SOLVING NON-LINEAR EQUATIONS BY FIXED POINT ITERATION METHOD AND ITS ACCELERATING APPROACH

by

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Due to the inherent complexity of non-linear equations, finding their analytical solutions is often challenging, leading to a reliance on numerical solutions. This article examines key methods for solving these solutions, beginning with an introduction to the fixed point iteration method. Through examples, we explore the nuances of this technique in solving non-linear equations. We then examine Aitken's accelerated method and Steffensen's iteration approach, and discuss their convergence properties. By applying these methods to a consistent example, we compare their effectiveness in achieving similar error accuracy. The analysis shows that both the Aitken method and the Steffensen method require significantly fewer iterations than the fixed point iteration approach for solving non-linear equations, with the Steffensen method showing superior performance over the Aitken method. Finally, we have applied the Steffensen method to the flash evaporation problem, and the computational results are highly efficient.

Key words: *non-linear equation, fixed point iteration method, astringency, Aitken accelerated method, Steffensen iteration method*

Introduction

Non-linear algebraic equations are prevalent in a wide range of engineering and theoretical disciplines, including thermal science, electronic circuits, power systems, agricultural irrigation systems, neural networks, and manufacturing engineering. The complexity of solving these equations has made them a focus of non-linear science, and various methods have been developed for this purpose: the artificial parameter method [1], the homotopy perturbation method [2-4], the variational iteration method [5, 6], the integral method [7, 8], the Newton iterative method, popularized by Chun [9], and Darvishi *et al.* [10]. Each method has its unique strengths, demonstrating the diverse challenges and solutions associated with non-linear equations.

The field of fixed point theory in operator analysis, a vital area in non-linear studies, stands as one of the most influential and dynamic tools in contemporary mathematics. With the advancement of science and technology, the iterative fixed point technique for solving non-linear equations has received increasing attention. Lotfi and Tavakoli [11] detailed the Steffensen technique, which is an effective linear interpolation method for enhancing iterative con-

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vergence in solving equations. Finding roots in algebraic equations is a long-standing mathematical challenge. Altaee *et al.* [12] explored the application of Newton iteration approach, bisection approach, and chord cut approach for solving non-linear equations. In addition, Sihwail *et al.* [13] presented a memoryless Steffensen-type method with an excellent fourth--order convergence rate, which was specifically designed to solve root-finding problems in non-linear equations. Taken together, these studies underscore the ongoing evolution and diversity of approaches in the field of non-linear equation solving, demonstrating the versatility and effectiveness of fixed point methods and related techniques in various mathematical disciplines.

In the field of computational mathematics, determining the fastest convergence within a given iteration scheme is a topic of considerable interest. This article will focus on exploring this aspect, beginning with an introduction to the Aitken method and the Steffensen iteration method. Our discussion begins with an in-depth look at the Aitken method, which is known for its efficiency in accelerating the convergence of iterative processes. We will then delve into the Steffensen iteration method, which is known for its fast convergence rate when dealing with non-linear equations. After introducing these methods, we will perform a comparative analysis between the fixed point iteration approach and the Steffensen iteration approach. Through this article, we aim to provide valuable insights into the effectiveness of different iterative methods in computational mathematics, emphasizing in particular the importance of choosing an appropriate method for efficient and rapid solution of non-linear equations.

Fixed point method

The fixed point problem finds diverse applications in everyday life. As a prevalent approach for tackling non-linear equations, the fixed point iteration method recognizes the various external constraints encountered in real-world scenarios, but these constraints often pose challenges in the search for fixed points.

Remarks of fixed point iterative methods

For functions $\phi(x)$ and $g(x)$, as long as there is a real number $\bar{x}$ such that $g(\bar{x})=0,$ there is $\bar{x}=\varphi(\bar{x}),$ and $\bar{x}$ is described as a fixed point of $\phi(x)$. Therefore, solving the root of the non-linear equation $g(x) = 0$ is equivalent to finding the fixed point of the function $\phi(x)$. To solve this problem, only an initial value x_0 needs to be determined, and the iterative function $\bar{x}_{g+1}=\varphi(\bar{x}_g)$ (where $g = 0, 1,\ldots$) can be used for several iterations. At this point, if a sequence $\{\bar{x}_g\}$ is obtained through the iterative process for all $\bar{x}_0 \in [\mu,\lambda]$, and there is a limit $\lim_{g\to\infty} \bar{x}_g = \bar{x},$ it can indicate that the iterative equation itself is convergent, where $\bar{x}=\varphi(\bar{x})$ is the fixed point of the function $\phi(x)$. This method is called fixed point iteration.

