# APPROXIMATION OF BLOCK NUMERICAL RANGE FOR HAMILTONIAN OPERATOR MATRIX

by

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*The aim of this paper is to derive approximations for the block numerical range of unbounded block operator matrices that are block dominant. To illustrate our approach, we calculate the quartic numerical range of a concrete Hamiltonian operator matrix.*

Key words: *Hamiltonian operator matrix, block numerical range, spectrum*

## Introduction

The spectra of linear operators play a pivotal role in numerous branches of mathematics and thermal science. In the context of numerical simulation, a complex heat equation can be recast as a spectral problem. The conventional methodology for determining the spectrum of a linear operator is through the numerical range, as elucidated in [1, 2]. In [3, 4], the notion of quadratic numerical range was introduced, which may provide a more precise localization of the spectrum than the usual numerical range. In Muhammad and Marletta [5], the quadratic numerical range of a finite block matrix was approximated by projection methods. This concept was subsequently generalized to the block numerical range in [6]. The refinement of the decomposition of the space allows for the block numerical range to provide a more precise localization of the spectrum than the usual numerical range. In Yu *et al.* [7], the approximations of the block numerical range of unbounded block operator matrices were established, classified as either diagonally dominant or off-diagonally dominant.

In Salemi *et al.* [8], the authors introduced two novel concepts: total decomposition and estimable decomposition. An estimable decomposition permits the approximation of the spectrum of a block operator matrix by its block numerical ranges. Nevertheless, the existence of an estimable decomposition is, in general, challenging to ascertain. Furthermore, numerical approximations for the spectra may be unreliable, particularly in the case of an operator that is not self-adjoint or normal.

This paper was motivated by an attempt to enhance one's comprehension of the block numerical range. We investigate the potential of projection methods for computing the block numerical range, which reduces the problem to that of computing the block numerical range of a finite block matrix. In the case of an unbounded block operator matrix, it is as-

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sumed that it is block dominant. The Hamiltonian operator matrix is a non-self-adjoint operator matrix derived from a linear Hamiltonian system, with a rich history in mechanics. From the perspective of numerical analysis, an understanding of the block numerical range of the Hamiltonian operator matrix can assist in estimating the location of its spectrum. A substantial body of literature exists on the spectral properties of Hamiltonian operator matrices. For further reading, please refer to [9-13] and the references therein.

## Preliminaries

Let $\mathcal{H}$ be a Hilbert space. For an unbounded linear operator $\mathcal{A}\in\mathcal{H}$ which admits a so-called block operator matrix representation:

$$\mathcal{A}:=\begin{pmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{n1} & \cdots & A_{nn} \end{pmatrix} \tag{1}$$

where $A_{ij}:\mathcal{H}_j\to\mathcal{H}_i$ is closable operators with dense domains $\mathcal{D}_{ij}\subseteq\mathcal{H}_j\,(i,j=1,\ldots,n)$. We always suppose that $\mathcal{A}$ with its natural domain $\mathcal{D}(\mathcal{A}):=\mathcal{D}_1\oplus\cdots\oplus\mathcal{D}_n$, where $\mathcal{D}_j:=\bigcap_{i=1}^{n}\mathcal{D}_{ij}\in\mathcal{H}_j$ $(i,j=1,\ldots,n)$ is also densely defined.

*Remark 1* It should be noted that, unlike bounded operators, unbounded linear operators, in general, do not admit a matrix representation (1), with respect to a given decomposition $\mathcal{H}=\mathcal{H}_1\oplus\cdots\oplus\mathcal{H}_n$.

The definition of the block numerical range for bounded linear operators (see [4], *Definition 1.11.12*), generalizes as follows to unbounded block operator matrices $\mathcal{A}$ of the form eq. (1) with dense domain $\mathcal{D}(\mathcal{A})$.

