EXACT SOLUTIONS AND NON-LINEAR ADAPTIVE BOUNDARY CONTROL PROBLEM OF THE FRACTIONAL MODIFIED GENERALIZED KdV-BURGERS EQUATION

by

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This article extends a modified generalized KdV-Burgers equation to the conformable fractional version with the aim of exploring novel solution structures and non-linear adaptive boundary control problems for fractional-order models. The results obtained include hyperbolic functional solutions of the fractional modified generalized KdV-Burgers equation and a non-linear adaptive boundary control law designed by attaching initial and boundary value conditions. It is demonstrated that the obtained hyperbolic functional solutions exhibit novel spatial structures, and that the solution of the fractional initial and boundary value problem is globally exponential stability.

Key words: *fractional modified generalized KdV-Burgers equation, hyperbolic function solution, initial and boundary value problem, non-linear adaptive boundary control law, auxiliary equation, global exponential stability*

Introduction

Fractional-order models have facilitated the advancement of numerous fields, as evidenced by the numerous citations in the literature, and fractal calculus [1, 2] becomes a useful tool for various engineering problems in human travel [3], thermal science [4], skin electrical impedance [5], fractal spacetime [6], fractal moisture permeability [7], fractal electrochemistry [8], and fractal thermodynamics [9]. This can be attributed to the fractional/fractal derivatives that they are equipped with. This also presents a compelling rationale for researchers to pursue the extension of traditional models to fractional-order partners, with the objective of achieving novel results or discoveries. Based on this starting point, Lu [10], Lu *et al.* [11], and Lu and Ma [12] studied the fractional-order versions of potential Yu-Toda-Sasa-Fukuyama, Bogoyavlenskii, and Benjamin-Bona-Mahony equations using the fractional complex transformation, the variational principle, and the homotopy perturbation method. A natural question that arises is whether the existing methods for integer-order equations can be extended to fractional differential equations. In general, the answer to this question depends not only on the equations being considered but also on the fractional derivatives used. For non-linear fractional differential equations, it is often challenging, if not impossible, to solve them by generalizing the known analytical methods due to the complexity of their fractional deriva-

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tives. Recently, the conformable fractional derivative proposed by Khalil *et al.* [13] has been employed to extend non-linear evolution equations to fractional-order cases [14]. Moreover, the extended fractional-order equations can still be solved based on the traditional analytical methods. This is largely due to the fact that Khalil *et al.*'s [13] derivatives possess properties similar to those of classical integer-order derivatives, and Leibniz's rule and the chain rule are still valid. Although the differences in the solution process are not significant, there are astonishing discoveries in the fractional power law results containing spatiotemporal variables. These results exhibit evolution characteristics that are consistent with the actual physical background. These characteristics include propagation with variable speed and wave width, as well as asymmetric spatial solution structures. These structures are not present in integer-order cases. Our findings indicate that variable speed propagation and asymmetric structures have already been observed in the deceleration propagation of anomalous diffusion in fractal dimensional media, the asymmetric structure of rogue waves, and the reverse tilting of solitons. Wang and He found the fractional spatio-temporal relation for fractal solitary waves [15], and now fractal soliton theory becomes a useful mathematical tool to solitary waves travelling along an unsmooth boundary [16-18].

This article mainly focuses on two parts of work. The first task is to exactly solve the fractional modified generalized KdV-Burgers (MGKdV-B) equation we propose for the first time:

$$\mathrm{D}_t^\theta u+\gamma_1 u^\alpha \mathrm{D}_x^\rho u-\nu \mathrm{D}_x^\rho \mathrm{D}_x^\rho u+\mu \mathrm{D}_x^\rho \mathrm{D}_x^\rho \mathrm{D}_x^\rho u+\gamma_2 \mathrm{D}_x^\rho \mathrm{D}_x^\rho \mathrm{D}_x^\rho \mathrm{D}_x^\rho u=0 \quad (0<\theta,\rho\leq 1) \tag{1}$$