The fixed point iteration method is a successive approximation method, whose basic idea is to reformulate the implicit equation $g(x) = 0$ into a comparable version of $x = \phi(x)$, and thus construct an explicit calculation formula $\bar{x}_{g+1}=\varphi(\bar{x}_g)$ (where $g = 0, 1,\ldots$). Then, by selecting an initial approximate value $\bar{x}_0$ and substituting it into the formula, iteratively correct the estimated value of the root until the desired precision is reached. Its geometric significance is shown in fig. 1.

Case study

Following this, we will delve into resolving the non-linear eq. (1) near point $\bar{x}_0$ = 1.5 by fixed point iteration method:

$$g(x) = x^3 - x^2 - 1 = 0 \tag{1}$$

The iterative process is shown in tab. 1.

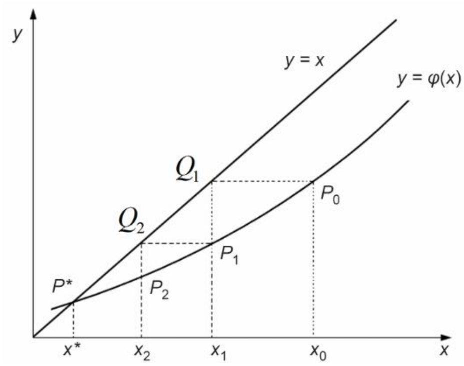

Figure 1. The geometric meaning of the iterative method

Table 1. Iteration results

g	$\overline{x}_g$	g	$\overline{x}_g$
0	1.5	9	1.465600
1	1.481248	10	1.465584
2	1.472705	11	1.465577
3	1.468817	12	1.465574
4	1.467048	13	1.465572
5	1.466243	14	1.465572
6	1.465877	15	1.465571
7	1.465710	16	1.465571

When we take 7 significant digits, the number after $\overline{x}_{15}$ tends to stabilize and can be taken as the root of the equation. Assuming again that the iterative formula is constructed:

$$\overline{x}_{g+1} = \sqrt{\overline{x}_g^3 - 1}, \quad g = 0,1,2,\ldots$$

and choose the same initial value $\overline{x}_0 = 1.5$, then the iterative process is shown in tab. 2.

Table 2. Iteration results

g	$\overline{x}_g$	g	$\overline{x}_g$
0	1.5	4	2.25839
1	1.54110	5	3.24322
2	1.63099	6	5.75444
3	1.82719	7	13.76772

Our observations reveal that as g increases, $\overline{x}_g$ value not only increases but does so at an accelerating rate. This pattern negates the need for further iteration, as such iterations evidently do not converge towards a specific limit, indicating divergence.

Convergence of fixed point iteration method

Generally speaking, when there is a singular fixed point for $\phi(x)$, a condition that guarantees the convergence of the iterative technique $\overline{x}_{k+1} = \varphi(\overline{x}_k)\left(k = 0,1,2,\cdots\right)$ can be obtained.