*Definition 1* [7] Let $\mathcal{S}^n=\{(x_1,\cdots,x_n)^t\in\mathcal{D}_1\oplus\cdots\oplus\mathcal{D}_n:\|x_1\|=\cdots=\|x_n\|=1\}$. For $x=(x_1,\cdots,x_n)^t\in\mathcal{S}^n$, define the $n\times n$ matrix $\mathcal{A}_x$:

$$\mathcal{A}_x:=\begin{pmatrix} (A_{11}x_1,x_1) & \cdots & (A_{1n}x_n,x_1) \\ \vdots & \ddots & \vdots \\ (A_{n1}x_1,x_n) & \cdots & (A_{nn}x_n,x_n) \end{pmatrix} \tag{2}$$

Let $W^n(\mathcal{A}):=\{\lambda\in\mathbb{C}:\lambda\in\sigma(\mathcal{A}_x),x\in\mathcal{S}^n\}$ be block numerical range of the unbounded block operator matrix $\mathcal{A}$, which is defined by (1).

*Remark 2* For $n=1$, the block numerical range is just the usual numerical range, for $n=2$, it is the quadratic numerical range, as the bounded case.

*Definition 2* The block operator matrix $\mathcal{A}$ in (1) is called

- diagonally dominant if $A_{ij}$ is $A_{jj}$-bounded (see [4], *Definition 2.1.2*), where $i,j=1,\ldots,n$;
- off-diagonally dominant if $A_{ij}$ is $A_{n+1-j,j}$-bounded, where $i,j=1,\ldots,n$;
- block dominant if, for each $j$, there exists $i_j$ ($0\le i_j\le n$), such that $A_{ij}$ is $A_{i_j,j}$-bounded, where $i=1,\ldots,i_j-1,i_j+1,\ldots,n;\ j=1,\ldots,n$.

*Remark 3* In fact, block dominant is the dominance of one element (operator) in every column of the block operator matrix $\mathcal{A}$, that is, there is an element (operator) in every column with respect to which the other operators in the column are relatively bounded. Obviously, the diagonal dominant or off-diagonal dominant are special cases of block dominance where $i_j=j$ or $i_j=n+1-j$, respectively.

The following result shows some important properties of the block numerical range of unbounded block operator matrix [7].

*Proposition 1* For an unbounded block operator matrix $\mathcal{A}$, we have:
- $\sigma_p(\mathcal{A}) \subseteq W^n(\mathcal{A})$, where $\sigma_p(\mathcal{A})$ is the point spectrum of $\mathcal{A}$.
- $W^n(\mathcal{A}) \subseteq W(\mathcal{A})$.
- $W^{\hat{n}}(\mathcal{A}) \subseteq W^n(\mathcal{A})$, where $\mathcal{D}_1 \oplus \cdots \oplus \mathcal{D}_{\hat{n}}$ is a refinement (see [4], *Definition 1.11.12*) of $\mathcal{D}_1 \oplus \cdots \oplus \mathcal{D}_n$.

*Definition 3* [9] The block operator matrix:

$$H = \begin{pmatrix} A & B \\ C & -A^* \end{pmatrix} : \mathcal{D}(H) \subseteq \mathcal{H} \oplus \mathcal{H} \to \mathcal{H} \oplus \mathcal{H}$$

is called Hamiltonian operator matrix, if the closed densely defined operators $A, B, C$ satisfy $B$, $C$ are self-adjoint and $H$ is densely defined.

## Convergence theorem for unbounded operator matrix

In order to study the Hamiltonian operator matrix more effectively, we will first focus on a broader class of unbounded operators. The following lemmas were introduced by Yu *et al.* [7].

