THE MIXED NEGATIVE BINOMIAL PROCESS RISK MODEL WITH SMALL CLAIMS
Stochastic Process and Ruin Probability

by

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If the random variable changes with time, we can consider it a stochastic process. The stochastic claims process is particularly important in insurance, where the frequency of claims is a random variable. Classical risk models typically assume that the number of claims by insurance companies follows an (a, b, 0) type distribution. In practice, however, the number of claims is often an over-dispersed or heavy-tailed phenomenon. To compensate for this deficiency, mixed distributions have been proposed. This article discusses the lapse probability of a general compound mixed negative binomial small claims process risk model based on a negative binomial mixture distribution.

Key words: *over-dispersed, compound mixed negative binomial process, heavy-tailed, ruin probability, small claims*

Introduction

Random process theory, as an important tool for studying random phenomena, is used in many fields. For example, He and Qian [1] proposed a fractal approach to a stochastic diffusion process with great success. This paper mainly considers the problem of bankruptcy probability in the stochastic claim process risk model with small claims.

The calculation of bankrupt probability is a crucial issue in actuarial science, the canonical risk model [2] is a classical model of non-life insurance theory. Here, the ruin problem refers to a state in which the surplus process falls below the level of 0 at a certain time. When the random variable of claim amount follows a light-tailed (small claims) and heavy-tailed (extremely large claims) distribution in the homogeneous Poisson claim process, the discussion of ruin probability is carried out. At present, relevant theories can be found in the literature [2, 3], *etc.* The asymptotic ruin probability can be discussed. For example, Hipp [4] obtained the bankruptcy probability according to the controlled risk model with small claims. Grigori [5] obtained an approximate expression for the bankruptcy probability of the discrete Brownian risk model. Albrecher [6] considered the bankruptcy probability of classical risk model with claim sizes that are mixtures of phase-type and sub-exponential variables, and so on. But the claim numbers of these risk models are based on Poisson distribution, negative binomial distribution or geometric distribution, the over-dispersion of claim numbers is not considered.

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In practice, the variance in the claims counting process is often greater than the mean, which usually shows the characteristics of over-dispersion and even heavy tails. Traditional distributions in the (a, b, 0) model cannot fit claims well. To compensate for this deficiency, mixed distributions have been successively proposed, see [7-11]. Since the variance of mixed negative binomial processes is larger than the mean, and the heavy tail index tends to 1, the risk model with mixed negative binomial counting process studied in this paper has a practical application background.

Compound mixed negative binomial process risk model

Mixed negative binomial (MNB) distribution is the combination of negative binomial distribution and other distributions. It has the characteristics of over-dispersion and heavy tail. This part mainly introduces the definitions of MNB distribution and compound MNB process risk model.

The pmf of classical negative binomial distribution can be expressed:

$$P(T=t)=\binom{\delta+t-1}{t}q^{\delta}(1-q)^{t},\quad t=0,1,2,\cdots$$

where parameters $\delta>0$, $0<q<1$. It can be noted that T follows a negative binomial distribution, that is, $T\sim NB(\delta,q)$. The first moment, variance, moment generating function (mgf) and generating function (gf) are, respectively:

$$E(T)=\frac{\delta(1-q)}{q},\ Var(T)=\frac{\delta(1-q)}{q^{2}},\ M_{T}(s)=\left[\frac{q}{1-(1-q)e^{s}}\right]^{r},\ G_{T}(s)=(1-q+qs)^{\delta}$$

If the pdf of the non-negative random variable Ξ is assumed to be $f_{\Theta}(\varsigma)$, mgf is $M_{\Xi}(\cdot)$, where Θ is a parameter vector and each element is greater than 0 , then the MNB distribution can be defined directly.