Lemma 1 [14] Let $\varphi(x) \in C[\alpha,\beta]$ and $\varphi(x)$ satisfies the following two conditions: For any $x \in [\alpha,\beta]$, $\alpha \le \varphi(x) \le \beta$ and exist a normal number L, satisfying $L < 1$. For any $\overline{x}, \overline{y} \in [\alpha,\beta]$, there is $\left|\varphi(\overline{x}) - \varphi(\overline{y})\right| \le L\left|\overline{x} - \overline{y}\right|$, then for any $\overline{x}_0 \in [\alpha,\beta]$, by $\overline{x}_{k+1} = \varphi(\overline{x}_k)$, $k = 0,1,\cdots$, the iterative sequence $\{\overline{x}_k\}$ can be obtained which will converge to the fixed point $\overline{x}$ of $\phi(x)$, and there is an error estimate:

$$\left|\overline{x}_k - \overline{x}\right| \le \frac{L^k}{1-L}\left|\overline{x}_1 - \overline{x}_0\right|$$

Lemma 2 [14] Let $\bar{x}^*$ be a fixed point of $\phi(x)$, $\phi'(x)$ remains continuous within a specific vicinity of $\bar{x}^*$ and $|\varphi'(\bar{x}^*)|<1$, then the iterative method $\bar{x}_{k+1}=\varphi(\bar{x}_k)$, $k=0,1,\cdots$ locally converges.

Lemma 3 [14] Let the iterative process $\bar{x}_{k+1}=\varphi(\bar{x}_k)$ converge to the root $\bar{x}^*$ of equation $x = \phi(x)$. If the iteration error $\bar{e}_k=\bar{x}_k-\bar{x}^*$ satisfies the asymptotic relationship $(\bar{e}_{k+1})/(\bar{e}_k^p)\to C$ when $k\to\infty$, then the iteration process is said to be p^{th} order convergent, where constant $C\neq 0$.

Case study

Next, we use the fixed point iteration approach to consider the solutions to the algebraic equation $x^2 - 2 = 0$, it has a sole root $\bar{x} = (2)^{1/2}$ within the range [1, 3]. We will introduce four methods for expressing the equation previously mentioned in the form $x = \phi(x)$. This will serve to illustrate the fixed point iteration technique.

Let $g(x) = x^2 - 2$. By rewriting it as the equivalent form $x = \phi(x)$, different iterative formulas can be constructed:

(i) $$\bar{x}_{k+1}=\bar{x}_k^2+\bar{x}_k-2,\quad \phi(x)=x^2+x-2$$

(ii) $$\bar{x}_{k+1}=\frac{2}{\bar{x}_k},\quad \phi(x)=\frac{2}{x}$$

(iii) $$\bar{x}_{k+1}=\bar{x}_k-\frac{1}{4}(\bar{x}_k^2-2),\quad \phi(x)=x-\frac{1}{4}(x^2-2)$$

(iv) $$\bar{x}_{k+1}=\frac{1}{2}\left(\bar{x}_k+\frac{2}{\bar{x}_k}\right),\quad \phi(x)=\frac{1}{2}\left(x+\frac{2}{x}\right)$$

Using an initial value $x_0 = 2.0$, we conducted a five-step calculation for each of the four iterative methods discussed earlier. The outcomes of these computations are displayed in tab. 3.

Table 3. Iteration results

k	$\bar{x}_k$	Formula (i)	Formula (ii)	Formula (iii)	Formula (iv)
0	$\bar{x}_0$	2	2	2	2
1	$\bar{x}_1$	4	1	1.5	1.5
2	$\bar{x}_2$	18	2	1.4375	1.416667
3	$\bar{x}_3$	340	1	1.420898	1.414216
4	$\bar{x}_4$	115938	2	1.416160	1.414214
5	$\bar{x}_5$	13441735780	1	1.414782	1.414214

From tab. 3, it can be observed that neither formula (i) nor formula (ii) converges because they do not satisfy the local convergence conditions. Both formula (iii) and (iv) satisfy local convergence conditions, thus they are convergent. Moreover, the convergence speed of formula (iv) is faster than that of formula (iii), because $\varphi'(\bar{x}^*) = 0$ in formula (iv) is smaller than $\varphi'(\bar{x}^*) \approx 0.293$ in formula (iii).

Acceleration of fixed point iteration method

When employing fixed point iteration methods to solve non-linear equations, it is not uncommon to encounter instances where the convergence rate is notably slow throughout the iterative process. To address this challenge, we will delve into two fixed point iteration acceleration methods in the following section.

