ANALYTICAL STUDY OF FRACTAL MODIFIED DEGASPERIS-PROCESI EQUATION INVOLVING BETA-DERIVATIVE

by

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This paper considers the fractal modified Degasperis-Procesi type equation involving a Beta-derivative as a generalized form of the standard ones. The approximate analytical solutions for the new model were obtained by employing the modified homotopy perturbation method coupled Laplace transformation, which is also called as He-Laplace method in literature. The presented example demonstrates the efficacy of the applied method in solving non-linear equations.

Key words: *modified Degasperis-Procesi equation, fractal derivative, Beta-derivative, homotopy perturbation method, fractal dimension*

Introduction

The following modified Degasperis-Procesi (MDP) equation is employed in the modeling of dispersive water wave propagation, as evidenced in [1-3].

$$\frac{\partial u}{\partial t}-\frac{\partial^3 u}{\partial x^2 \partial t}+4u^2\frac{\partial u}{\partial x}=3\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}+u\frac{\partial^3 u}{\partial x^3} \tag{1}$$

This equation also serves as a model for non-linear thermal waves in cylindrical hyper-elastic rods [4-9]. Due to the use of the classical derivative, the model is limited in its ability to describe the local characteristics of a system in a discontinuous medium.

In recent years, there has been a surge of interest in fractional and fractal derivatives due to their applications in various scientific, engineering, and technological fields. A considerable number of authors have conducted research into non-linear differential equations involving fractional and fractal derivatives. In general, the fractional derivative contains parameters that afford it greater flexibility than the classical derivative in modeling diverse behaviors [10-13]. In certain instances, this can result in more accurate models. The fractal derivative represents an extension of the traditional derivative concept, with the objective of addressing the specific characteristics of discontinuous media, for examples, the fractal convection-diffusion problem [14], fractal vibration systems [15, 16], fractal MEMS system [17-20], the fractal thermal conduction [21], the fractal Zhiber-Shabat oscillator [22], the fractal fluidity [23], the fractal Chen-Lee-Liu equation [24], and the fractal Boussinesq equation [25]. In the present study, we examine the following Cauchy problem of fractal MDP equation involving a Beta-derivative:

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$$ {}_{0}^{A}\mathrm{D}_{t}^{\beta}\left(u-\frac{\partial u}{\partial x^{2\alpha}}\right)+au^{2}\frac{\partial u}{\partial x^{\alpha}}=b\frac{\partial u}{\partial x^{\alpha}}\frac{\partial u}{\partial x^{2\alpha}}+cu\frac{\partial u}{\partial x^{3\alpha}} \tag{2} $$

with initial condition:

$$ u(x,0)=\phi\left[\frac{x^{\alpha}}{\Gamma(1+\alpha)}\right] \tag{3} $$

where $0<\alpha,\beta\le 1$, and *a*, *b*, *c* are constants, ${}_{0}^{A}\mathrm{D}_{t}^{\beta}$ is the Beta-derivative operator with respect to time variable [26], $\partial/\partial x^{\alpha}$ the He's fractal derivative operator with respect to space variable [27].

Equation (2) is a generalized form of the standard MDP equation. Usually, there is no general method to find exact solution for the non-linear PDE involving fractional derivatives and fractal derivatives. Thus, several analytical methods, *e.g.*, the homotopy perturbation method [28-30] and the variational iteration method [31], have been applied to obtain approximate solutions of such problems. Formerly, many authors have studied the approximate analytical solutions for non-linear MDP equation by using different analytical methods by the homotopy perturbation method [32, 33], the Cole-Hopf method [34], the q-homotopy analysis method and Sumudu transform [35, 36] or the Jacobi wavelet collocation method [37]. We mention He's polynomials and the He-Laplace method [38], which couples He's homotopy perturbation method and Laplace transform. Motivated by these works, in this paper, we derive the approximate analytical solutions for the problem (2)-(3) by using homotopy perturbation transform method.

Basic definitions and properties

In this section, we recall some basic definitions and properties of Beta-derivative, Laplace transform and fractal derivative, for more details see [11, 39].