*Lemma 1* [7] Let:

$$\mathcal{A} := \begin{pmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{n1} & \cdots & A_{nn} \end{pmatrix}$$

be an unbounded operator in $\mathcal{H} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n$. Let $(U_{k_i}^i)_{k_i=1}^{\infty}, (i = 1, \cdots, n)$ be nested families of space in $\mathcal{D}_i$, where $U_{k_i}^i := \text{span}(\alpha_1^i, \ldots, \alpha_{k_i}^i), (\alpha_k^i)_{k=1}^{\infty} \in \mathcal{D}_i$ is orthonormal. Let $\mathbb{N}_+ = \{1, 2, 3, \ldots\}$, and multi-index $k := (k_1, \ldots, k_n) \in \mathbb{N}_+^n$. Consider:

$$\mathbb{A}_k := \begin{pmatrix} A_{k_1 \times k_1} & \cdots & A_{k_1 \times k_n} \\ \vdots & \ddots & \vdots \\ A_{k_n \times k_1} & \cdots & A_{k_n \times k_n} \end{pmatrix} \tag{3}$$

where $(A_{k_p \times k_q})_{st} = (A_{pq}\alpha_t^q, \alpha_s^p)$, $s = 1, \ldots, k_p; t = 1, \ldots, k_q; p, q = 1, \ldots, n.$ Then $W^n(\mathbb{A}_k) \subseteq W^n(\mathcal{A})$.

*Lemma 2* [7] Let $(U_{k_i}^i)_{k_i=1}^{\infty}$ and $\mathbb{A}_k$ be as in *Lemma 1*. Suppose that $k, k \in \mathbb{N}_+^n$, and $\hat{k} \geq k$, in the sense that, $\hat{k}_i \geq k_i, i = 1, \ldots, n$. Then $W^n(\mathbb{A}_k) \subseteq W^n(\mathbb{A}_{\hat{k}})$.

In the following result, we obtain the approximations of the block numerical range of unbounded block operator matrices which is block dominant.

*Theorem 1* Let $\mathcal{A}$, $\mathbb{A}_k$ and $(U_{k_i}^i)_{k_i=1}^{\infty}$ be as in *Lemma 1*. Suppose that $\mathcal{A}$ is block dominant and $(U_{k_i}^i)_{k_i=1}^{\infty}$ is a core (see [14], Section III.3) of $A_{j_i i}, (i = 1, \ldots, n; 1 \leq j_i \leq n)$, where $A_{j_i i}$ is the dominant element in the $i^{\text{th}}$ column of $\mathcal{A}$. Then:

$$\overline{\bigcup_{k \in \mathbb{N}_+^n} W^n(\mathbb{A}_k)} = \overline{\bigcup_{m^n \in \mathbb{N}_+^n} W^n(\mathbb{A}_{m^n})} = \overline{W^n(\mathcal{A})},$$

where $m^n := (m, \ldots, m) \in \mathbb{N}_+^n$.

*Proof.* By *Lemma 2*, it is immediate that:

$$\overline{\bigcup_{k\in\mathbb{N}_+^n} W^n(\mathbb{A}_k)} = \overline{\bigcup_{m^n\in\mathbb{N}_+^n} W^n(\mathbb{A}_{m^n})}$$

if we take $m := \max\{k_1,\ldots,k_n\}$ and $m := \min\{k_1,\ldots,k_n\}$, respectively. By *Lemma 1*, it therefore now remains to prove that:

$$W^n(\mathcal{A}) \subseteq \overline{\bigcup_{k\in\mathbb{N}_+^n} W^n(\mathbb{A}_k)}$$

Let $\lambda \in W^n(\mathcal{A})$. There then exists $x \in \mathcal{S}^n$, such that $\lambda$ is an eigenvalue of $\mathcal{A}_x$ as defined in eq. (2). Since $(U_{k_i}^i)_{k_i=1}^\infty$ is a core of $A_{j_i i}, (i=1,\ldots,n)$, there exists a sequence $(x_k^i)_{k=1}^\infty$, with each $x_k^i \in \operatorname{span}(\alpha_1^i,\ldots,\alpha_{k_i}^i)$ for some $k_i > 0$, such that $\| x^i - x_k^i \| \to 0$ and $\| A_{j_i i} x^i - A_{j_i i} x_k^i \| \to 0$ as $k \to \infty$, where $x^i$ denotes the $i^{\text{th}}$ component of $x$ and $j = 1,\ldots,n$. Because $A_{ji}$ is $A_{j_i i}$-bounded for $j = 1,\ldots,n$, we have that $\| A_{ji} x^i - A_{ji} x_k^i \| \to 0$ as $k \to \infty$. Let $x_k = (x_k^1,\ldots,x_k^n)^t$, by a simple calculation, we then obtain that:

$$\left\| \mathcal{A}_{x_k} - \mathcal{A}_x \right\| \to 0, \quad \text{as} \quad k \to \infty$$