Definition 1 A random variable T is said to have a MNB distribution with the non--negative parameters (δ,Θ) if its pmf is provided by:

$$P(T=t)=\int_{0}^{+\infty}\binom{\delta+t-1}{t}e^{-\varsigma\delta}(1-e^{-\varsigma})^{t}f_{\Theta}(\varsigma)\mathrm{d}\varsigma,\quad t=0,1,2,\cdots \tag{1}$$

where $T\,|\,\Xi\sim NB(\delta,e^{-\varsigma})$. We denote the MNB distribution by $T\sim MNB(\delta,\Theta)$. According to the formula of conditional expectation, the expectation, variance, mgf and gf can be obtained:

$$E(T)=\delta[M_{\Xi}(1)-1],\qquad Var(T)=\delta\{M_{\Xi}(2)-M_{\Xi}(1)+\delta[M_{\Xi}(2)-M_{\Xi}^{2}(1)]\},$$
$$M_{T}(s)=E_{\Xi}\left\{\frac{e^{-\varsigma\delta}}{[1-(1-e^{-\varsigma})e^{s}]^{\delta}}\right\},\quad G_{T}(s)=E_{\Xi}\{[(1-s)e^{\varsigma}+s]^{-\delta}\}$$

where $E_{\Xi}(\cdot)$ is the expectation of Ξ .

Over-dispersion is an important feature of the number of insured losses. This paper discusses the over-dispersion and heavy tail of MNB distribution through the following theorem.

Theorem 1 Let $\boldsymbol{T\sim MNB(\delta,\Theta)}$ and assume (δ,Θ) is a non-negative parameter vector, then:

– Over-dispersion coefficient $DI = \dfrac{Var(T)}{E(T)} > 1$

– Heavy tail index $\lim\limits_{t\to\infty} HT = \lim\limits_{z\to\infty} \dfrac{P(T=t+1)}{P(T=t)} = 1$

Proof Let:

$$\tilde{T} \sim NB\left(\delta, \frac{1}{M_\Xi(1)}\right), \quad \text{note that} \quad 0 < \frac{1}{M_\Xi(1)} < \frac{1}{M_\Xi(0)} = 1$$

The expectation and variance of $\tilde{T}$ are:

$$E(\tilde{T}) = \delta[M_\Xi(1) - 1] = E(T)$$

$$Var(\tilde{T}) = \delta M_\Xi(1)[M_\Xi(1) - 1]$$

Then:

$$Var(T) - Var(\tilde{T}) = \delta(1+\delta)[M_\Xi(2) - M_\Xi(1)] > 0$$

We can obtain:

$$\frac{Var(T)}{E(T)} > \frac{Var(\tilde{T})}{E(T)} = M_\Xi(1) > 1$$

Meanwhile:

$$HT = \frac{\delta + t}{t+1} \frac{\int\limits_0^{+\infty} e^{-\varsigma\delta}(1-e^{-\varsigma})^{t+1} f_\Theta(\varsigma)\mathrm{d}\varsigma}{\int\limits_0^{+\infty} e^{-\varsigma\delta}(1-e^{-\varsigma})^{t} f_\Theta(\varsigma)\mathrm{d}\varsigma} \to 1, \quad t \to +\infty$$

So the MNB distribution has both heavy tail and over-dispersion characteristics.

According to the definition of the MNB distribution, the MNB random sequence, the MNB processes, compound MNB processes and MNB process risk model can be defined.

Definition 2 The non-negative integer valued random variables sequence $\{M(\iota),\ \iota \ge 0\}$ is MNB random sequence, if $[M(\iota_2) - M(\iota_1)] \sim MNB[\delta(\iota_2 - \iota_1), \Theta]$ for all $\iota_2 > \iota_1$, that is:

$$P[M(\iota_2) - M(\iota_1) = \kappa] = \int\limits_0^{+\infty} \begin{bmatrix} \delta\left(w_2 - w_1\right) + \kappa - 1 \\ \kappa \end{bmatrix} e^{-\varsigma\delta\left(\iota_2 - \iota_1\right)} (1 - e^{-\varsigma})^\kappa f_\Theta(\varsigma)\mathrm{d}\varsigma, \quad \kappa = 0, 1, 2\cdots \quad (2)$$

Theorem 2 The MNB random sequence has stationary independent increments.

