VARIATIONAL APPROACH TO TIME-SPACE FRACTIONAL COUPLED BOITI-LEON-PEMPINELLI EQUATION

by

Jun-Feng LU* and Meng-Jie CHEN

Department of Statistics, Zhejiang Gongshang University Hangzhou College of Commerce, Hangzhou, China

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This paper focuses on the variational approach to a time-space fractional coupled Boiti-Leon-Pempinelli equation. The fractional system can be transformed into the original coupled Boiti-Leon-Pempinelli equation by using the fractional complex transformation. The variational approach provides three new types of soliton solutions.

Key words: *variational approach, fractional complex transformation, soliton*

Introduction

Fractional calculus has been widely used to model various types of interdisciplinary problems in engineering and science [1-4]. Various fractional derivatives have been proposed for fractional calculus, including Riesz fractional derivative, Riemann-Liouville fractional derivative, Caputo fractional derivative, He's fractional derivative, and Jumarie's fractional derivative and others [3-7]. Fractional PDE based on the combination of fractional derivatives and differential equations have received much attention in recent decades due to their efficiency in modeling various phenomena in plasma physics, fluid mechanics, electrochemistry, optics, bioinformatics, and finance and other fields [4, 8-13]. Due to the non-local nature of fractional operators, it is difficult to directly obtain the exact solutions of PDE equations. Recently, some analytical and numerical methods have been presented to solve linear and nonlinear fractional differential equations [6, 10, 13-16]. In this paper, we consider the following time-space fractional coupled Boiti-Leon-Pempinelli (BLP) equation:

$$\begin{gathered}\frac{\partial^\gamma}{\partial y^\gamma}\left(\frac{\partial^\alpha u}{\partial t^\alpha}\right)=\frac{\partial^\gamma}{\partial y^\gamma}\left[\frac{\partial^\beta}{\partial x^\beta}\left(u^2-\frac{\partial^\beta u}{\partial x^\beta}\right)\right]+2\frac{\partial^{3\beta} v}{\partial x^{3\beta}}\\ \frac{\partial^\alpha v}{\partial t^\alpha}=\frac{\partial^{2\beta} v}{\partial x^{2\beta}}+2u\frac{\partial^\beta v}{\partial x^\beta}\end{gathered}\tag{1}$$

where α, β, and γ are given constants in (0, 1], the fractional operators:

$$\frac{\partial^\alpha u}{\partial t^\alpha},\ \frac{\partial^\beta u}{\partial x^\beta},\ \text{and}\ \frac{\partial^\gamma u}{\partial y^\gamma}$$

* Corresponding author, e-mail: ljfblue@hotmail.com

are defined by He's fractional derivatives [4, 14-17]:

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}}=\frac{1}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\int_{t_0}^{t}(\zeta-t)^{n-\alpha-1}[u_0(x,y,\zeta)-u(x,y,\zeta)]\mathrm{d}\zeta \tag{2}$$

$$\frac{\partial^{\beta} u}{\partial x^{\beta}}=\frac{1}{\Gamma(n-\beta)}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\int_{x_0}^{x}(\xi-x)^{n-\beta-1}[u_0(\xi,y,t)-u(\xi,y,t)]\mathrm{d}\xi \tag{3}$$

$$\frac{\partial^{\gamma} u}{\partial y^{\gamma}}=\frac{1}{\Gamma(n-\gamma)}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\int_{y_0}^{y}(\eta-y)^{n-\gamma-1}[u_0(x,\eta,t)-u(x,\eta,t)]\mathrm{d}\eta \tag{4}$$