Aitken acceleration method

The Aitken acceleration method [11] is an iterative technique employed until convergence is achieved. In the Aitken method, the initial non-linear function is linearized using two mapping points acquired after two consecutive iterations. The solution of this linearized approximation function is then utilized as the new approximate solution, effectively completing one iteration cycle.

Let $\bar{x}^* = \varphi(\bar{x}^*)$ be the fixed point of $\phi(x)$, and $\bar{x}_0$ be the approximation point x^*, then there is an iterative formula $\bar{x}_{m+1} = \varphi(\bar{x}_m)$. According to the differential mean value theorem, we have eqs. (2) and (3):

$$\bar{x}_1 = \varphi(\bar{x}_0) = \bar{x}_1 - \bar{x}^* = \varphi(\bar{x}_0) - \varphi(\bar{x}^*) = \varphi'(\xi_1)(\bar{x}_0 - \bar{x}^*) \tag{2}$$

$$\bar{x}_2 = \varphi(\bar{x}_1) = \bar{x}_2 - \bar{x}^* = \varphi(\bar{x}_1) - \varphi(\bar{x}^*) = \varphi'(\xi_2)(\bar{x}_1 - \bar{x}^*) \tag{3}$$

If the fluctuation of $\phi'(x)$ is not significant, it can be determined $\phi'(\xi_1) \approx \phi'(\xi_2)$. By combining eqs. (2) and (3), we can deduce:

$$\bar{x}^* \approx \frac{\bar{x}_0\bar{x}_2 - \bar{x}_1^2}{\bar{x}_2 - 2\bar{x}_1 + \bar{x}_0} = \bar{x}_0 - \frac{(\bar{x}_1 - \bar{x}_0)^2}{\bar{x}_2 - 2\bar{x}_1 + \bar{x}_0} \tag{4}$$

Using eq. (4) as a new approximate iteration and recording it as $\bar{w}_1$, we have eq. (5):

$$\bar{w}_{m+1} = \bar{x}_m - \frac{(\bar{x}_{m+1} - \bar{x}_m)^2}{\bar{x}_m - 2\bar{x}_{m+1} + \bar{x}_{m+2}} = \bar{x}_m - \frac{(\Delta\bar{x}_m)^2}{\Delta^2\bar{x}_m}, \quad m = 0,1,\ldots \tag{5}$$

With $\lim_{m\to 0}(\bar{w}_{m+1} - \bar{x}^*)/(\bar{x}_m - \bar{x}^*) = 0$, the convergence efficiency of sequence $\{\bar{w}_m\}$ is much higher than that of sequence $\{\bar{x}_m\}$.

Stephenson iteration method

The Steffensen iteration method is a method for accelerating convergence of fixed point iteration methods, commonly used for accelerating linear convergence iteration methods. Let:

$$\begin{gathered} \bar{y}_n = \varphi(\bar{x}_n), \quad \bar{z}_n = \varphi(\bar{y}_n)(\bar{y}_n) \\ \bar{x}_{n+1} = \bar{x}_n - \frac{(\bar{y}_n - \bar{x}_n)^2}{\bar{z}_n - 2\bar{y}_n + \bar{x}_n} \end{gathered}, \quad n = 0,1,\ldots$$

Let $\bar{x}^* = \varphi(\bar{x}^*)$ be the fixed point of $\phi(x)$, and the error be $\varepsilon(x) = \phi(x) - x$, $\varepsilon(\bar{x}^*) = \varphi(\bar{x}^*) - \bar{x}^* = 0$, then $\varepsilon(\bar{x}_n) = \varphi(\bar{x}_n) - \bar{x}_n = \bar{y}_n - \bar{x}_n$, $\varepsilon(\bar{y}_n) = \varphi(\bar{y}_n) - \bar{y}_n = \bar{z}_n - \bar{y}_n$.

