A NOVEL METHODOLOGY FOR ESTABLISHING A FRACTAL-FRACTIONAL OSCILLATOR AND ITS APPLICATION TO THE AGGREGATION OF GRASS CARP'S ROES

by

Jing-Yan NIU *

Jiaozuo Normal College, Jiaozuo, China

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This article examines the aggregation model of grass carp roe. This model is analogous to the Toda-like oscillator model, as any oscillation caused by perturbations will decay immediately. In general, the water environment on which fish depend for their survival does not need to be considered as a porous medium. However, in certain special water environments, the properties of the water may exhibit porous characteristics due to the presence of solid particles. In such instances, the consideration of porous medium characteristics of the water may assist in the more accurate description and comprehension of the aggregation model of grass carp. The appropriate methodology is employed in this study in the traditional case. An alternative fractal system is proposed as a means of establishing the roe aggregation system in fractal space. This system is based on a Toda-like fractal-fractional system and has been demonstrated to exhibit analogous properties to integer order systems.

Key words: *grass carp, roe, Toda-like oscillator, frequency formula fractal space*

Introduction

The spatial distribution of spawning fish is constrained by a number of environmental factors, with water velocity playing a dominant role in the drift of spawning fish [1]. During the spawning period, it is essential to maintain a minimum flow rate to ensure the safe drifting of fish eggs and to facilitate the natural reproduction of drifting spawning fish. This is in accordance with the findings of studies in [2-4]. It is recommended that grass carp roe be aggregated in order to maximize their survival against various predators. It is imperative to minimize vibration caused by environmental disturbances. Should the vibration be excessive, there is a risk that some roes may become detached from the cluster, which could have fatal consequences. Authors in [5, 6] proposed an Euler-Lagrange system as a means of addressing the drift of roes. The roes are described in detail as spherical particles dispersed within the flow field. The horizontal position change within a time step is determined using Newton's second law to create a motion equation that accounts for the effects of convection and turbulence. The grass carp roe aggregation model can be represented by a particle system with non-linear spring connections, which is a method of using physical models to explain biological phenomena. In this model, each particle represents a roe, while the non-linear springs represent the interaction forces between the roes, such as attraction or repulsion. This model facili-

* Author’s e-mail: 115475601@qq.com

tates an understanding of the aggregation behavior of grass carp roes under specific environmental conditions. These include the aggregation or dispersion of roes in response to water flow, temperature, salinity, and other factors. This model is analogous to the Toda oscillator [7-9], which is of significant value for elucidating the underlying mechanism of roe condensation.

In the environment in which fish grow, water is typically the primary medium through which they live. In the majority of cases, water does not possess the characteristics of a porous medium, as it is not a pore space composed of a solid skeleton. Instead, it exists as a fluid in its own right. Fish live in water by utilizing the dissolved oxygen and other nutrients in the water, rather than through fluid exchange in porous media. However, in certain special circumstances, such as wetlands, mudflats, or water bodies with large amounts of sediment, the properties of the water can be influenced by these solid particles, forming structures similar to porous media [10]. In such instances, the consideration of porous media characteristics in water may assist in the more accurate description and comprehension of the water quality and ecological processes occurring within fish growth environments.

A substantial body of research exists on fractional-order differential systems in the existing literature, for examples, fractal oscillators [11, 12], fractal variational principles [13-17], fractal MEMS systems [18-20]. El-Dib [21, 22] and El-Dib *et al.* [23] proposed a relation between these two-scale fractal derivatives. In light of these considerations, this paper employs a novel approach to develop a Toda-like fractal-fractional oscillator to elucidate the aggregation of grass carp roe. This method of converting integer-order differential equations into fractal-fractional-order differential equations has proven to be an effective approach for obtaining an alternative system and elucidating the aggregation of grass carp roes.

Toda-like fractal-fractional oscillator

As with the nanobeam system, the vibration problem of roe agglomeration can be described as a system of particles connected by non-linear springs [24-29]. The roe with mass, m, is connected to the other roe by springs, as illustrated in fig. 1. The adsorption force is high near the equilibrium position throughout the entire system. As the displacement, x, increases, the adsorption force decreases. If the displacement exceeds a certain threshold, the adsorption force is lost, resulting in roe diffusion.