Proof For:

$$0=\iota_0<\iota_1<\iota_2<\cdots<\iota_m$$

$$P[M(\iota_i)-M(\iota_{i-1})=\kappa_i]=\int_0^{+\infty}\begin{bmatrix}\delta(\iota_i-\iota_{i-1})+\kappa_i-1\\ \kappa_i\end{bmatrix}e^{-\varsigma\delta(\iota_i-\iota_{i-1})}(1-e^{-\varsigma})^{\kappa_i}f_\Theta(\varsigma)\mathrm{d}\varsigma,\quad \kappa_i=0,1,2\cdots$$

It is easy to know that $M(\iota_1)-M(\iota_0),M(\iota_2)-M(\iota_1),\cdots,M(\iota_\kappa)-M(\iota_{\kappa-1})$ are independent, then the MNB random sequence has the independent increments. There have the same distribution for any $\iota>0$, $M(\iota+\kappa)-M(\iota)$, then $\{M(\iota),\ \iota\geq 0\}$ has stationary increments. Therefore, the MNB random sequence $\{M(\iota),\ \iota\geq 0\}$ has a stationary independent increment.

Definition 3 $\{\boldsymbol{M(\iota)},\ \boldsymbol{\iota}\geq 0\}$ is called the MNB process with parameters $(\delta\iota,\Theta)$, if:

(1) $$M(0) = 0$$

(2) $\{\boldsymbol{M(\iota)},\ \boldsymbol{\iota}\geq 0\}$ has a stationary independent increment

(3) $M(\iota)\sim MNB(\delta\iota,\ \Theta)$ for $\forall\,\iota>0$ and

$$E[M(\iota)]=\delta\iota[M_\Lambda(1)-1]$$

$$Var[M(\iota)]=\delta\iota\{M_\Xi(2)-M_\Xi(1)+\delta\iota[M_\Xi(2)-M_\Xi^2(1)]\}$$

where $\delta>0,\ \Theta\in R^+$.

Definition 4 $\Sigma_{i=1}^{M(\iota)}T_i$ is a compound MNB process if $\{M(\iota),\ \iota\geq 0\}$ is a MNB process with parameters $(\delta\iota,\Theta)$, $\{Z_i, i=1,2,\cdots\}$ are i.i.d. random variables and independent of $M(\iota)$. .

Definition 5 Given the measurable space $(\boldsymbol{\Omega},\boldsymbol{F},\boldsymbol{P})$, the risk model surplus process $\{\boldsymbol{\Pi(\iota)},\ \boldsymbol{\iota}\geq 0\}$ is defined:

$$\Pi(\iota)=x+c\iota-\sum_{i=1}^{M(\iota)}T_i,\qquad \iota\geq 0 \tag{3}$$

where $\boldsymbol{x}=\boldsymbol{\Pi}(0)\geq 0$ represents initial capital, $c>0$ is premium in unit time. The $S(\iota)=\Sigma_{i=1}^{M(\iota)}T_i$ is the claim amount up to the time of ι, $\{M(\iota)\}_{\iota\geq 0}$ is the claim amount that have occurred up to time υ and it is the MNB process of parameter $(\delta\iota,\Theta)$, $T_i(i\geq 1)$ represents the i^{th} claim amount. The $\{T_i\}_{i\geq 1}$ is an i.i.d. random variables, $\{M(\iota)\}_{\iota\geq 0}$ and $\{T_i\}_{i\geq 1}$ are independent of each other. If $S=\min\{\iota\mid\iota\geq 0,\ \Pi(\iota)\leq 0\}$ represents the moment of ruin, then $\Psi(x)=P(S<+\infty)$ is the ruin probability when the initial capital is *x*.

Definition 6

$$\rho=\frac{c}{u_1[\delta M_\Xi(1)-1]}-1>0$$

is called safety coefficient. When $\rho\leq 0$, bankruptcy inevitably occurs, that is, $\Psi(u)=1.$. Where u_1 is the expectation of claim *T*.