When $\alpha=\beta=\gamma=1$ eq. (1) reduces to the conventional coupled BLP equation:

$$\begin{gathered} u_{ty}=(u^2-u_x)_{xy}+2v_{xxx} \\ v_t=v_{xx}+2uv_x \end{gathered} \tag{5}$$

which can be used to simulate the interactions of two waves with different dispersion relations [18, 19]. The localized structure on the periodic background wave of the coupled BLP equation was obtained by using an object reduction in [18]. The mapping method was used to study the Jacobian elliptic wave structure and the periodic wave evolution behavior of (5) [19]. Some types of solutions of (5) have been provided in [20-24]. Explicit exact solutions of the coupled BLP equation were given in [20] using the extended tanh method. Feng *et al.* [21] obtained symmetry reduction solutions of the (2+1)-D BLP equation. Kumar *et al.* [22] applied similarity transformation method to obtain some more similarity solutions of (5). The Khater method was used to obtain the elliptic and solitary wave solutions [23]. The (1/G') expansion method was considered in [24] for solving the coupled BLP system, and the hyperbolic type solutions were further given. The study of different wave structures and solutions of the coupled BLP equation is useful. However, when the wave behavior of this non-linear equation is observed from a small time scale, or the solutions to (5) depend on the time history, the variables may become discontinuous about the time variable. To solve this problem, the coupled BLP equation can be considered in fractional time space. Due to the storage property of He's fractional derivative [4, 14, 15], we'll consider the coupled BLP equation with He's fractional operators. As mentioned in the previous paragraph, the non-local property and the complexity of the fractional operators in (1) lead to the difficulty of obtaining different types of solutions. To overcome this difficulty, the fractional complex transformation proposed by He is used to transform (1) into the original BLP eq. (5). Different from the existing approaches in [18-24], the variational approach is proposed to find the soliton solutions of the fractionally coupled BLP equation. Through the stationary conditions from the variational formulations [25, 26], three new types of soliton solutions are given in detail, including bright soliton solution, kinky-bright soliton solution, and bright-like soliton solution. Finally, some conclusions are drawn.

Fractional complex transformation for fractional equations

For illustrating the efficiency of fractional complex transformation, we consider the fractional partial differential equation:

$$f(u, u_t^{\alpha}, u_x^{\beta}, u_y^{\gamma}, u_t^{2\alpha}, u_x^{2\beta}, u_y^{2\gamma}, \cdots) = 0 \tag{6}$$

where the fractional derivatives in (6) are defined by (2)-(4), and $0 < \alpha, \beta, \gamma \le 1$ [4, 14-17].

The fractional complex transformation proposed by He can be formulated:

$$T = \frac{rt^{\alpha}}{\Gamma(1+\alpha)}, \quad X = \frac{px^{\beta}}{\Gamma(1+\beta)}, \quad Y = \frac{qy^{\gamma}}{\Gamma(1+\gamma)} \tag{7}$$

with three constants *r*, *p,* and *q*. The physical understanding of the transformations can be seen in [14-17, 27-31]. By (7), the fractional eq. (6) can be rewritten as an ordinary non-linear partial differential equation:

$$f(u, u_T, u_X, u_Y, u_{TT}, u_{XX}, u_{YY}, \cdots) = 0 \tag{8}$$

where

$$u_T = \frac{\partial u}{\partial T}, \quad u_X = \frac{\partial u}{\partial X}, \quad \text{and} \quad u_Y = \frac{\partial u}{\partial Y}$$

Variational approach for fractional coupled BLP equation

By the following fractional complex transformation:

$$T = \frac{t^{\alpha}}{\Gamma(1+\alpha)}, \quad X = \frac{x^{\beta}}{\Gamma(1+\beta)}, \quad Y = \frac{y^{\gamma}}{\Gamma(1+\gamma)} \tag{9}$$

we equivalently rewrite eq. (1) as the couple BLP equation:

$$\begin{aligned} u_{TY} &= (u^2 - u_X)_{XY} + 2v_{XXX} \\ v_T &= v_{XX} + 2uv_X \end{aligned} \tag{10}$$