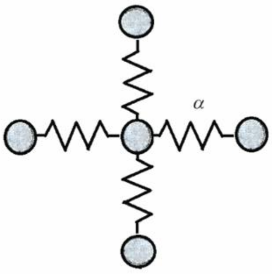

Figure 1. Aggregation system of grass carp roes

Newton's second law allows us to derive the equation of motion:

$$x'' + \frac{k}{m}x + \frac{b}{m(\alpha + x)} = 0, \quad x(0) = A, \quad x'(0) = 0 \tag{1}$$

where k, b, and α represent the adsorption parameter, elasticity coefficient and distance, respectively.

Equation (1) is designated as a Toda-like oscillator. Upon expansion of the non-linear term using the Taylor series, it can be seen as a Duffing oscillator with quadratic non-linearity [30]. In the study of natural phenomena, fractals play a significant role. To understand the non-linear vibrations of rods, it is necessary to consider the fractal aggregation in discontinuous spaces.

The classical approach of replacing t by t^β was unsuccessful in obtaining a solution. Subsequently, it is necessary to determine the manner in which traditional differential equations can be transformed into corresponding fractional-order differential equations.

The general form of differential equations can be expressed:

$$f[t, x, x', x'', \cdots, x^{(n)}] = 0 \tag{2}$$

The definition of fractional derivatives is primarily based on mainly the Riemann--Liouville fractional derivative [31], Caputo fractional derivative [31], He fractional derivative [32, 33], and the two-scale fractal derivative [34, 35]. He's fractional derivative is a widely utilized tool in dynamic and physical systems, and has led to the development of numerous analytical techniques for the solution of effective approximate solutions for fractional non-linear systems. It has been extensively discussed in the literature.

The definition of He's fractal derivative is [36]:

$$\frac{\mathrm{d}x}{\mathrm{d}t^\beta} = \Gamma(1+\beta) \lim_{\substack{t-t_0 \to \Delta t \\ \Delta t \neq 0}} \frac{x(t) - x(t_0)}{(t-t_0)^\beta} \tag{3}$$

In this context, β represents the fractal dimension in the t-direction. The fractal derivative has been adapted for discontinuous fractal media quality and is regarded as a natural evolution of the Leibniz derivatives. In the context of models in porous media, the most effective method for representing systems in real-world scientific and engineering applications is the use of fractal derivatives. The method of converting fractal space into continuous fractal space was proposed by El-Dib [21, 22], and El-Dib *et al.* [23], which represents a significantly advantageous new technique. The objective of this study is to propose a novel method for converting traditional derivatives into fractal derivatives in discontinuous space.

For the fractal dimension $0 < \beta_1 < 1$, the following fractional function is proposed by El-Dib [21, 22], and El-Dib *et al.* [23]:

$$\frac{\mathrm{d}x}{\mathrm{d}t^{\beta_1}} = \mathrm{S}^{\beta_1} \cos\frac{\pi\beta_1}{2} x + \mathrm{S}^{\beta_1 - 1} \sin\frac{\pi\beta_1}{2} x' \tag{4}$$

where S is a real parameter and can be obtained by the frequency of the oscillator system.

We also have:

$$\frac{\mathrm{d}x}{\mathrm{d}t^{\beta_2}} = S^{\beta_2 - 1} \cos\frac{\pi(\beta_2 - 1)}{2} x' + S^{\beta_2 - 2} \sin\frac{\pi(\beta_2 - 1)}{2} x'', \quad 1 < \beta_2 < 2 \tag{5}$$

For the fractal dimension β, the general form is:

$$\frac{\mathrm{d}x}{\mathrm{d}t^\beta} = S^{\beta - [\beta]} \cos\frac{\pi(\beta - [\beta])}{2} x^{([\beta])} + S^{(\beta - ([\beta]+1))} \sin\frac{\pi(\beta - [\beta])}{2} x^{([\beta]+1)}, \quad [\beta] < \beta < [\beta] + 1 \tag{6}$$