which is the fractional extension of the known MGKdV-B equation [19]. Here x and t are all real variables, α is positive integer, D_x^ρ and D_t^θ are Khalil *et al.*'s [13] conformable fractional derivatives with the domains extended from $[0, \infty)$ to $(-\infty, +\infty)$, γ_1, ν, μ, and γ_2 are arbitrary constants. The second task is to design a non-linear adaptive boundary control (ABC) law for eq. (1) when $x \in (0, 1)$, $t \in [0, +\infty)$, u is real-valued function, γ_1, ν, μ, and γ_2 are all positive, and the initial and boundary value (IBV) conditions are taken as follows:

$$u(0,t)=0,\quad \mathrm{D}_x^\rho \mathrm{D}_x^\rho u(0,t)=0,\quad \mathrm{D}_x^\rho u(\sqrt{2},t)=\phi_1(t),\quad \mathrm{D}_x^\rho \mathrm{D}_x^\rho u(\sqrt{2},t)=\phi_2(t) \tag{2}$$

$$u(x,0)=u_0(x) \tag{3}$$

In 2020, Chentouf *et al.* [19] designed four different non-linear ABC laws for the similar integer-order case of the IBV problems (1)-(3).

Exact solutions

Through the transformation of spatiotemporal variables [20]:

$$u=u(\xi),\quad \xi=k\frac{x^\rho}{\rho}+l\frac{t^\theta}{\theta}+c \tag{4}$$

where k and l are constants that we will determine later, c is an arbitrary constant, eq. (1) can be written as:

$$l\frac{\mathrm{d}u}{\mathrm{d}\xi}+k\gamma_1 u^\alpha\frac{\mathrm{d}u}{\mathrm{d}\xi}-k^2\nu\frac{\mathrm{d}^2u}{\mathrm{d}\xi^2}+k^3\mu\frac{\mathrm{d}^3u}{\mathrm{d}\xi^3}+k^4\gamma_2\frac{\mathrm{d}^4u}{\mathrm{d}\xi^4}=0 \tag{5}$$

We assume that u has a polynomial form:

$$u = \sum_{j=0}^{n} a_j \phi^j \tag{6}$$

where $\phi = \phi(\xi)$ satisfies the following auxiliary equation [21, 22] with five special solutions and an arbitrary constant R:

$$\frac{\mathrm{d}\phi}{\mathrm{d}\xi} = \phi^2 + R \tag{7}$$

Then the integer n can be determined as:

$$n + 1 + n\alpha = n + 4 \Rightarrow n = \frac{3}{\alpha} \tag{8}$$

Therefore, we have $n = 3$ when $\alpha = 1$, or $n = 1$ when $\alpha = 3$.

For the case of $n = 1$ and $\alpha = 3$, the constants a_0, a_1, k, and l can be determined:

$$a_1 = \mp \frac{1}{2}\sqrt{\frac{\mu^2 + 4\gamma_2 \nu}{2R}} \left(\frac{3}{\gamma_1 \gamma_2^2}\right)^{\frac{1}{3}}, \quad a_0 = -\frac{\mu}{2}\left(\frac{1}{9\gamma_1 \gamma_2^2}\right)^{\frac{1}{3}} \tag{9}$$

$$k = \pm \frac{1}{4\gamma_2}\sqrt{\frac{\mu^2 + 4\gamma_2 \nu}{2R}}, \quad l = \mp \frac{\mu(7\mu^4 + 64\gamma_2 \mu^2 \nu + 144\gamma_2^2 \nu^2)}{576\gamma_2^3 \sqrt{2R(\mu^2 + 4\gamma_2 \nu)}} \tag{10}$$

Thus, we obtain the hyperbolic function solutions of eq. (1):

$$u = \pm \frac{1}{2}\sqrt{-\frac{\mu^2 + 4\gamma_2 \nu}{2}} \left(\frac{3}{\gamma_1 \gamma_2^2}\right)^{\frac{1}{3}} \tanh(\sqrt{-R}\xi) - \frac{\mu}{2}\left(\frac{1}{9\gamma_1 \gamma_2^2}\right)^{\frac{1}{3}} \tag{11}$$

with:

$$\xi = \pm \frac{1}{4\rho\gamma_2}\sqrt{\frac{\mu^2 + 4\gamma_2 \nu}{2R}} x^{\rho} \mp \frac{\mu(7\mu^4 + 64\gamma_2 \mu^2 \nu + 144\gamma_2^2 \nu^2)}{576\theta\gamma_2^3 \sqrt{2R(\mu^2 + 4\gamma_2 \nu)}} t^{\theta} + c \tag{12}$$