Fix $x_k$ as previously. Define isometries:

$$\pi_{k_i}^i : U_{k_i}^i \to \mathbb{C}^{k_i}$$

by

$$\pi_{k_i}^i(\beta_1^i \alpha_1^i + \cdots + \beta_{k_i}^i \alpha_{k_i}^i) := (\beta_1^i,\ldots,\beta_{k_i}^i)^t$$

for $i = 1, \ldots, n$.

Take $\beta_i \in \mathbb{C}^{k_i}, i = 1,\ldots,n$, by:

$$\beta_i = \frac{\pi_{k_i}^i(x_k^i)}{\left\| \pi_{k_i}^i(x_k^i) \right\|}$$

Consider the matrix:

$$M_k := \begin{pmatrix} (A_{k_1\times k_1}\beta_1, \beta_1) & \cdots & (A_{k_1\times k_n}\beta_n, \beta_1) \\ \vdots & \ddots & \vdots \\ (A_{k_n\times k_1}\beta_1, \beta_n) & \cdots & (A_{k_n\times k_n}\beta_n, \beta_n) \end{pmatrix}$$

A simple calculation yields that $M_k = \mathcal{A}_{x_k}$. Since $\| \mathcal{A}_{x_k} - \mathcal{A}_x \| \to 0$ as $k \to \infty$, this entails that $\| M_k - \mathcal{A}_x \| \to 0$ as $k \to \infty$. Obviously, the eigenvalues of $M_k$ are elements of $W^n(\mathbb{A}_k)$, where $k := (k_1,\ldots,k_n) \in \mathbb{N}_+^n$. There hence exists $\lambda_k \in W^n(\mathbb{A}_k)$ such that $\lambda_k \to \lambda$, as $k \to \infty$. It then follows from *Lemma 2* that:

$$\lambda \in \overline{\bigcup_{k\in\mathbb{N}_+^n} W^n(\mathbb{A}_k)}$$

*Remark 4* Note that, the result of *Theorem 4.6* from [7], is the special case of *Theorem 1*.

**Block numerical range of Hamiltonian operator matrix**

*Example 1.* Consider the rectangular thin plate bending problem with two opposite edges simply supported in region $\{(x,y): 0 \le x \le 1, 0 \le y \le 1\}$. The governing equation in terms of displacement is:

$$D\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)^2 w=0 \tag{4}$$

where $D>0$ is a constant, the boundary conditions for simply supported edges are:

$$w(x,0)=w(x,1)=0, \quad \frac{\partial^2 w}{\partial y^2}=0, \quad y=0,1 \tag{5}$$

We introduce the rotation, $\theta$, the Lagrange parametric function, $q$, and the moment, $m$:

$$\theta=\frac{\partial w}{\partial x}, \quad q=D\left(\frac{\partial^3}{\partial x^3}+\frac{\partial^3}{\partial x \partial y^2}\right), \quad m=-D\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)$$

Then eqs. (4) and (5) become [15-17]:

$$\frac{\partial}{\partial x}\begin{pmatrix} w \\ \theta \\ q \\ m \end{pmatrix}=\begin{pmatrix} 0 & 1 & 0 & 0 \\ -\frac{\partial^2}{\partial y^2} & 0 & 0 & -\frac{1}{D} \\ 0 & 0 & 0 & \frac{\partial^2}{\partial y^2} \\ 0 & 0 & -1 & 0 \end{pmatrix}\begin{pmatrix} w \\ \theta \\ q \\ m \end{pmatrix} \tag{6}$$