We introduce an auxiliary variable $\xi = X + Y - cT$ with a constant *c*, and transform (10) as the following system:

$$\begin{aligned} -cu'' &= (u^2 - u')'' + 2v''' \\ -cv' &= v'' + 2uv' \end{aligned} \tag{11}$$

By integrating the first equation of (11), and substituting the formulation of v' and v'' into the second equation of (11), we have the following:

$$u'' - 2u^3 - 3cu^2 - c^2 u = 0 \tag{12}$$

The variational formulation for (12) can be given by the semi-inverse method [10, 32-37], which is defined by:

$$J(u) = \int \left[-\frac{1}{2}(u')^2 - \frac{1}{2}u^4 - cu^3 - \frac{c^2}{2}u^2 \right] \mathrm{d}\xi \tag{13}$$

We show that how to obtain the soliton-like solutions to (1). Three types of the soliton solutions will be given by the variational principles.

Bright soliton solution

According to the variational theory [38, 39], the bright soliton solution to (12) is assumed in the form:

$$u = p_1 \operatorname{sech}(\xi) \tag{14}$$

with a unknown constant p_1 determined later.

The following variational principle can be followed by substituting (14) into (13):

$$J(p_1) = \int_0^{\infty} \left\{ -\frac{1}{2}[p_1 \operatorname{sech}(\xi) \tanh(\xi)]^2 - \frac{1}{2} p_1^4 \operatorname{sech}^4(\xi) - c p_1^3 \operatorname{sech}^3(\xi) - \right.$$

$$\left. - c p_1^3 \operatorname{sech}^3(\xi) - \frac{c^2}{2} p_1^2 \operatorname{sech}^2(\xi) \right\} \mathrm{d}\xi = -\frac{1}{12} p_1^2 (2 + 6c^2 + 4p_1^2 + 3c\pi p_1) \tag{15}$$

The stationary condition for previous variational formulation can be given by:

$$\frac{\mathrm{d}J(p_1)}{\mathrm{d}p_1} = 0 \tag{16}$$

which results in the following root:

$$p_1 = -\frac{9c\pi}{32} \pm \frac{\sqrt{81c^2\pi^2 - 768c^2 - 256}}{32} \tag{17}$$

By (15), the bright soliton solution to (12) is formulated by:

$$u = \left(-\frac{9c\pi}{32} \pm \frac{\sqrt{81c^2\pi^2 - 768c^2 - 256}}{32} \right) \operatorname{sech}(\xi) \tag{18}$$

By the fractional complex transformation, we have the following fractional bright soliton solution to (1):

$$u = \left(-\frac{9c\pi}{32} \pm \frac{\sqrt{81c^2\pi^2 - 768c^2 - 256}}{32} \right) \operatorname{sech}\left[\frac{x^\beta}{\Gamma(1+\beta)} + \frac{y^\gamma}{\Gamma(1+\gamma)} - \frac{ct^\alpha}{\Gamma(1+\alpha)} \right] \tag{19}$$

Kinky-bright soliton solution

The kinky-bright soliton solution to (12) is given by:

$$u = p_2 \operatorname{sech}^2(\xi) \tag{20}$$

where p_2 is a unknown constant.

By (13) and (20), we have:

$$J(p_2)=\int_0^\infty\left\{-\frac{1}{2}[p_2\operatorname{sech}^2(\xi)\tanh(\xi)]^2-\frac{1}{2}p_2^4\operatorname{sech}^8(\xi)-cp_2^4\operatorname{sech}^6(\xi)-\right.$$

$$\left.-\frac{c^2}{2}p_2^2\operatorname{sech}^4(\xi)\right\}\mathrm{d}\xi=-\frac{1}{105}p_2^2(28+35c^2+24p_2^2+56cp_2) \tag{21}$$