We can see from fig. 1 that when the parameters are taken to the same values, the solution (11) with fractional orders has a steeper kink structure compared to the corresponding integer-order case.

For the case where $n = 3$ and $\alpha = 1$, the constants a_0, a_1, a_2, and a_3 can be determined:

$$a_3 = -\frac{120k^3\gamma_2}{\gamma_1}, \quad a_2 = -\frac{15k^2\mu}{\gamma_1}, \quad a_1 = -\frac{15(608k^3 R\gamma_2^2 - k\mu^2 - 16k\gamma_2 \nu)}{76\gamma_1 \gamma_2} \tag{13}$$

$$a_0 = -\frac{608l\gamma_2^2 + 6080k^3 R\gamma_2^2 \mu + 13k\mu^3 + 56k\gamma_2 \mu\nu}{608k\gamma_1 \gamma_2^2} \tag{14}$$

where l is an arbitrary constant, k satisfies the algebraic equations:

$$15k^2R\mu(3040k^2R\gamma_2^2-13\mu^2-56\gamma_2\nu)(\mu^2+16\gamma_2\nu)=0 \tag{15}$$

$$15k^3R[369664k^4R^2\gamma_2^4-131\mu^4-696\gamma_2\mu^2\nu-704\gamma_2^2\nu^2-3040k^2R\gamma_2^2(\mu^2+16\gamma_2\nu)]=0 \tag{16}$$

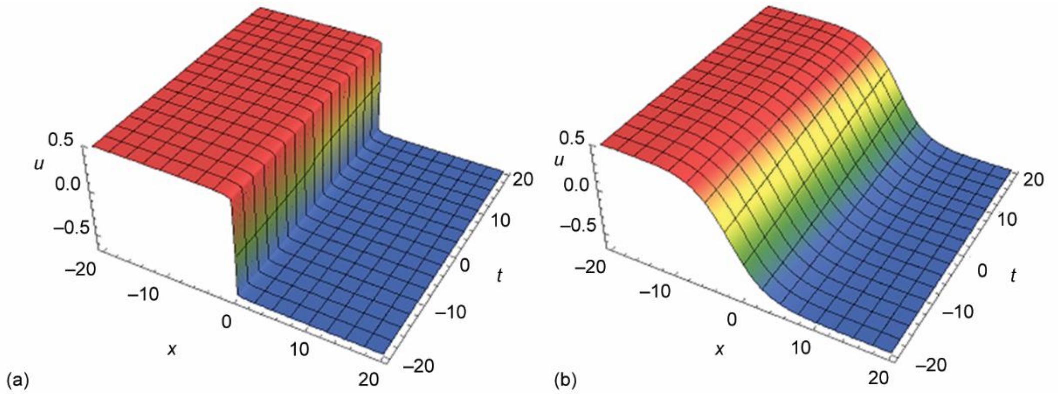

Figure 1. Solution (11) with R = –1, γ_1 = 3, γ_2 = –2,5, μ = 1, ν = 1, c = 0; (a) ρ = 1/9, θ = 1/3, and (b) ρ = 1, θ = 1 (*for color image see journal web site*)

Solving eqs. (15) and (16), we arrive at three sets of solutions for k combined with the constraints from ν, which are:

Case 1: $$k=\pm\frac{\mu}{24\sqrt{-R}\gamma_2} \text{ or } k=\pm\frac{\mu}{32\sqrt{-R}\gamma_2}, \quad \nu=\frac{3040k^2R\gamma_2^2-13\mu^2}{56\gamma_2} \tag{17}$$

