\eta w^\eta)\kappa^{2\eta}}{15 q^\eta} \tag{16}$$

According to He's variational method [44], taking the partial derivatives for κ and q, respectively, we get:

$$-\frac{8}{15}u^{4\eta}\kappa^\eta q^\eta + \frac{8}{5q^\eta}u^{2\eta}\kappa^{2\eta} - \frac{2u^\eta w^\eta \kappa^\eta}{3q^\eta} + \frac{2av^{2\eta}\kappa^\eta}{3q^\eta} = 0$$
$$\frac{4u^{2\eta}\kappa^{2\eta}}{15} + \frac{8u^{2\eta}\kappa^{3\eta}}{15q^{2\eta}} - \frac{u^\eta w^\eta \kappa^{2\eta}}{3q^{2\eta}} + \frac{av^{2\eta}\kappa^{2\eta}}{3q^{2\eta}} = 0 \tag{17}$$

Solving eq. (17), we can determine the values of κ and q:

$$\kappa^\eta = \frac{1}{2}\left(\frac{w^\eta}{u^\eta} - \frac{av^{2\eta}}{u^{2\eta}}\right) \tag{18}$$

$$q^\eta = \frac{1}{2u^{2\eta}}\sqrt{u^\eta w^\eta - av^{2\eta}} \tag{19}$$

So eq. (15) can be represented as:

$$\Xi(\varepsilon^{\eta}) = \frac{1}{2}\left(\frac{w^{\eta}}{u^{\eta}} - \frac{av^{2\eta}}{u^{2\eta}}\right)\mathrm{sech}^2\left(\frac{\varepsilon^{\eta}}{2u^{2\eta}}\sqrt{u^{\eta}w^{\eta} - av^{2\eta}}\right) \tag{20}$$

With the fractal variational formula, the solitary wave solution of the fractal generalized Kadomtsev-Petviashvili equation can be approximated:

$$\Xi(x^{\alpha}, y^{\beta}, t^{\gamma}) = \frac{1}{2}\left(\frac{w^{\eta}}{u^{\eta}} - \frac{av^{2\eta}}{u^{2\eta}}\right)\mathrm{sech}^2\left[\frac{1}{2u^{2\eta}}\sqrt{u^{\eta}w^{\eta} - av^{2\eta}}\,(u^{\eta}x^{\alpha} + v^{\eta}y^{\beta} - w^{\eta}t^{\gamma})\right] \tag{21}$$

Two examples

In this section, we will show two numerical examples and give the corresponding images to show the dynamics of eq. (21) in the space of different fractal dimensions. Considering the properties of the fractal GKPE, solitary wave solutions in single direction are considered.

Example 1. Consider variables in eq. (21), let $u = 4$, $v = 1$, $w = 3$, $\varepsilon_0 = 1$, $\alpha = 1$, $\beta = 1$, $\gamma = 1$, $y = 1$, and $a = 3$. Selecting different fractional order dimension values η, we obtain the solitary wave solution shown in fig. 1.