If the difference in error $\varepsilon(x)$ is very small, by connecting point $[\bar{x}_n, \varepsilon(\bar{x}_n)]$ to point $[\bar{y}_n, \varepsilon(\bar{y}_n)]$, eq. (6) can be obtained:

$$\varepsilon(\bar{x}_n) + \frac{\varepsilon(\bar{y}_n) - \varepsilon(\bar{x}_n)}{\bar{y}_n - \bar{x}_n}(x - \bar{x}_n) = 0 \tag{6}$$

Then:

$$x = \bar{x}_n - \frac{\varepsilon(\bar{x}_n)}{\varepsilon(\bar{y}_n) - \varepsilon(\bar{x}_n)}(\bar{y}_n - \bar{x}_n) = \bar{x}_n - \frac{(\bar{y}_n - \bar{x}_n)^2}{\bar{z}_n - 2\bar{y}_n + \bar{x}_n} = \bar{x}_{n+1} \tag{7}$$

Therefore, we have $x_{n+1} = \varphi(x)$, $n = 0, 1,\ldots$ The essence of the Stephenson method lies in the fusion of two calculations into a single computation step. This innovative approach results in a novel fixed point iteration method, and:

$$\phi(x) = x - \frac{[\varphi(x) - x]^2}{\varphi[\varphi(x)] - 2\varphi(x) + x} \tag{8}$$

Lemma 4 If $\bar{x}^*$ is the fixed-point of the iterative function $\varphi(x)$ defined by eq. (8), then $\bar{x}^*$ is the fixed point of $\phi(x)$. On the contrary. If $\bar{x}^*$ is a fixed point of $\phi(x)$, assuming $\phi''(x)$ exists and $\phi'(x) \neq 1$, then $\bar{x}^*$ is a fixed point of $\varphi(x)$, and the Stephenson iteration method is second-order convergent.

Case study

Now we will use the moving point iteration acceleration method to solve eq. (9):

$$f(x) = x^3 - x^2 - 1 = 0 \tag{9}$$

Firstly, we use the Aitken acceleration method to solve the algebraic eq. (9). We choose a following convergent iterative formula:

$$x_{m+1} = \sqrt[3]{x_m^2 + 1}, \quad g = 0,1,2,\ldots,$$

Let $\varphi(x) = \sqrt[3]{x_m^2 + 1}$ and select $\bar{x}_0 = 1.5$, then the iteration results are shown in tab. 4.

Table 4. Iteration results

m	$\bar{x}_m$	$\bar{x}_{m+1}$	$\bar{x}_{m+2}$	$\bar{w}_m$
0	1.5	1.481248	1.472706	1.465558
1	1.481248	1.472706	1.468817	1.465568
2	1.472706	1.468817	1.467048	1.465571
3	1.468817	1.467048	1.466243	1.465571

A comparison between tabs. 3 and 2 reveals a notable difference. When solving the same equation, the fixed point iteration method necessitates 15 iterations to attain the approximate fixed point values, whereas the Aitken method only requires three iterations. This stark contrast underscores the significant improvement in computational speed achieved by the Aitken method. However, it's important to note that the Aitken method does come with a drawback – it has a conditional limitation on convergence within the iterative formula.

Now we employ the Stephenson method to tackle eq. (9). Let:

$$\bar{x}_{m+1} = \sqrt[3]{\bar{x}_m^2 + 1}, \quad m = 0,1,2,\ldots$$

$$\phi(x) = \sqrt[3]{x^2 + 1}$$

and select $\bar{x}_0 = 1.5$, then the iteration results are shown in tab. 5.

Next, we select the divergent iterative formula:

$$\bar{x}_{m+1} = \sqrt{\bar{x}_m^3 - 1}, m = 0,1,2,\ldots \quad \text{Let} \quad \phi(x) = \sqrt{x^3 - 1}$$

and still select $\bar{x}_0 = 1.5$ as the initial value, then the iteration results are shown in tab. 6.

Table 5. Iteration results

m	$\bar{x}_m$	$\bar{y}_m$	$\bar{z}_m$
0	1.5	1.481248	1.472706
1	1.465558	1.465565	1.465569
2	1.465571	1.465571	1.465571
3	1.465571	1.465571	1.465571

Table 6. Iteration results

m	$\bar{x}_m$	$\bar{y}_m$	$\bar{z}_m$
0	1.5	1.541104	1.630988
1	1.465365	1.465119	1.464577
2	1.465571	1.465571	1.465571
3	1.465571	1.465571	1.465571

Observing tab. 6, it becomes evident that the calculation process exhibits convergence, irrespective of whether the chosen iterative formula itself converges. In essence, the Stephenson method consistently achieves convergence, regardless of the convergence characteristics of the underlying iterative formula.