Case 2: $$k=\pm\frac{1}{4\gamma_2}\sqrt{\frac{13\mu^2+56\gamma_2\nu}{190R}}, \quad \nu=-\frac{73\mu^2}{256\gamma_2} \text{ or } \nu=-\frac{47\mu^2}{144\gamma_2} \tag{18}$$

Case 3: $$k=\pm\frac{\mu}{8\sqrt{R}\gamma_2} \text{ or } k=\pm\frac{\mu}{8\sqrt{-R}\gamma_2}, \quad \nu=-\frac{\mu^2}{16\gamma_2} \tag{19}$$

By utilizing eqs. (4), (6), and (13)-(19), we can obtain some hyperbolic function solutions of eq. (1). For example, one of these solutions reads:

$$u=\frac{120k^3\gamma_2}{\gamma_1}(\sqrt{-R})^3\tanh^3(\sqrt{-R}\xi)-\frac{15k^2\mu}{\gamma_1}R\tanh^2(\sqrt{-R}\xi)+$$
$$+\frac{15(608k^3R\gamma_2^2-k\mu^2-16k\gamma_2\nu)}{76\gamma_1\gamma_2}\sqrt{-R}\tanh(\sqrt{-R}\xi)-$$
$$-\frac{608l\gamma_2^2+6080k^3R\gamma_2^2\mu+13k\mu^3+56k\gamma_2\mu\nu}{608k\gamma_1\gamma_2^2} \tag{20}$$

where k and ν satisfy any one of eqs. (17)-(19). In fig. 2, it can be seen that the solution (20) with fractional orders has symmetry about spatiotemporal variables, but the corresponding integer-order case does not have such a symmetrical structure.

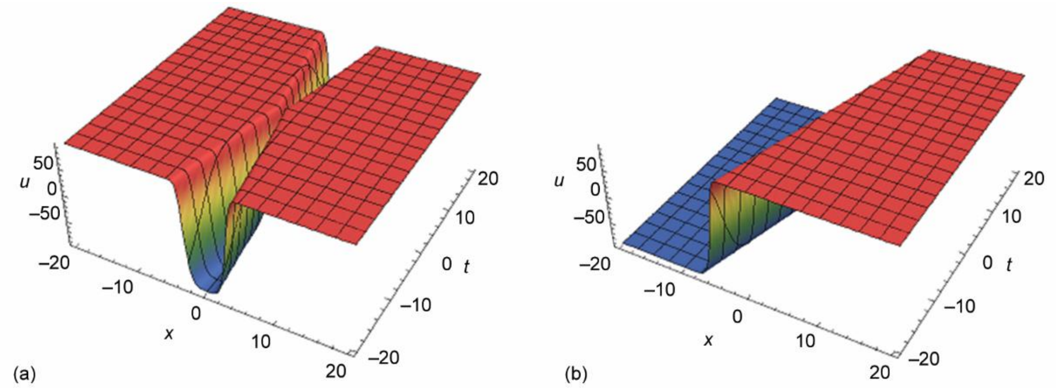

Figure 2. Solution (20) with R = –1, L = –0.5, γ_1 = –3, γ_2 = –0,3, μ = –5, c = 0; (a) ρ = 8/9, θ = 4/5, and (b) ρ = 1, θ = 1 (*for color image see journal web site*)

Non-linear ABC law

Based on the work [19], let's start with the fractional Lyapunov function:

$$V(t)=\frac{1}{2}I^{\rho}_{0,\sqrt{2}}u^2(x,t) \tag{21}$$

where $I^{\rho}_{0,\sqrt{2}}$ represents Khalil *et al.*'s [13] conformable fractional integral of the affected function with respect to the variable x in the interval $[0,\sqrt{2}]$. Then we have:

$$\mathrm{D}^{\theta}_{t}V(t)=I^{\rho}_{0,\sqrt{2}}(u\mathrm{D}^{\theta}_{t}u)=$$

$$=I^{\rho}_{0,\sqrt{2}}u(-\gamma_1u^{\alpha}\mathrm{D}^{\rho}_{x}u+\nu\mathrm{D}^{\rho}_{x}\mathrm{D}^{\rho}_{x}u-\mu\mathrm{D}^{\rho}_{x}\mathrm{D}^{\rho}_{x}\mathrm{D}^{\rho}_{x}u-\gamma_2\mathrm{D}^{\rho}_{x}\mathrm{D}^{\rho}_{x}\mathrm{D}^{\rho}_{x}\mathrm{D}^{\rho}_{x}u) \tag{22}$$