The relationship between integer order derivatives and fractional order derivatives is:

$$x^{(n)} = -S\cot\frac{\pi[\beta(n-1)]}{2} x^{(n-1)} + S^{n-\beta} \csc\frac{\pi[\beta(n-1)]}{2} \frac{\mathrm{d}x}{\mathrm{d}t^\beta}, \quad n - 1 < \beta < n \tag{7}$$

Substitute eq. (7) into eq. (2), and the corresponding fractional differential equation has the form:

$$f\left[t, x, \frac{\mathrm{d}x}{\mathrm{d}t^{\beta-(n-1)}}, \frac{\mathrm{d}x}{\mathrm{d}t^{\beta-n}}, \cdots, \frac{\mathrm{d}x}{\mathrm{d}t^{\beta}}\right]=0 \tag{8}$$

For a second-order differential equation with the following form:

$$x''+g(x)=0 \tag{9}$$

Its corresponding fractional differential system is:

$$\frac{\mathrm{d}x}{\mathrm{d}t^{\beta}}+S\tan\frac{\pi\beta}{2}\frac{\mathrm{d}x}{\mathrm{d}t^{\beta-1}}-xS^{\beta}\sin\frac{\pi\beta}{2}\tan\frac{\pi\beta}{2}-g(x)S^{\beta-2}\cos\frac{\pi\beta}{2}=0, \quad 1<\beta<2 \tag{10}$$

The fractal-fractional Toda-like oscillator corresponding to eq. (1) has the following form:

$$\frac{\mathrm{d}x}{\mathrm{d}t^{\beta}}+S\tan\frac{\pi\beta}{2}\frac{\mathrm{d}x}{\mathrm{d}t^{\beta-1}}-xS^{\beta}\sin\frac{\pi\beta}{2}\tan\frac{\pi\beta}{2}-\left[\frac{k}{m}x+\frac{b}{m(\alpha+x)}\right]S^{\beta-2}\cos\frac{\pi\beta}{2}=0 \tag{11}$$

It can also be written as:

$$\frac{\mathrm{d}x}{\mathrm{d}t^{\beta}}+\delta\frac{\mathrm{d}x}{\mathrm{d}t^{\beta-1}}-x\left(\delta\sigma+\frac{k\sigma}{m\delta}\right)-\frac{b\sigma}{m\delta(\alpha+x)}=0 \tag{12}$$

where

$$\delta=S\tan\frac{\pi\beta}{2}, \quad \sigma=S^{\beta-1}\sin\frac{\pi\beta}{2} \tag{13}$$

At the same time, the initial conditions of eq. (1) become:

$$x(0)=A, \quad \frac{\mathrm{d}}{\mathrm{d}t^{\beta-1}}x(0)=\sigma A \tag{14}$$

It is straightforward to demonstrate that eq. (11) reduces to eq. (1) when the fractional order parameter β is set to 2. At present, there is no universal method for converting all kinds of integer-order differential equations into fractional-order differential equations. In the current literature, integer-order derivative terms are typically replaced by corresponding fractional-order derivative terms, and the transformed equation may no longer possess the same physical meaning or interpretation as the original equation. To illustrate, the rank of the decay term in an oscillator system is equal to half the rank of the upper-order term in continuous space. However, this may not be the case in fractal space. The proposed method ensures that the original system physical meaning is preserved, and that the overall frequency of the system remains unchanged.

Seeking parameters *S* and discussing

It is important to note that the method employed in the literature to transform the traditional system into a fractional order system is simply to convert integer order derivatives into fractional order derivatives. However, this tool is based on the idea of transforming the traditional system into the equivalent one in fractal space. The method does not include an explicit damping term, whereas the current approach reveals the role of a hidden damping term, which is reflected by the fractional parameter *S*. This is of significant importance in physical and engineering applications and cannot be overlooked. The fractional parameter S can be determined by comparing the restoring force of eq. (1) with the corresponding force for the fractal system described by eq. (12).