A comparison between tabs. 4 and 5 highlights a noteworthy distinction. When employing the same convergent iterative formula for iteration, the fixed point iteration method demands 15 iterations to obtain the approximate fixed point values. However, when utilizing the Stephenson method for the solution, merely three iterations suffice. This indicates that the Stephenson method has significant value in improving the effectiveness of fixed point iterative methods.

An application of accelerated iteration method

Now the real-world usage of the Stephenson iteration method in the context of flash evaporation will be given. The Rachford-Rice eq. (13) [15] during the flash evaporation process is:

$$f(\sigma) = \sum_{i=1}^{n} \frac{z_i(1-k_i)}{1+\sigma(k_i-1)} = 0 \tag{10}$$

where z_i is the equilibrium ratio of species i, σ – the proportion of feed that turns into vapor, and k_i – the balance constant of the component j. Once the parameter σ is established, it becomes possible to compute the liquid and vapor constituents.

Equation (10) is a non-linear equation, and we will use the Stephenson iterative method to solve it. We first selected the data z_i, k_i(i = 1, 2, 3, 4) given in tab. 1 [15] and rewritten eq. (10) into eq. (11):

$$f(\sigma)=\frac{-0.32}{1+3.2\sigma}+\frac{-0.15}{1+0.75\sigma}+\frac{0.078}{1-0.26\sigma}+\frac{0.264}{1-0.66\sigma}=0 \quad (11)$$

Since the effective value in eq. (10) must fall within the interval (0, 1) [15], we take the initial value σ = 0.7. Next, using MATLAB software, we will employ the Stephenson iteration technique to resolve the eq. (11). After three iterations, the solutions to eq. (11) were obtained as: 0.023029877444744, 0.120894714757108, and 0.121883818984502. That is to say, after three iterations, we obtained the solution of the eq. (11) as $\sigma = 0.121883818984502$ Compared with $\sigma_{\text{exact}} = 0.12188396426827$ of eq. (11), its error reaches 1.5×10^{-7}.

Now, we again selected the data z_i, k_i(i = 1, 2, 3, 4, 5) given in tab. 3 and rewritten eq. (10) into eq. (12):

$$f(\sigma)=\frac{-0.305}{1+1.525\sigma}+\frac{0.06876}{1-0.2292\sigma}+\frac{-0.0264}{1+0.066\sigma}+\frac{0.037995}{1-0.7599\sigma}+\frac{0.00343}{1-0.686\sigma}=0 \quad (12)$$

We take the initial value σ = 0.6 and use MATLAB software to solve eq. (12). After three iterations, the results of eq. (12) are: 0.517524029740291, 0.524669844579032, and 0.524720645053340. Compare the solution of three iterations $\sigma = 0.524720645053340$ with $\sigma_{\text{exact}} = 0.524720647638382$ of eq. (12), and the error is 2.6×10^{-9}. The previous two examples demonstrate that the Stephenson iteration method is highly efficient. The same problem was solved by the ancient Chinese algorithm in [16], which was also used to solve an important pull-in voltage of an MEMS system [17]. The present method might be more powerful to find the pull-in voltage for a more complex MEMS system, *e.g.* [18].

Conclusion

The solution of non-linear equations is an important and versatile topic, characterized by its robust derivability and scalability. In this article, we introduce the Aitken accelerated method and the Stephenson iteration method, along with an exploration of their convergence characteristics. To illustrate the effectiveness of these methods, we apply them to solve the same non-linear equation and perform a comparative analysis of the solution process and results. In particular, we observe that the fixed point iteration method requires significantly more iterations than the Aitken method and the Stephenson method. Finally, we demonstrate an application of the Stephenson iterative method. The results of our research indicate that this iterative method outperforms the method mentioned in the study of Fatoorehchi *et al.* [15], demonstrating its effectiveness in real-world applications.

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