Directly calculating the fractional integral in eq. (22) yields:

$$\mathrm{D}^{\theta}_{t}V(t)=-\frac{\gamma_1}{\alpha+2}u^{\alpha+2}(\sqrt{2},t)+\nu u(\sqrt{2},t)\mathrm{D}^{\rho}_{x}u(\sqrt{2},t)-\nu I^{\rho}_{0,\sqrt{2}}(\mathrm{D}^{\rho}_{x}u)^2-$$

$$-\mu u(\sqrt{2},t)\mathrm{D}^{\rho}_{x}\mathrm{D}^{\rho}_{x}u(\sqrt{2},t)+\frac{\mu}{2}(\mathrm{D}^{\rho}_{x}u)^2-\gamma_2u\mathrm{D}^{\rho}_{x}\mathrm{D}^{\rho}_{x}\mathrm{D}^{\rho}_{x}u(\sqrt{2},t)+$$

$$+\gamma_2\mathrm{D}^{\rho}_{x}u(\sqrt{2},t)\mathrm{D}^{\rho}_{x}\mathrm{D}^{\rho}_{x}u(\sqrt{2},t)-\gamma_2I^{\rho}_{0,\sqrt{2}}(\mathrm{D}^{\rho}_{x}u)^2 \tag{23}$$

By considering eq. (2), it can be concluded from eq. (23) that:

$$\mathrm{D}^{\theta}_{t}V(t)\le-\nu I^{\rho}_{0,\sqrt{2}}(\mathrm{D}^{\rho}_{x}u)^2-(\nu\varphi_1+\mu\varphi_2)u^2(\sqrt{2},t)\le-\nu I^{\rho}_{0,\sqrt{2}}u^2-(\nu\varphi_1+\mu\varphi_2)u^2(\sqrt{2},t) \tag{24}$$

where we have proposed the non-linear control law:

$$\phi_1=-\varphi_1u(\sqrt{2},t) \tag{25}$$

$$\phi_2 = -\frac{\gamma_1}{(\alpha+2)(\mu+\gamma_2\varphi_1)}u^{\alpha+1}(\sqrt{2},t) + \frac{\mu(\varphi_1^2+2\varphi_2)}{2(\mu+\gamma_2\varphi_1)}u(\sqrt{2},t) - \frac{\gamma_2}{\mu+\gamma_2\varphi_1}\mathrm{D}_x^\rho \mathrm{D}_x^\rho \mathrm{D}_x^\rho u(\sqrt{2},t) \tag{26}$$

Suppose that the non-negative energy function $E(t)$ has the form [19]:

$$E(t) = V(t) + \frac{1}{2\nu r_1}(\nu\varphi_1 - a)^2 + \frac{1}{2\mu r_2}(\nu\varphi_2 - b)^2,\ (a \ge 0, b \ge 0) \tag{27}$$

which tells:

$$\mathrm{D}_t^\theta E(t) = \mathrm{D}_t^\theta V(t) + \frac{1}{r_1}(\nu\varphi_1 - a)\mathrm{D}_t^\theta \varphi_1 + \frac{1}{r_2}(\nu\varphi_2 - b)\mathrm{D}_t^\theta \varphi_2 \tag{28}$$

We further let:

$$\mathrm{D}_t^\theta \varphi_1 = r_1 u^2(\sqrt{2},t), \qquad \mathrm{D}_t^\theta \varphi_2 = r_2 u^2(\sqrt{2},t), \qquad (r_1 > 0, r_2 > 0) \tag{29}$$

Taking into account eq. (24), from eq. (28), we know that for any $t \ge 0$, there is:

$$\mathrm{D}_t^\theta E(t) = -\nu I_{0,1}^\rho u^2 - (a+b)u^2(\sqrt{2},t) \le 0 \tag{30}$$

which hints $E(t) \ge E(0)$ for any $t \ge 0$. Thus, we arrive at the fact that the integral $I_{0,\infty}^\theta u^2(\sqrt{2},t)$ is bounded.