The corresponding $4\times 4$ Hamiltonian operator matrix is given by:

$$H=\begin{pmatrix} 0 & 1 & 0 & 0 \\ -\frac{\mathrm{d}^2}{\mathrm{d}y^2} & 0 & 0 & -\frac{1}{D} \\ 0 & 0 & 0 & \frac{\mathrm{d}^2}{\mathrm{d}y^2} \\ 0 & 0 & -1 & 0 \end{pmatrix} := \begin{pmatrix} A & B \\ 0 & -A^* \end{pmatrix}$$

the domain is $\mathcal{D}(A)\oplus\mathcal{D}(A)\subseteq\mathcal{H}\oplus\mathcal{H}$, where $\mathcal{H}=\mathcal{L}_2(0,1)\oplus\mathcal{L}_2(0,1)$, and:

$$A=\begin{pmatrix} 0 & 1 \\ -\frac{\mathrm{d}^2}{\mathrm{d}y^2} & 0 \end{pmatrix}, \quad B=\begin{pmatrix} 0 & 0 \\ 0 & -\frac{1}{D} \end{pmatrix}$$

$$\mathcal{D}(A):=\left\{\begin{pmatrix} w \\ \theta \end{pmatrix}\in\mathcal{H}: w(0)=w(1)=0, \quad w'\in AC[0,1], \quad w''\in\mathcal{L}_2(0,1)\right\}$$

After calculation, we obtain the characteristic equation [16]: $\sin^2\lambda=0$. Therefore, the eigenvalues of Hamiltonian operator matrix $H$ are $\lambda_j = j\pi$, where $j=\pm 1,\pm 2,\cdots$.

Consider $4\times 4$ Hamiltonian operator matrix:

$$H = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\dfrac{\mathrm{d}^2}{\mathrm{d}y^2} & 0 & 0 & -\dfrac{1}{D} \\ 0 & 0 & 0 & \dfrac{\mathrm{d}^2}{\mathrm{d}y^2} \\ 0 & 0 & -1 & 0 \end{pmatrix} := \begin{pmatrix} 0 & 1 & 0 & 0 \\ A_1 & 0 & 0 & -\dfrac{1}{D} \\ 0 & 0 & 0 & -A_1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$$\mathcal{D}(H) = \mathcal{D}(A_1) \oplus \mathcal{H} \oplus \mathcal{H} \oplus \mathcal{D}(A_1) \subseteq \mathcal{H} \oplus \mathcal{H} \oplus \mathcal{H} \oplus \mathcal{H} \quad \text{where } \mathcal{H} = \mathcal{L}_2(0,1),$$

$$\mathcal{D}(A_1) := \{ w \in \mathcal{H} : w(0) = w(1) = 0, \quad w' \in AC[0,1], \quad w'' \in \mathcal{L}_2(0,1) \}$$

By *Theorem 1*, the quadratic and quartic numerical ranges of Hamiltonian operator matrix $H$ can be approximated using the projection method. Since $\sigma(H) = \sigma_p(H)$ [18] and $\sigma_p(H) \subseteq W^4(H) \subseteq W^2(H)$, we can roughly estimate the spectrum of Hamiltonian operator matrix $H$ using its quartic numerical range.

Next, we will utilize the projection method to compute:

$$\overline{\bigcup_{m^4 \in \mathbb{N}_+^4} W^4(\mathbb{H}_{m^4})}$$

of Hamiltonian operator matrix $H$ and subsequently estimate the spectrum. As is well known, the eigenvalues and normalized eigenvectors of the operator $A_1$ are given by:

$$\lambda_j = (j\pi)^2, \quad w_j(y) = \sqrt{2}\sin(j\pi y), \quad j = 1, 2, \ldots.$$