Definition 1. Let $a \in R$ and f:$[a, \infty)\to R$ Then Beta-derivative of f is defined:

$$ {}_{0}^{A}\mathrm{D}_{t}^{\beta} f(t)=\lim_{\varepsilon\to 0}\frac{f\left\{t+\varepsilon\left[t+\frac{1}{\Gamma(\beta)}\right]^{1-\beta}\right\}-f(t)}{\varepsilon} \tag{4} $$

for all $t\ge a, \beta\in(0,1]$.

If the limit of the previous exists, then we say that f is Beta-differentiable.

Theorem 1. Let $\beta \in (0,1]$ and assume f, g to be Beta-differentiable. Then:

$$ {}_{0}^{A}\mathrm{D}_{t}^{\beta}(af+bg)=a\,{}_{0}^{A}\mathrm{D}_{t}^{\beta}f+b\,{}_{0}^{A}\mathrm{D}_{t}^{\beta}g \quad \text{for all } a,b\in R \tag{5} $$

$$ {}_{0}^{A}\mathrm{D}_{t}^{\beta}(fg)=g\,{}_{0}^{A}\mathrm{D}_{t}^{\beta}f+f\,{}_{0}^{A}\mathrm{D}_{t}^{\beta}g \tag{6} $$

$$ {}_{0}^{A}\mathrm{D}_{t}^{\beta}\left(\frac{f}{g}\right)=\frac{g\,{}_{0}^{A}\mathrm{D}_{t}^{\beta}f-f\,{}_{0}^{A}\mathrm{D}_{t}^{\beta}g}{g^{2}} \quad (g\ne 0) \tag{7} $$

Theorem 2. Assume that $f(t)$ is differentiable and also Beta-differentiable. Let $g(t)$ be a differentiable function, then we have:

$$ {}_{0}^{A}\mathrm{D}_{t}^{\beta}[g\circ f(t)]=\left[t+\frac{1}{\Gamma(\beta)}\right]^{1-\beta}\frac{\mathrm{d}g}{\mathrm{d}f}\frac{\mathrm{d}f}{\mathrm{d}t} \tag{8} $$

Definition 2. Let $f:[a, \infty)\to R$ be a function and Beta-differentiable. Then the Beta-integral of the function f is given:

$$ {}_0^A I_t^\beta[f(t)] = \int_a^t \left[\tau + \frac{1}{\Gamma(\beta)}\right]^{\beta-1} f(\tau)\,\mathrm{d}\tau \tag{9}$$

Theorem 3. Let $f:[a, \infty)\to R$ be a continuous and differentiable function. Then for all $t \geq a$ we have:

$$ {}_0^A\mathrm{D}_t^\beta\, {}_0^A I_t^\beta[f(t)] = f(t) \tag{10}$$

Definition 4. Let $f:[0, \infty)\to R$ be real valued-function. Then the Laplace transform of f is defined by:

$$L[f(t)](s) = \int_0^\infty \exp(-st) f(t)\mathrm{d}t \tag{11}$$

The following properties can be easily derived:

$$L[f'(t)](s) = sL[f(t)] - f(0) \tag{12}$$

$$L(t^\alpha)(s) = \frac{\Gamma(1+\alpha)}{s^{1+\alpha}} \tag{13}$$

Definition 5. The He's fractal derivative of $f(x)$ is defined:

$$\frac{\partial f(x)}{\partial x^\alpha} = \Gamma(1+\alpha) \lim_{\substack{x_1 - x \to \Delta x \\ \Delta x \neq 0}} \frac{f(x_1) - f(x)}{(x_1 - x)^\alpha} \quad (0 < \alpha \leq 1) \tag{14}$$

and

$$\frac{\partial f(x)}{\partial x^{2\alpha}} = \frac{\partial}{\partial^\alpha}\left[\frac{\partial f(x)}{\partial x^\alpha}\right], \quad \frac{\partial f(x)}{\partial x^{3\alpha}} = \frac{\partial}{\partial^\alpha}\left[\frac{\partial f(x)}{\partial x^{2\alpha}}\right] \tag{15}$$

where α is relative to the two-scale fractal dimensions, reflecting the porosity [40].