It is easy to see from eqs. (21) and (24) that:

$$\mathrm{D}_t^\theta V(t) \le -2\nu V(t) - (\nu\varphi_1 + \mu\varphi_2)u^2(\sqrt{2},t) \tag{31}$$

Solving eq. (31) yields:

$$V(t) \le V(0)\exp\left(-\frac{2\nu}{\theta}t^\theta\right) + \sup_{t>0}\left|\nu\varphi_1(t) + \mu\varphi_2(t)\right| I_{0,t}^\theta \left\{\exp\left[-\frac{2\nu}{\theta}(t-\tau)^\theta\right]u^2(\sqrt{2},\tau)\right\} \tag{32}$$

Therefore, we have the exponential approximation:

$$u(x,t) \to 0, \qquad (t \to \infty) \tag{33}$$

Then we can conclude that when $u_0(x)$ is ρ-fractional square integrable on the interval $[0,\sqrt{2}]$, under the non-linear ABC law provided by eqs. (24), (25), and (29), eq. (1) with the IBV conditions (2) and (3) is globally stable in $[0,\sqrt{2}]$. As an example of $u_0(x)$, we take it:

$$u_0 = \sin\left(\frac{\sqrt{2}\pi}{2\rho}x^\rho\right) \tag{34}$$

Then the ρ-fractional integral of $u_0(x)$ can be obtained:

$$I = I_{0,\sqrt{2}}^\rho u_0^2(x) = I_{0,\sqrt{2}}^\rho \sin^2\left(\frac{\sqrt{2}\pi}{2\rho}x^\rho\right) = \frac{2^{\rho/2-1}}{\rho} - \frac{1}{2\sqrt{2}\pi}\sin\left(\frac{2^{\rho/2-1/2}\pi}{\rho}\right) \tag{35}$$

which is bounded for the fractional order $0 < \rho \le 1$. It shows in fig. 3 that the integral (35) decreases as fractional order ρ increases.

Conclusion

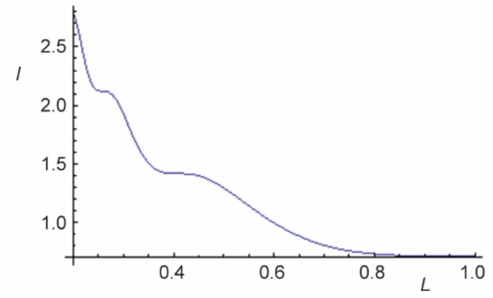

Figure 3. The integral curve of eq. (35) that decreases with the increase of fractional order ρ in the interval [0.2, 1]

We have presented the fractional MGKdV-B eq. (1) with conformable derivatives and obtained its hyperbolic functional solutions (11) and (20). These solutions were combined with the IBV conditions (2) and (3) to design a non-linear ABC law of eq. (1) through the use of eqs. (25), (26), and (29) for the purpose of global stability within the interval $[0,\sqrt{2}]$. When the parameter $R > 0$ and $R = 0$, solving eq. (1) with the auxiliary eq. (7) is effective for other exact solutions in both trigonometric and rational forms. For the integer-order case of eq. (1), the globally stable interval [0, 1] has been designed in [19], which is different from the interval $[0,\sqrt{2}]$ in this article. The main reason for deriving such a different interval $[0,\sqrt{2}]$ can be attributed to our derivation of the fractional-order integral inequality $I^{\rho}_{0,\sqrt{2}}u^2 \le I^{\rho}_{0,\sqrt{2}}(\mathrm{D}^{\rho}_{x}u)^2$. The present technology can be easily extended to fractional differential equations with the M-fractional derivatives [23] and the two-scale fractal derivatives [24-26].

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