Since the operator $A_1$ is self-adjoint, these eigenvectors can be used as a basis for $\mathcal{L}_2(0,1)$. Define $(U_m)_{m=1}^{\infty}$ is nested families of space in $\mathcal{L}_2(0,1)$ as in *Theorem 1*, where:

$$U_m := \mathrm{span}\{w_1, w_2, \cdots, w_\mathrm{m}\}$$

Therefore, the matrix $\mathbb{H}_{m^4}$ is:

$$\mathbb{H}_{m^4} := \begin{pmatrix} H_{11} & H_{12} & H_{13} & H_{14} \\ H_{21} & H_{22} & H_{23} & H_{24} \\ H_{31} & H_{32} & H_{33} & H_{34} \\ H_{41} & H_{42} & H_{43} & H_{44} \end{pmatrix} = \begin{pmatrix} 0_m & I_m & 0_m & 0_m \\ T_m & 0_m & 0_m & -\dfrac{1}{D} I_m \\ 0_m & 0_m & 0_m & -T_m \\ 0_m & 0_m & -I_m & 0_m \end{pmatrix}$$

where $H_{ij}, (i, j = 1, \cdots, 4)$ is $m \times m$ matrix; $I_m$ and $0_m$ denote the $m \times m$ identity and zero matrix, respectively:

$$H_{24} = -\frac{1}{D} I_m, \quad H_{21} = -H_{34} = \mathrm{diag}\{\pi^2, 4\pi^2, \ldots, m^2\pi^2\} := T_m$$

It is easy to see that:

$$W^4(\mathbb{H}_{m^4}) = W^2\left(\begin{pmatrix} 0_m & I_m \\ T_m & 0_m \end{pmatrix}\right)$$

Let $\mathcal{S}_{\mathbb{C}}^{m} := \{x \in \mathbb{C}^m, \|x\| = 1\}$, then we have:

$$W^4(\mathbb{H}_{m^4}) = \{\lambda \in \mathbb{C} : \lambda^2 = (x, y)(T_m y, x), \forall x, y \in \mathcal{S}_{\mathbb{C}}^{m}\}$$

Furthermore, since $\sigma(\mathbb{H}_{m^4}) = \{\pm\pi, \pm 2\pi, \ldots, \pm m\pi\}$. Specifically, in the previous equation, when $x = y$, then we obtain that:

$$\{\lambda \in \mathbb{C} : \lambda^2 = (x, x)(T_m x, x), \forall x \in \mathcal{S}_{\mathbb{C}}^{m}\} := \mathrm{co}\{-\pi, -2\pi, \cdots, -\mathrm{m}\pi\} \cup \mathrm{co}\{\pi, 2\pi, \cdots, \mathrm{m}\pi\}$$

Hence, $\sigma(\mathbb{H}_{m^4}) \to \sigma_p(H)$ as $m \to \infty$, *i. e.*,

$$\sigma_p(H) \subseteq \overline{\bigcup_{m^4 \in \mathbb{N}_+^4} W^4(\mathbb{H}_{m^4})}$$

On the other hand, eqs. (4) and (5) can become [19]:

$$\frac{\partial}{\partial x}\begin{pmatrix} w \\ -m \\ q \\ \theta \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -\dfrac{\partial^2}{\partial y^2} & 0 & 0 \\ -\dfrac{\partial^2}{\partial y^2} & \dfrac{1}{D} & 0 & 0 \end{pmatrix}\begin{pmatrix} w \\ -m \\ q \\ \theta \end{pmatrix} \tag{7}$$

The corresponding $4 \times 4$ Hamiltonian operator matrix $\tilde{H}$ is given by:

$$\tilde{H} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -\dfrac{\mathrm{d}^2}{\mathrm{d}\,y^2} & 0 & 0 \\ -\dfrac{\mathrm{d}^2}{\mathrm{d}\,y^2} & \dfrac{1}{D} & 0 & 0 \end{pmatrix} := \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}$$

Notice that the Hamiltonian operator matrix derived from eq. (7) is off-diagonally dominant, while the case from eq. (6) is block dominant. Similarly, we can obtain the following related conclusions:

$$\sigma(H) = \sigma_p(H) = \sigma(\tilde{H}) = \sigma_p(\tilde{H})$$

$$\tilde{\mathbb{H}}_{m^4} = \begin{pmatrix} 0_m & 0_m & 0_m & I_m \\ 0_m & 0_m & I_m & 0_m \\ 0_m & T_m & 0_m & 0_m \\ T_m & \dfrac{1}{D} & 0_m & 0_m \end{pmatrix}$$

$$W^4(\tilde{\mathbb{H}}_{m^4}) = \{\lambda \in \mathbb{C} : \lambda^2 = (x, y)(T_m y, x), \forall x, y \in \mathcal{S}_{\mathbb{C}}^{m}\}$$