Description of method

To solve the problem (2)-(3), we will use He-Laplace method [38], which is to decompose the non-linear equation into a series linear equations by the homotopy perturbation method, and then the linear equations are solved by using Laplace transform. The method can be applied to solve a wide range of non-linear problems, see for examples [41, 42]. In literature, it was also called as the He-Laplace algorithm [43].

In this section, to describe the solution procedure, we consider the following non-linear partial differential equation:

$$\frac{\partial w}{\partial t} + Rw(x,t) + Nw(x,t) = \Psi(x,t) \tag{16}$$

with initial condition $w(x, 0) - \varphi(x)$, where $w(x, t)$ is a function of x and t, R – the bounded linear operator, N – the general non-linear operator, which is Lipschitz continuous and $\Psi(x, t)$ – the source term.

Employing Laplace transform on eq. (16), we obtain:

$$L_t\left[\frac{\partial w}{\partial t}+Rw(x,t)+Nw(x,t)\right]=L_t[\Psi(x,t)] \tag{17}$$

By (12) and (13), we get:

$$L_t[w(x,t)](s)=\frac{\phi(x)}{s}+\frac{1}{s}L_t[\Psi(x,t)]-\frac{1}{s}L_t[Rw(x,t)+Nw(x,t)] \tag{18}$$

Operating the inverse Laplace transform on eq. (18), we have:

$$w(x,t)=\Phi(x,t)-L_t^{-1}\left(\frac{1}{s}\{L_t[Rw(x,t)+Nw(x,t)]\}\right) \tag{19}$$

where $\Phi(x, t)$ stands for the term appearing from the initial condition and the source term.

Now, we implement the homotopy perturbation method [44]:

$$w(x,t)=\sum_{k=0}^{\infty}w_k(x,t)q^k \tag{20}$$

and the non-linear term can be decomposed as:

$$Nw(x,t)=\sum_{k=0}^{\infty}H_k(w)q^k \tag{21}$$

by using the He's polynomials [45] $H_k(w)$ that are given:

$$H_k(w_0,w_1,\cdots,w_k)=\frac{1}{k!}\frac{\partial^k}{\partial\lambda^k}\left[N\left(\sum_{n=0}^{k}\lambda^n u_n\right)\right]_{\lambda=0} \quad k=0,1,2,\cdots \tag{22}$$

Substituting eqs. (20) and (21) into (19) gives:

$$\sum_{k=0}^{\infty}q^k w_k=\Phi(x,t)-q\left\{L_t^{-1}\left[\frac{1}{s}L_t\left(\sum_{k=0}^{\infty}q^k Rw_k+\sum_{k=0}^{\infty}q^k H_k\right)\right]\right\} \tag{23}$$

Equating the coefficients of like powers of q, we get:

$$q^0: \ w_0(x,t)=\Phi(x,t) \tag{24}$$

$$q^1: \ w_1(x,t)=L_t^{-1}\left[\frac{1}{s}L_t(Rw_0+H_0)\right] \tag{25}$$

$$q^2: \ w_2(x,t)=L_t^{-1}\left[\frac{1}{s}L_t(Rw_1+H_1)\right] \tag{26}$$

and so on.

Finally, the k-term approximate solution of (16) is:

$$u=u_0+u_1+\cdots+u_{k-1} \tag{27}$$

Solution of MDP equation

In this section, we solve the problem (2)-(3) by using He-Laplace method.