Main results

Lemma 1 If $S(t)=\Sigma_{i=1}^{M(\iota)}T_i$ is compound MNB process, per claim $T_i \geq 0$. The expectation, variance and mgf are, respectively, given by $\mu_1 = E(T), D_1^2 = Var(T),$ and $M_T(\iota) = E(e^{\iota T}),$ then:
– Claim process $\{S(\iota), \iota \geq 0\}$ has the stationary independent increment.
– The expectation, variance and mgf of $S(\iota)$ are as shown.

$$E[S(\iota)] = \mu_1 E[M(\iota)], \quad Var[S(\iota)] = D_1^2 E[M(\iota)] + \mu_1^2 Var[M(\iota)]$$

$$M_{S(\iota)}(\iota) = E_\Xi \left\{ \left[\frac{e^{-\varsigma}}{1-(1-e^{-\varsigma})M_T(\iota)} \right]^{\delta\iota} \right\}$$

Proof (1) Let $0 \leq \iota_1 < \iota_2 < \cdots$, then $S(\iota_\omega) - S(\iota_{\omega-1}) = \Sigma_{i=M(\iota_{\omega-1})+1}^{M(\iota_\omega)} T_i$. From the independent of $\{T_i\}_{i\geq 1}$:

$$\sum_{i=M(\iota_1)+1}^{M(\iota_2)} T_i, \quad \sum_{i=M(\iota_2)+1}^{M(\iota_3)} T_i, \quad \cdots$$

are Mutually independent. So $\{S(\iota),\ \iota \geq 0\}$ has the independent increment. And:

$$E\left(e^{\sum_{i=M(\iota_{m-1})+1}^{M(\iota_m)} T_i}\right) = \sum_{l<n} E[e^{\sum_{i=M(\iota_{m-1})+1}^{M(\iota_m)} T_i} \mid M(\iota_{\omega-1}) = l, \quad M(\iota_\omega) = \kappa] = E\left(e^{\sum_{i=1}^{M(\iota_\omega)-M(\iota_{\omega-1})} T_i}\right)$$

Then:

$$\sum_{i=M(\iota_{\omega-1})+1}^{M(\iota_\omega)} T_i \quad \text{and} \quad \sum_{i=1}^{M(\iota_\omega)-M(\iota_{\omega-1})} T_i$$

have the same characteristic function. So $S(\iota_\omega) - S(\iota_{\omega-1})$ and $S(\iota_\omega - \iota_{\omega-1})$ have the equal distribution. Then $\{S(\iota),\ \iota \geq 0\}$ has the stationary increment and the result (1) is obtained.

(2) The expectation and variance of $S(\iota)$ can be easily calculated by the conditional expectation formula. The mgf of $S(\iota)$ is:

$$M_{S(\iota)}(\iota) = M_{M(\iota)}[\log M_T(\iota)] = E_\Xi \left\{ \left[\frac{e^{-\varsigma}}{1-(1-e^{-\varsigma})M_T(\iota)} \right]^{\delta\iota} \right\}$$

where $M_S(\cdot)$ is the mgf of S and $E_\Xi(\cdot)$ is the expectation of Ξ.

Lemma 2 For the profit process $\{Z(\iota),\ \iota \geq 0\}$, there exists a function $g_\iota(\varepsilon)(\varepsilon > 0)$ such that $E(e^{-\varepsilon Z(\iota)}) = e^{g_\iota(\varepsilon)}$, where $Z(\iota) = c\iota - S(\iota)$ and:

$$g_\iota(\varepsilon) = -\varepsilon c \iota + \log\{M_{M(\iota)}[\log M_T(\varepsilon)]\} \tag{4}$$

Proof It is easy to infer from the *Lemma 1*.

Lemma 3 There is only one positive number Λ such that $g_\iota(\varepsilon) = 0,$ and Λ is referred to as the adjustment coefficient.