The stationary condition:

$$\frac{\mathrm{d}J(p_2)}{\mathrm{d}p_2}=0$$

for (21) implies that:

$$p_2=-\frac{7}{8}c\pm\frac{\sqrt{21c^2-336}}{24} \tag{22}$$

By (20) and (9) together with (22), we have the following fractional kinky-bright soliton solution:

$$u=\left(-\frac{7}{8}c\pm\frac{\sqrt{21c^2-336}}{24}\right)\operatorname{sech}^2\left[\frac{x^\beta}{\Gamma(1+\beta)}+\frac{y^\gamma}{\Gamma(1+\gamma)}-\frac{ct^\alpha}{\Gamma(1+\alpha)}\right] \tag{23}$$

Bright-like soliton solution

Assume that the bright-like solition solution to (12) is defined by:

$$u=\frac{p_3}{1+\cosh(\xi)} \tag{24}$$

By (13), we have the following formulation:

$$J(p_3)=\int_0^\infty\left\{-\frac{1}{2}p_3^2\frac{\sinh^2(\xi)}{[1+\cosh(\xi)]^4}-\frac{1}{2}p_3^4\frac{1}{[1+\cosh(\xi)]^4}-\frac{cp_3^3}{[1+\cosh(\xi)]^3}-\right.$$

$$\left.-\frac{c^2p_3^2}{2[1+\cosh(\xi)]^2}\right\}\mathrm{d}\xi=-\frac{1}{210}p_3^2(7+35c^2+6p_3^2+28cp_3) \tag{25}$$

By

$$\frac{\mathrm{d}J(p_3)}{\mathrm{d}p_3}=0$$

it follows that:

$$p_3=-\frac{7}{4}c\pm\frac{\sqrt{21c^2-84}}{12} \tag{26}$$

Together with (24) and (26), we obtain the following bright-like soliton solution to (12):

$$u = \frac{-\frac{7}{4}c \pm \frac{\sqrt{21c^2 - 84}}{12}}{1 + \cosh(\xi)} \tag{27}$$

Then the fractional bright-like soliton solution to (1) can be written:

$$u = \frac{-\frac{7}{4}c \pm \frac{\sqrt{21c^2 - 84}}{12}}{1 + \cosh\left[\frac{x^\beta}{\Gamma(1+\beta)} + \frac{y^\gamma}{\Gamma(1+\gamma)} - \frac{ct^\alpha}{\Gamma(1+\alpha)}\right]} \tag{28}$$

Numerical results

In this section, some 3-D graphs of the obtained wave-type solutions are presented to show the numerical behavior of the time-space fractional coupled BLP equation. We consider the parameter $c = 5$ and study the propagation of three types of soliton solutions.

We first consider the bright soliton solutions to the fractionally coupled BLP eq. (1). Figure 1 plots the propagation of the classical BLP eq. (1) with $\alpha = \beta = \gamma = 1$ at space co-ordinate $y = 10$ or time co-ordinate $t = 1$. The behavior of the fractional BLP equation is different, the bright soliton solutions along the x and y space directions are plotted in fig. 2. The fractional dimensions for the left, middle, and right sides of fig. 2 are 0.3, 0.5, and 0.8, respectively. The kinky-bright soliton solutions to the classical BLP eq. (1) are shown in fig. 3, where the co-ordinates $y = 10$ and $t = 1$ are used in the left and right sides of fig. 2, respectively. The numerical results for the fractional space cases with $t = 1$ are shown in fig. 4. We note that the behavior of the kinky-bright soliton solutions is similar to that of the bright soliton solutions. The propagation of the bright-like soliton solutions with integer or fractional dimensions is shown in figs. 5 and 6. By comparing the results in these figures, the propagation behaviors become much more complicated and strongly non-linear as the fractional dimension approaches a small constant.