Using the two-scale transform [46, 47]:

$$X=\frac{x^{\alpha}}{\Gamma(1+\alpha)},\quad T=\beta\left\{\left[t+\frac{1}{\Gamma(\beta)}\right]^{\beta}-\left[\frac{1}{\Gamma(\beta)}\right]^{\beta}\right\} \tag{28}$$

and the properties of Beta-derivative and fractal derivative, eq. (2) can be converted into the following form:

$$\frac{\partial u}{\partial T}-\frac{\partial^3 u}{\partial T\partial X^2}+au^2\frac{\partial u}{\partial X}=b\frac{\partial u}{\partial X}\frac{\partial^2 u}{\partial X^2}+c\frac{\partial^3 u}{\partial X^3} \tag{29}$$

and the initial condition becomes:

$$u(x,0)=\phi(X)$$

The two-scale transform [46, 47] is a modification of the fractional complex transform [48], it is a good tool to solving fractional differential equations. It is proved that only when a, b, and c satisfy certain conditions, eq. (29) has exact solutions [49]. In the general case, we can only find approximate solutions. Next, we apply the method proposed in the last section to solve eq. (29).

Firstly, taking the Laplace transform on both the sides of eq. (29), we have:

$$L_T\left(\frac{\partial u}{\partial T}\right)=L_T\left(\frac{\partial^3 u}{\partial X^2\partial T}-au^2\frac{\partial u}{\partial X}+b\frac{\partial u}{\partial X}\frac{\partial^2 u}{\partial X^2}+cu\frac{\partial^3 u}{\partial X^3}\right) \tag{30}$$

By (12), we get:

$$L_T[u(X,T)]=\frac{1}{s}\phi(X)+\frac{1}{s}L_T\left(\frac{\partial^3 u}{\partial X^2\partial T}-au^2\frac{\partial u}{\partial X}+b\frac{\partial u}{\partial X}\frac{\partial^2 u}{\partial X^2}+cu\frac{\partial^3 u}{\partial X^3}\right) \tag{31}$$

Then, operating with the Laplace inverse transform on both sides of eq. (31) gives:

$$u(X,T)=\phi(X)+L^{-1}\left[\frac{1}{s}L_T\left(\frac{\partial^3 u}{\partial X^2\partial T}-au^2\frac{\partial u}{\partial X}+b\frac{\partial u}{\partial X}\frac{\partial^2 u}{\partial X^2}+cu\frac{\partial^3 u}{\partial X^3}\right)\right] \tag{32}$$

Employing the homotopy perturbation method, we obtain:

$$\sum_{k=0}^{\infty}q^k u_k(X,T)=\phi(X)+qL^{-1}\left\{\frac{1}{s}L_T\left[\sum_{k=0}^{\infty}q^k\frac{\partial^3 u_k}{\partial X^2\partial T}+\sum_{k=0}^{\infty}q^k\left(bK_k+cG_k-aH_k\right)\right]\right\} \tag{33}$$

where H_k, K_k, and G_k are respectively He's polynomials of the following non-linear terms:

$$u^2\frac{\partial u}{\partial X},\ \frac{\partial u}{\partial X}\frac{\partial^2 u}{\partial X^2}\quad\text{and}\quad u\frac{\partial^3 u}{\partial X^3}$$

On equating the coefficients of like powers of q we get:

$$q^0: \quad u_0(X,T)=\phi(X) \tag{34}$$

$$q^1: \quad u_1(X,T)=L_T^{-1}\left[\frac{1}{s}L_T\left(\frac{\partial^3 u_0}{\partial T\partial X^2}+bK_0+cG_0-aH_0\right)\right] \tag{35}$$

$$q^2: \quad u_2(X,T)=L_T^{-1}\left[\frac{1}{s}L_T\left(\frac{\partial^3 u_1}{\partial T\partial X^2}+bK_1+cG_1-aH_1\right)\right] \tag{36}$$

and we can find rest of the components in the similar way.

Hence, we get the k-term approximate solutions of (29):

$$u=u_0+u_1+\cdots+u_{k-1}.$$

Finally, by (28), we can obtain the solution of the problem (2)-(3).

In order to elucidate the solution procedure of He-Laplace method, we give an example.