Proof From the *Lemma 2*:

$$1 < M_T(\varepsilon) < \frac{1}{1-e^{-\varsigma}}$$

The mgf continues monotonically increasing on the interval $\left[0,\varepsilon_1\right)$ with respect to ε. The uniqueness of the positive solution is proved below.

Firstly,

$$g_\iota(0)=\log\int_0^{+\infty}e^{-\varsigma\delta\iota}f_\Theta(\varsigma)[1-(1-e^{-\varsigma})]^{-\delta\iota}\mathrm{d}\varsigma=0$$

When:

$$\varepsilon\to\varepsilon_1,\quad M_T(\varepsilon)\to\frac{1}{1-e^{-\varsigma}}$$

then:

$$\lim_{\varepsilon\to\varepsilon_1}g_\iota(\varepsilon)=+\infty$$

Secondly, the first and second order derivatives of $g_\iota(\varepsilon)$ with respect to ε are respectively:

$$g'_\iota(\varepsilon)\to -c\iota+\mu\delta\iota[M_\Xi(1)-1]<0,\quad \varepsilon\to 0$$

$$g''_\iota(\varepsilon)=\frac{\delta\iota E(T^2e^{\varepsilon T})I_1L_3+\delta\iota E^2(Te^{\varepsilon T})I_2L_2+\delta^2\iota^2E^2(Te^{\varepsilon T})(I_2L_2-I_1^2)}{L_2^2}$$

Then $\lim_{\varepsilon\to 0}g'_\iota(\varepsilon)<0.$ Let $L_1=\delta\iota E(Te^{\varepsilon T})I_1$ and:

$$L_2=\int_0^{+\infty}e^{-\varsigma\delta\iota}f_\Theta(\varsigma)[1-(1-e^{-\varsigma})M_T(\varepsilon)]^{\delta\iota}\mathrm{d}\varsigma$$

then the first order derivatives of L_1, L_2 with respect to ε are following:

$$L'_1=\delta\iota E(T^2e^{\varepsilon T})I_1+\delta\iota\left(\delta\iota+1\right)E^2(Te^{\varepsilon T})I_2$$

where $L'_2=L_1$:

$$I_1=\int_0^{+\infty}(1-e^{-\varsigma})e^{-\varsigma\delta\iota}f_\Theta(\varsigma)[1-(1-e^{-\varsigma})M_T(\varepsilon)]^{-\delta\iota-1}\mathrm{d}\varsigma$$

$$I_2=\int_0^{+\infty}(1-e^{-\varsigma})^2e^{-\varsigma\delta\iota}f_\Theta(\varsigma)[1-(1-e^{-\varsigma})M_T(\varepsilon)]^{-\delta\iota-2}\mathrm{d}\varsigma$$

The contradiction method can be used to prove $I_2L_2-I_1^2=0$. Let:

$$J_1=e^{\varsigma\delta\iota}f_\Theta(\varsigma)>0\quad J_2=1-e^{-\varsigma}>0$$

$$J_3=1-(1-e^{-\varsigma})M_T(\varepsilon)>0$$

then:

$$I_1=\int_0^{+\infty}J_1J_2J_3^{-\delta\iota-1}\mathrm{d}\varsigma,\quad I_2=\int_0^{+\infty}J_1J_2^2J_3^{-\delta\iota-2}\mathrm{d}\varsigma,\quad L_2=\int_0^{+\infty}J_1\mathrm{d}\varsigma$$

We can obtain:

$$I_2 - I_1 = \int_0^{+\infty} J_1 J_2 J_3^{-\delta\iota-2}(J_2 - J_3)\mathrm{d}\varsigma, \quad L_2 - I_1 = \int_0^{+\infty} J_1 J_3^{-\delta\iota-1}(J_3 - J_2)\mathrm{d}\varsigma$$

If $I_2 L_2 - I_1^2 \neq 0$, we could let $I_2 L_2 - I_1^2 > 0$. Then there exists I_1, I_2, I_3 makes inequality $I_2 > I_1, L_2 > I_1$. Two inequalities hold simultaneously and contradict each other, therefore $I_2 L_2 - I_1^2 = 0$ and $g_\iota''(\varepsilon) > 0.$