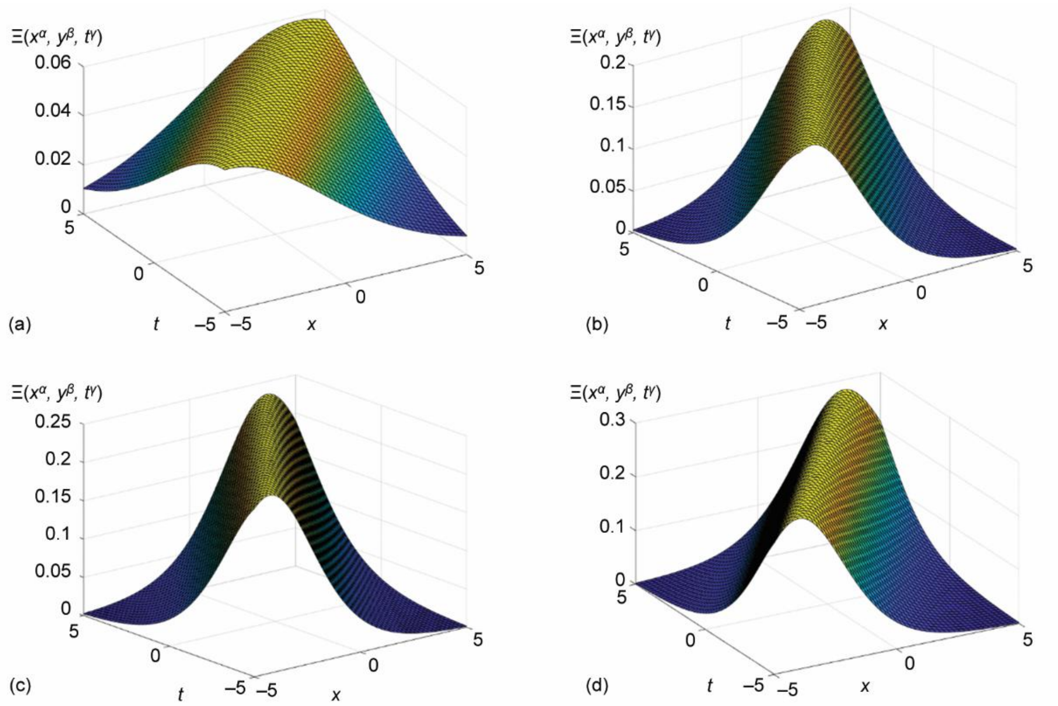

Figure 1. The 3-D images with different fractional dimension values, which (a) η = 0.5, (b) η = 0.7, (c) η = 0.8, (d) η = 1

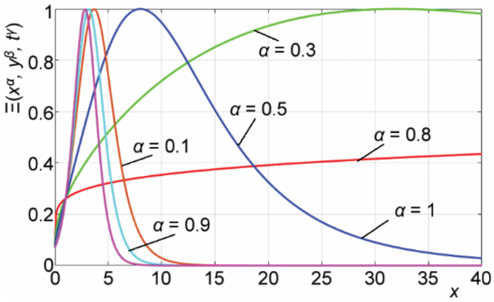

Figure 2. Comparison of different fractal dimensions

Example 2. In this example, we use $u^\eta = 1$, $v^\eta = 1$, $w^\eta = 3$, $a = 1$, $y^\beta = 1$, $t^\gamma = 1$. Take the values $\alpha = 0.1$, 0.3, 0.5, 0.8, 0.9, and 1 to represent different fractal dimensions, and draw 2-D graphs with different fractal dimensions as in fig. 2.

In fig. 1, we use the example in *Example 1* to draw 3-D images of isolated waves in different fractal dimensions. From fig. 1, we can see that the isolated wave receives the influence of different fractal dimensions, and the morphology of the wave and the position of the wave peaks change in the same variation range. It is worth noting that the total variation of the 3-D image of the isolated wave becomes progressively smaller as the fractal dimension becomes progressively larger. In fig. 2 we consider the variation of waves in a single direction. In this case, the change of the isolated wave in a single dimension is more drastic as the fractal dimension increases, but the value of the wave peak does not change. The final result shows that the non-smooth boundary does not affect the peak value of the isolated wave.

Conclusion

In this study, we use the FVP method to obtain fractal isolated wave solutions of the GKPE and present them in different fractal dimensions. The FVP method is an efficient and simple way to deal with non-linear PDE of wave motion and to find new exact solutions for these non-linear evolution equations.

Acknowledgment

This article was supported by the Provincial Cultivation Fund for the high-level scientific research project of Jiaozuo Normal College (Grant No. GPY 2022-07 and Grant No. GPY2020-05) and Henan Provincial Key Research Project Fund for Higher Education Institutions (Grant Nos. 22B110007 and 20B110009 24B110008 and 22B110007). We are particularly grateful to the anonymous reviewers for their constructive and many critical comments.

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Paper submitted: August 11, 2023
Paper revised: February 23, 2024
Paper accepted: March 1, 2024