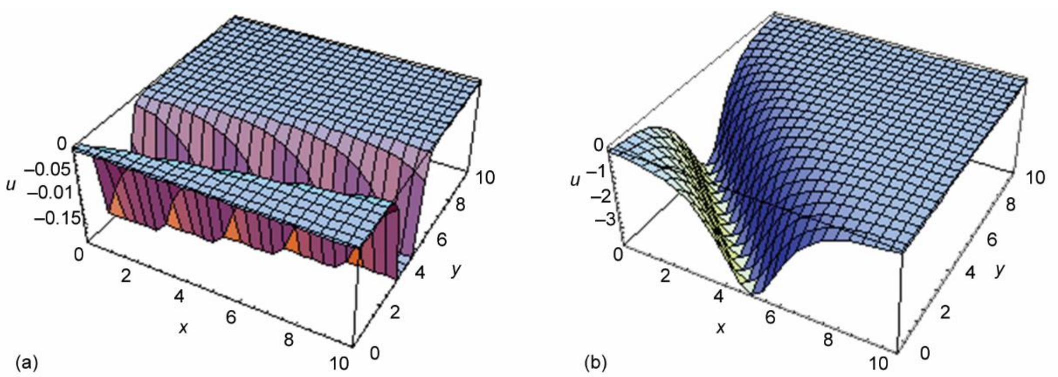

Figure 1. Numerical behavior of bright soliton solutions to (1); (a) y = 10 and (b) t = 1

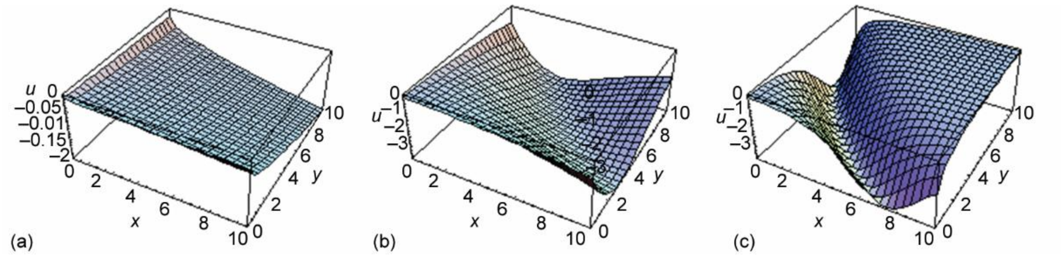

Figure 2. Numerical behavior of bright soliton solutions to (1) with fractional dimensions; (a) 0.3, (b) 0.5, and (c) 0.8

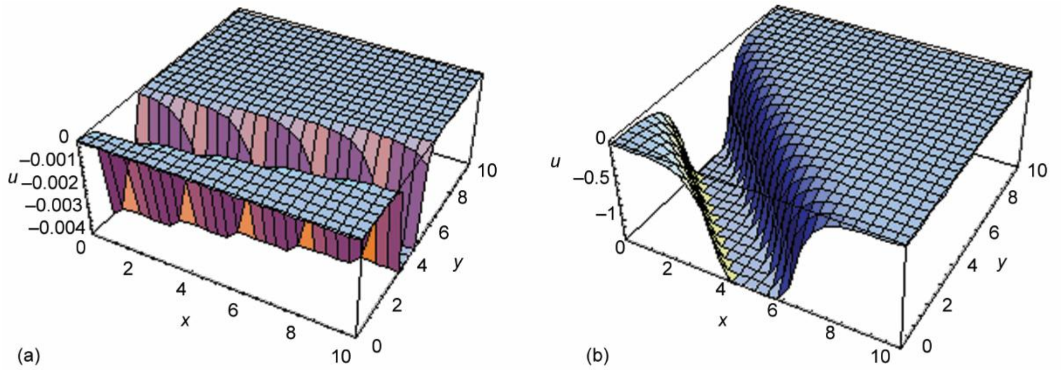

Figure 3. Numerical behavior of kinky-bright soliton solutions to (1); (a) $y = 10$ and (b) $t = 1$