So it can be seen that $g_\iota(\varepsilon) = 0$ has only one positive root Λ on $\varepsilon \in [0, \varepsilon_1)$ for any $\iota \geq 0$. At this point:

$$E[e^{-\Lambda\Pi(\iota)}] = E\{e^{-\Lambda[x+Z(\iota)]}\} = e^{-\Lambda x}$$

Therefore, if $F_\iota = \sigma[Z(\iota), \nu \leq \iota]$, then E is only positive number for martingales $e^{-\Lambda\Pi(\iota)}$.

Theorem 3 Under the assumption of *Lemma 3*, Λ is the adjustment coefficient and the finite time ruin probability of the general compound MNB process risk model is:

$$\Psi(x) = \frac{e^{-\Lambda x}}{E[e^{-\Lambda\Pi(S)} \mid S < +\infty]} \tag{5}$$

for $\Lambda > 0$ and any $\iota > 0$.

Proof The equation:

$$E\left[e^{-\Lambda\Pi(\iota)}\right] = E\left[e^{-\Lambda\Pi(\iota)} \mid S \leq \iota\right]P(S \leq \iota) + E\left[e^{-\Lambda\Pi(\iota)} | S > \iota\right]P(S > \iota)$$

another form is:

$$E\left[e^{-\Lambda\Pi(\iota)}\right] = E\left[e^{-\Lambda\left[\Pi(\iota)+Z(\iota)-Z(S)\right]}\right] = E\left[e^{-\Lambda\Pi(\iota)}\right]e^{g_\iota(\Lambda)}e^{g_S(\Lambda)} = E\left[e^{-\Lambda\Pi(\iota)}\right]$$

Then:

$$E\left[e^{-\Lambda\Pi(\iota)} \mid S \leq \iota\right]P(S \leq \iota) = E\left[e^{-\Lambda\Pi(\iota)} \mid S \leq \iota\right]P(S \leq \iota)$$

And:

$$E\left[e^{-\Lambda\Pi(\iota)} | S > \iota\right]P(S > \iota) = E\left[e^{-\Lambda\Pi(\iota)}, 0 \leq \Pi(\iota) \leq \Pi_0(\iota) | S > \iota\right]P[S > \iota, 0 \leq \Pi(\iota) \leq \Pi_0(\iota)] +$$

$$+ E\left[e^{-\Lambda\Pi(\iota)}, \Pi(\iota) > \Pi_0(\iota) | S > \iota\right]P[S > \iota, \Pi(\iota) > \Pi_0(\iota)] \leq P[\Pi(\iota) \leq \Pi_0(\iota)] + E\left[e^{-\Lambda\Pi_0(\iota)}\right]$$

When $\Pi_0(\iota) \to +\infty, E\left[e^{-\Lambda\Pi_0(\iota)}\right] \to 0.$ According to the Chebyshev inequality:

$$P[\Pi(\iota) \leq \Pi_0(\iota)] = P\left\{\frac{\Pi(\iota) - E[\Pi(\iota)]}{\sqrt{Var[\Pi(\iota)]}} \leq \frac{\Pi_0(\iota) - E[\Pi(\iota)]}{\sqrt{Var[\Pi(\iota)]}}\right\} \leq$$

$$\leq P\left\{\left|\frac{E[\Pi(\iota)] - \Pi(\iota)}{\sqrt{Var[\Pi(\iota)]}}\right| \geq \frac{E[\Pi(\iota)] - \Pi_0(\iota)}{\sqrt{Var[\Pi(\iota)]}}\right\} \leq \frac{1}{\left[\dfrac{\Pi_0(\iota) - E[\Pi(\iota)]}{\sqrt{Var[\Pi(\iota)]}}\right]^2}$$