Taking $a = 4$, $b = 3$, and $c = 1$, we consider the problem (2)-(3) in the form:

$${}_0^A\mathrm{D}_t^\beta\left(u-\frac{\partial u}{\partial x^{2\alpha}}\right)+4u^2\frac{\partial u}{\partial x^\alpha}=3\frac{\partial u}{\partial x^\alpha}\frac{\partial u}{\partial x^{2\alpha}}+u\frac{\partial u}{\partial x^{3\alpha}} \tag{37}$$

subject to the initial condition:

$$u(x,0)=\frac{1}{4}-\frac{15}{8}\sec h^2\left[\frac{x^\alpha}{2\Gamma(1+\alpha)}\right] \tag{38}$$

By eqs. (34)-(36), and (28), we can obtain:

$$u_0(x,t)=\frac{2\cosh^2\left[\frac{x^\alpha}{2\Gamma(1+\alpha)}\right]-15}{8\cosh^2\left[\frac{x^\alpha}{2\Gamma(1+\alpha)}\right]}$$

$$u_1(x,t)=\frac{-45\sinh\left[\frac{x^\alpha}{2\Gamma(1+\alpha)}\right]\{[t\Gamma(\beta)+1]^\beta-1\}}{16\beta\Gamma^\beta(\beta)\cosh^2\left[\frac{x^\alpha}{2\Gamma(1+\alpha)}\right]}$$

$$u_2(x,t)=\frac{\left\{135\cosh^2\left[\frac{x^\alpha}{2\Gamma(1+\alpha)}\right]-405\sinh^2\left[\frac{x^\alpha}{2\Gamma(1+\alpha)}\right]\right\}\{[t\Gamma(\beta)+1]^\beta-1\}^2}{128\beta^2\Gamma^{2\beta}(\beta)\cosh^4\left[\frac{x^\alpha}{2\Gamma(1+\alpha)}\right]}$$

and so on.

Thus, the approximate analytical solution is:

$$u(x,t)=\frac{2\cosh^2\left[\dfrac{x^\alpha}{2\Gamma(1+\alpha)}\right]-15}{8\cosh^2\left[\dfrac{x^\alpha}{2\Gamma(1+\alpha)}\right]}+\frac{-45\sinh\left[\dfrac{x^\alpha}{2\Gamma(1+\alpha)}\right][t\Gamma(\beta)-1]}{16\beta\Gamma^\beta(\beta)\cosh^2\left[\dfrac{x^\alpha}{2\Gamma(1+\alpha)}\right]}+$$

$$+\frac{\left\{135\cosh^2\left[\dfrac{x^\alpha}{2\Gamma(1+\alpha)}\right]-405\sinh^2\left[\dfrac{x^\alpha}{2\Gamma(1+\alpha)}\right]\right\}\{[t\Gamma(\beta)+1]^\beta-1\}^2}{128\beta^2\Gamma^{2\beta}(\beta)\cosh^4\left[\dfrac{x^\alpha}{2\Gamma(1+\alpha)}\right]}+\cdots$$

When $\alpha = \beta = 1$ we have:

$$u(x,t)=\frac{2\cosh^2\left(\dfrac{x}{2}\right)-15}{8\cosh^2\left(\dfrac{x}{2}\right)}-\frac{45t\sinh\left(\dfrac{x}{2}\right)}{16\cosh^2\left(\dfrac{x}{2}\right)}+\frac{135t^2\cosh^2\left(\dfrac{x}{2}\right)-405t^2\sinh^2\left(\dfrac{x}{2}\right)}{128\cosh^4\left(\dfrac{x}{2}\right)}+\cdots$$

which is different from the solution obtained by Wazwaz in [49]. This result shows that the solution of the non-linear MDP equation is very sensitive to small changes in the initial conditions.

Conclusion

The non-linear differential equation considered in this paper represents a generalized form of the standard MDP equation. The equation contains the Beta-derivative and fractal derivative, which can be employed in discontinuous media. The approximate analytical solutions for the new model were derived by employing the homotopy perturbation method and Laplace transform. The presented example demonstrates the efficacy of the applied method in solving non-linear problems, and can be a paradigm to develop new analytical method by coupling the homotopy perturbation method with other integral transforms, *e.g.*, He-transform [50, 51].

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