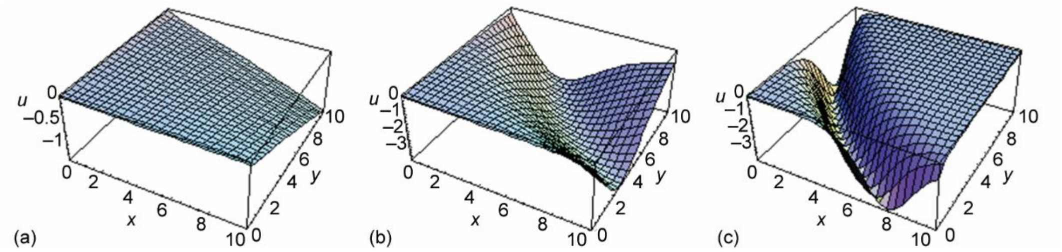

Figure 4. Numerical behavior of kinky-bright soliton solutions to (1) with fractional dimensions; (a) 0.3, (b) 0.5, and (c) 0.8

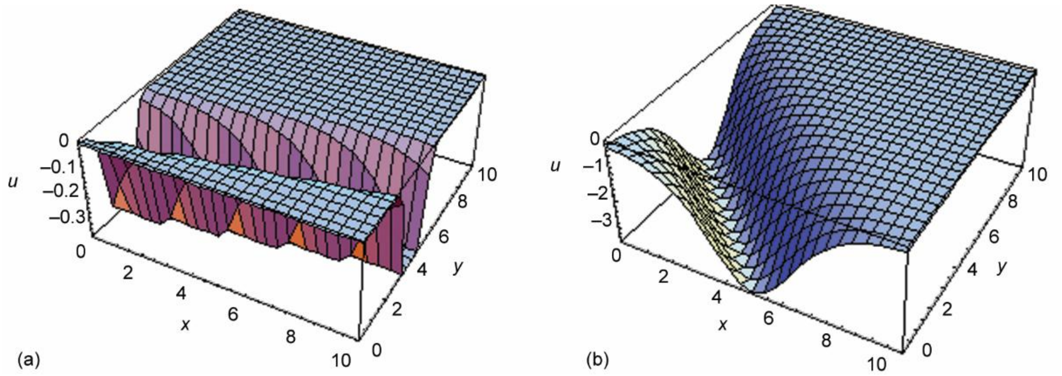

Figure 5. Numerical behavior of bright-like soliton solutions to (1); (a) $y = 10$ and (b) $t = 1$

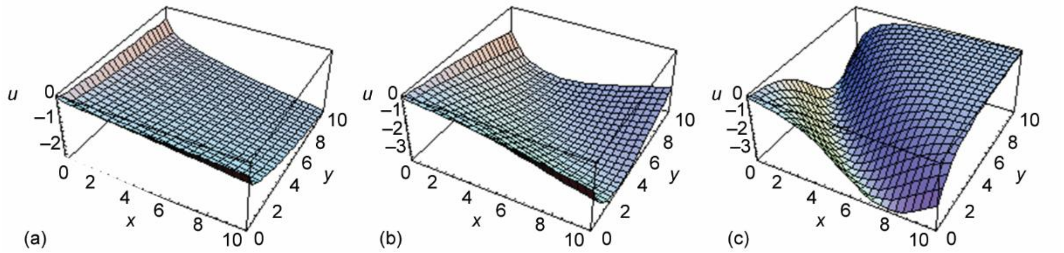

Figure 6. Numerical behavior of bright-like soliton solutions to (1) with fractional dimensions; (a) 0.3, (b) 0.5, and (c) 0.8

Conclusion

The variational approach together with the fractional complex transformation has been successfully used to solve the time-space fractional coupled BLP equation. Compared with the existing results in [18-24], we have two significant improvements: the variational formulation was provided for the time-space fractional coupled BLP equation and some types of soliton solutions were obtained for this fractional system, which was not touched in the existing literatures. The variational approach presented in this paper is available for other fractional non-linear PDE, and we will consider this topic in our future work.

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