If there is $\Pi_0(\iota)$, such as $\Pi_0(\iota)=\iota^{c_0}(c_0>1/4)$, , so that:

$$\left[\frac{\Pi_0(\iota)-E[\Pi(\iota)]}{\sqrt{Var[\Pi(\iota)]}}\right]^2\to+\infty, \text{, when } \iota\to+\infty$$

then:

$$E\left[e^{-\Lambda\Pi(\iota)}\right]=E[e^{-\Lambda\Pi(\iota)}\,|\,S\le+\infty]P(S\le+\infty),\quad \text{as}\,\iota\to+\infty$$

Morever:

$$E\left[e^{-\Lambda\Pi(\iota)}\right]=e^{-\Lambda x}$$

Then the ruin probability is given by:

$$\Psi(x)=\frac{e^{-\Lambda x}}{E(e^{-\Lambda\Pi(S)}\,|\,S<+\infty)}$$

and the ultimate ruin probability $\Psi(x)\le e^{-\Lambda x}$.

Corollary 1 Under the risk model $\{\boldsymbol{\Pi(\iota)},\,\boldsymbol{\iota}\ge 0\}$ with a total claim amount of compound MNB process, the claim amount T_i is independent and identically distributed and obeys exponential distribution with parameter τ . The final ruin probability is:

$$\Psi(x)=\frac{(\tau-\Lambda)e^{-\Lambda x}}{\tau}\tag{6}$$

where $\boldsymbol{\Lambda}$ is a unique positive solution with equation $g_\iota(\varepsilon)=0$.

Proof If bankruptcy occurs at a finite time of $S<\infty$, denote H as the surplus before the ruin time of S , and event $\{-\boldsymbol{\Pi(S)}>\boldsymbol{t}\,|\,\boldsymbol{S}<+\infty\}$ is equivalent to $\{\boldsymbol{T}>\boldsymbol{H}+\boldsymbol{t}|\boldsymbol{T}>\boldsymbol{H}\}$, then:

$$P(T>H+t|T>H)=P(-\Pi(S)>t\,|\,S<+\infty)=e^{\tau t}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}P(-\Pi(S)<t|S<+\infty)=\tau e^{\tau t}$$

where $\boldsymbol{E}$ is a unique positive solution with equation $g_\iota(\varepsilon)=0.$.

(1) If $\Xi\sim IG(\alpha,\theta)$, pdf is:

$$f_\Xi(\varsigma)=\frac{\alpha}{\sqrt{2\pi\theta\varsigma^3}}e^{-\frac{(\alpha-\theta\varsigma)^2}{2\theta\varsigma}}$$

then accommodation coefficient E is the solution of:

$$e^{-\varepsilon c\iota}\left(\frac{\alpha}{\sqrt{2\pi\theta}}\right)(\tau-\varepsilon)^{\delta\iota}\int_0^{+\infty}\frac{\varsigma^{\frac{3}{2}}e^{-\frac{(\alpha-\theta\varsigma)^2}{2\theta\varsigma}}}{(\pi-\varepsilon e^{\varsigma})^{\delta\upsilon}}\mathrm{d}\varsigma-1=0$$

(2) If $\Xi\sim Gamma(\alpha,\beta)$, pdf is:

$$f_\Xi(\varsigma)=\frac{\beta^\alpha\varsigma^{\alpha-1}}{\Gamma(\alpha)}e^{-\beta\varsigma}$$

then accommodation coefficient E is the solution of:

$$\frac{e^{-\varepsilon ct}\beta^{\alpha}(\tau-\varepsilon)^{\delta t}}{\Gamma(\alpha)}\int_0^{+\infty}\frac{\varsigma^{\alpha-1}e^{-\beta\varsigma}}{(\pi-\varepsilon e^{\varsigma})^{\delta t}}\mathrm{d}\varsigma-1=0$$

Conclusion

This paper considers the over-dispersion and heavy tail of the number of claims. Under the small claims conditions, the MNB claims process risk model is analyzed with different parameters to obtain the representation of the probability of bankruptcy.

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