A periodic motion of the Toda-like oscillator was generally assumed to exist, and the total frequency in the continuous space is represented by He's frequency formula [37-40]:

$$W^2 = \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{k}{m}x + \frac{b}{m(\alpha + x)}\right]\Big|_{x=\frac{A}{2}} = \frac{k}{m} - \frac{b}{m\left(\alpha + \frac{A}{2}\right)^2} \tag{15}$$

Due to its simplicity and reliability, He's frequency formula was successfully applied to fractal oscillators [41], MEMS systems [42-44], and others [45]. For the corresponding system in the fractal space, the natural frequency is:

$$\omega^2 = \frac{\mathrm{d}}{\mathrm{d}x}\left[-x\left(\delta\sigma + \frac{k\sigma}{m\delta}\right) - \frac{b\sigma}{m\delta(\alpha + x)}\right]\Big|_{x=\frac{A}{2}\sqrt{1+\frac{\sigma^2}{W^2}}} = -\left(\delta\sigma + \frac{k\sigma}{m\delta}\right) + \frac{b\sigma}{m\delta\left(\alpha + \frac{A}{2}\sqrt{1+\frac{\sigma^2}{W^2}}\right)^2} \tag{16}$$

where W is given by:

$$W^2 = \omega^2 - \frac{\delta^2}{4} \tag{17}$$

Using eqs. (15)-(17), it obtains:

$$\left[\frac{k}{b} + \frac{k\delta}{b\sigma} + \frac{m}{b}\delta^2 - \frac{m\delta^3}{4b\sigma} - \frac{\delta}{\sigma\left(\alpha + \frac{A}{2}\right)^2}\right]\left[\alpha + \frac{A}{2}\sqrt{\frac{(k + m\sigma^2)\left(\alpha + \frac{A}{2}\right)^2 - b}{k\left(\alpha + \frac{A}{2}\right)^2 - b}}\right]^2 = 1 \tag{18}$$

It is easy to find that S is a function of the fractional order β, if the system parameters are ascertained. The parameter β has a huge impact on the two trigonometric functions in $\delta = S\tan(\pi/2)$ and $\sigma = S^{\beta-1}\sin(\pi\beta/2)$. When β changes from 1 to 2, $\tan(\pi\beta/2)$ increases from the minimum value to zero, while $\sin(\pi\beta/2)$ decreases from the maximum value to zero. Changes in parameter S can be observed in tab. 1 and fig. 2. They all display the function S as an increasing function of the parameter α with the variation of the parameter β, but as β increases, S first increases until a critical point and then decreases. The closer β is to 2, the smaller S is to zero. When $\beta = 2$, the parameter S has no effect.

Table1. The fractional parameter *S* for eq. (18) when *k* = *m* = *b* = 1, *A* = 1

α \ β	1.1	1.3	1.5	1.7	1.9
0.01	0.068871	0.132623	0.128235	0.066391	0.002412
0.05	0.092547	0.170165	0.178815	0.115654	0.004929
0.1	0.161224	0.273494	0.315226	0.262455	0.041439

To provide insight into the fractal-fractional oscillator, a comparison between the periodic solutions of the integer-order differential eq. (1) and its corresponding fractional-

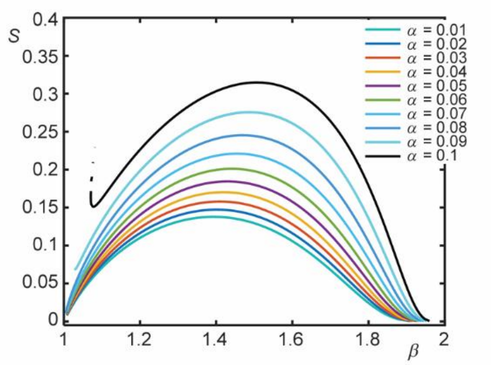

Figure 2. The fractional parameter *S* for eq. (18) when *k* = *m* = *b* = 1, *A* = 1

order differential eq. (11) is presented in fig. 3. Equation (1) lacks a damping term, whereas eq. (11), derived through our proposed fractional order method, incorporates a damping term. This indicates that the parameter S plays a novel role, counteracting the damping behavior and supporting non-damping motion until the fractional order β approaches 2.