The next step is to provide the figure at $m = 2, 4, 6, 8$ using the random vector method to approximate the quartic numerical range of the matrix $\mathbb{H}_{m^4}$. The red dots represent the point spectra, and the blue plots represent the quartic numerical range of the corresponding matrix $\mathbb{H}_{m^4}$.

Figure 1 shows the quartic numerical range of the matrix $\mathbb{H}_{m^4}$. From figs. 1(a) and 1(b), it can be seen that when $m = 2, 4$, the point spectra of the corresponding matrix almost falls within its quartic numerical range. However, from figs. 1(c) and 1(d), it can be observed that when $m = 6, 8$, some point spectra of the matrix falls outside its quartic numerical range. Since we used $5\times10^5$ random vectors to generate figs. 1(a)-1(d), the lack of coverage of the point spectra by the quartic numerical range in figs. 1(c) and 1(d) may be due to the smaller number of random vectors used. Therefore, we used $5\times10^6$ random vectors to obtain fig. 1(e). Although it did not achieve the expected effect, the quartic numerical range has slightly increased. As we see, the random vector method is very slow in filling the quartic numerical range of matrices with larger dimensions.

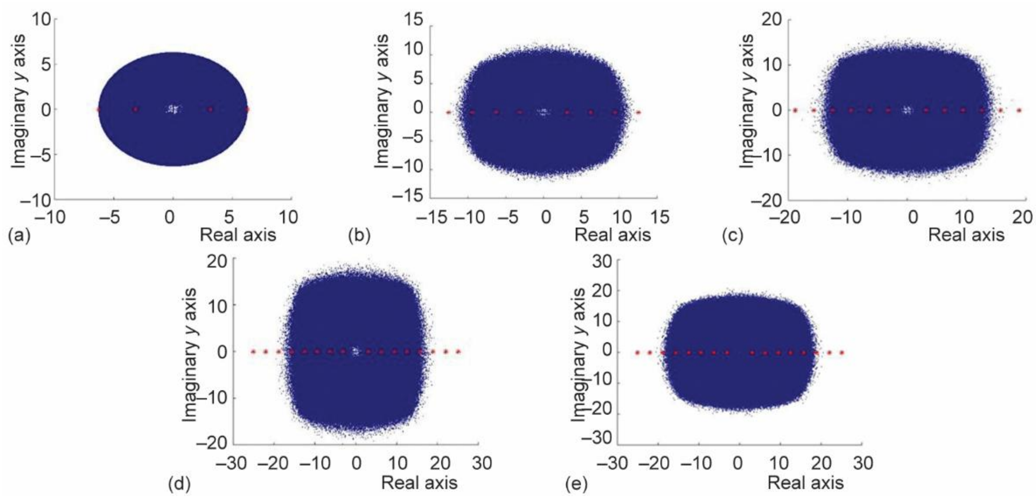


**Figure 1. Approximate the quartic numerical range of the matrix $\mathbb{H}_{m^4}$; (a) $m$ = 2, (b) $m$ = 4, (c) $m$ = 6, (d) $m$ = 8, and (e) $m$ = 8**

## Conclusions

In this paper, we approximate the block numerical range of unbounded block operator matrices that are block dominant. As a preliminary illustration, we compute the quartic numerical range of a concrete Hamiltonian operator matrix. An understanding of the block numerical range of the Hamiltonian operator matrix can facilitate the estimation of its spectrum location.

From the perspective of numerical approximation, we will investigate the drawing algorithm of the block numerical range of Hamiltonian operator matrices. This algorithm can efficiently represent the image of its block numerical range.

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