The results obtained provide an effective explanation of the phenomenon of aggregation of roes in nature. Upon release to the surface, the roes sink rapidly due to their higher density than water. However, the turbulence effect facilitates the suspension of the roes. Consequently, the roes are not entirely submerged beneath the bed, but rather remain in a stable position once a relatively balanced situation has been reached. Research has demonstrated that in fast-flowing areas of rivers, grass carp roes drift primarily passively. In areas of strong current, the diffusion effect of roes is significantly less pronounced than passive drift. Conversely, in areas of weak current, their diffusion range is greater. As the riverbed is elevated and incised, there is a reduction in the amplitude of fluctuations in the riverbed elevation during the downstream drift process. The incubation period for grass carp gametes is 30-40 hours when the temperature is suitable, and they are susceptible to external factors during this process. It is of paramount importance to maintain the stability of the grass carp roe aggregation system and enhance the hatching rate of fry by selecting appropriate fractal dimension parameters during the hatching process. The aggregation stability of grass carp roe can be enhanced through the use of a sticky attachment mechanism, which serves to mitigate the effects of perturbation oscillations. The stability of the system can be enhanced by the application of strong adhesion.

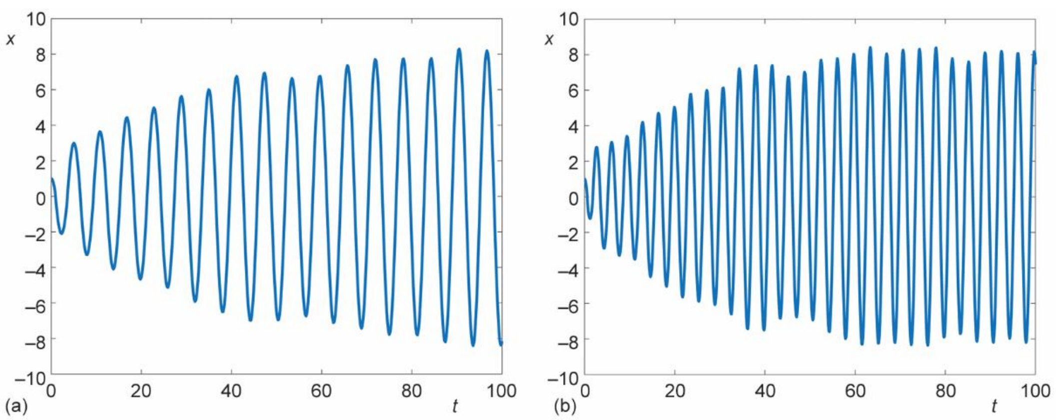

Figure 3. Periodic motion with *k* = *m* = *b* = *A* = 1, *α* = 0.01, and *β* = 1.9; (a) eq. (1) and (b) eq. (11)

Conclusion

This paper employs a Toda-like system to elucidate the aggregation of roe. In light of the contamination of the aquatic environment, which is essential for fish survival, it is necessary to consider the water as a porous medium. Consequently, a more realistic fractal-fractional model is required to study the aggregation of roe. The classical method of replacing t by t^{β} was unsuccessful in obtaining an appropriate fractional order model. An appropriate method was employed to attempt to establish this fractal-fractional model in fractal space, namely a Toda-like fractal-fractional system. Analogies were also made with integer order systems, with encouraging results. The phenomenon of grass carp roe on the water surface is analyzed with the help of the Toda-like fractal-fractional model. The dynamic characteristic of the fractal-fractional oscillator is low frequency. The influence of the fractal derivative order is presented in detail and illustrated in figures. The model offers a novel approach to biomechanics, providing a creative framework for biomimetic design of chatter vibration systems inspired by the agglomerated roes.

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