SOLITARY WAVE AND SINGULAR WAVE SOLUTIONS TO THE NEW (2+1)-D SHALLOW WATER WAVE EQUATION

by

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The new (2+1)-D shallow water wave equation is considered in this research. Two effective methods, namely the Kudryashov method and the Bernoulli sub-equation function method are used to construct the diverse exact wave solutions. The solitary wave and singular wave solutions are obtained, The dynamic behaviors of the extracted wave solutions are unveiled graphically via MAPLE.

Key words: *solitary wave solutions, Bernoulli sub-equation function method, Kudryashov method, shallow water wave equation*

Introduction

Non-linear problems have become an increasingly important discipline, and the continuous maturity of modern science and technology has directly promoted the development of non-linear partial differential equations (NPDE) in the field of physics and mathematics [1-3]. In practical applications, the NPDE can more accurately describe complex non-linear phenomena that exist in natural sciences and even social sciences. At present, theoretical knowledge of non-linear science is widely applied in the fields of fluid dynamics, crystals, non-linear fiber optic communication, and plasma research [4-6]. By establishing non-linear equations and analyzing the analytical solutions of equation models, we gradually understand the essence of these non-linear phenomena. Therefore, solving NPDE through appropriate methods has become increasingly important in the study of non-linear science. By now, different effective approaches have been presented to solve the NPDE, for instance the Hirota bilinear method [7, 8], trial equation approach [9, 10], general integral method [11, 12], variational method [13, 14], Backlund transformation [15, 16], unified method [17, 18] and many others [19-21]. In this work, we aim to probe the new (2+1)-D shallow water wave equation as [22]:

$$u_{yt}+u_{xxxy}-3u_{xx}u_y-3u_xu_{xy}+\alpha u_{xx}+\beta_{yy}+\gamma u_{xy}=0 \tag{1}$$

where $u=u(x, y, t)$, and α, β, and γ are non-zero constants. In this work, we will use two methods to explore the exact wave solutions of eq. (1).

The exact wave solutions

To develop the exact wave solutions of eq. (1), we take the transformation:

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$$u(x,y,t)=U(\xi),\ \xi=mx+ny+kt \tag{2}$$

where m, n, and k are the non-zero real constants. Applying eq. (2), eq.(1) becomes:

$$m^3nU^{(4)}+\left(nk+\alpha m^2+\beta n^2+\gamma mn\right)U''-6m^2nU''U'=0 \tag{3}$$

where

$$U^{(4)}=\frac{\mathrm{d}^4U}{\mathrm{d}\xi^4},\ U''=\frac{\mathrm{d}^2U}{\mathrm{d}\xi^2},\ U'=\frac{\mathrm{d}U}{\mathrm{d}\xi}$$

Integrating eq. (3) once and setting the constant of integration as zero gives:

$$m^3nU'''+\left(nk+\alpha m^2+\beta n^2+\gamma mn\right)U'-3m^2n\left(U'\right)^2=0 \tag{4}$$

The Kudryashov method

By the Kudryashov method [23, 24], there is the auxiliary function:

$$\Phi'=\Phi^2-\Phi \tag{5}$$

where $\Phi=\Phi(\xi)$, $\Phi'=\mathrm{d}\Phi/\mathrm{d}\xi$ and there is the solution:

$$\Phi=\frac{1}{1+\mathrm{e}^{\xi}} \tag{6}$$

It is assumed that eq. (4) admits the solution:

$$U(\xi)=\sum_{i=0}^{p}\varepsilon_i\Phi^i=\varepsilon_0+\varepsilon_1\Phi+\varepsilon_2\Phi^2+.....+\varepsilon_p\Phi^p \tag{7}$$

Inserting it into eq. (4) and balancing U''' and $(U')^2$ yields:

$$p+3=2p+2 \tag{8}$$

which leads to:

$$p=1 \tag{9}$$

Then there is:

$$U(\xi)=\varepsilon_0+\varepsilon_1\Phi(\xi) \tag{10}$$

Taking it into eq. (4) and collecting the coefficients of the different terms Φ^j (j = 0, 1,.., 4) to be zero give:

$$\varepsilon_1=2m,\ k=-\frac{\alpha m^2+m^3n+n^2\beta+mn\gamma}{n} \tag{11}$$

Then we can get the exact wave solution of eq. (1) as:

$$u(x,y,t)=\varepsilon_0+\frac{2m}{1+\mathrm{e}^{mx+ny-\frac{\alpha m^2+m^3n+n^2\beta+mn\gamma}{n}t}} \tag{12}$$

The Bernoulli sub-equation function method

By the Bernoulli sub-equation function method [25], it is assumed eq. (4) has the solution:

$$U=\sum_{i=0}^{p}\varepsilon_iS^i=\varepsilon_0+\varepsilon_1S+\varepsilon_2S^2+....+\varepsilon_pS^p \tag{13}$$

There is:

$$S' = bS + dS^K \tag{14}$$

which admits the solutions as:

$$S(\xi) = \left[-\frac{d}{b} + \frac{c}{\mathrm{e}^{b(K-1)\xi}}\right]^{\frac{1}{1-K}}, \; b \neq d \tag{15}$$

$$S(\xi) = \left[\frac{(c-1)+(c+1)\tanh\left(\dfrac{b(1-K)\xi}{2}\right)}{1-\tanh\left(\dfrac{b(1-K)\lambda}{2}\right)}\right]^{\frac{1}{1-K}} \tag{16}$$

where $c \in R$, $b \neq 0$, $d \neq 0$, $K \in R - \{0, 1, 2\}$. The $S = S(\xi)$ is the Bernoulli differential polynomial. The relation of K and p can be obtained through taking eqs. (13) and (14) into eq. (4) via balancing U'' and $(U')^2$ as:

$$K = p + 1 \tag{17}$$

When $K = 3$, $p = 2$, eq. (13) becomes:

$$U = \varepsilon_0 + \varepsilon_1 S + \varepsilon_2 S^2 \tag{18}$$

Substituting it into eq. (4) and setting the coefficients of S^j ($j = 0, 1,.., 8$) to be zero yields:

$$\varepsilon_1 = 0, \; \varepsilon_2 = 4dm, \; k = -\frac{\alpha m^2 + 4bm^3 n + n^2\beta + mn\gamma}{n}$$

Case 1: For $b \neq 0$, we get the exact wave solution as:

$$u(x,y,t) = \varepsilon_0 + \frac{4dm}{\left[-\dfrac{d}{b} + \dfrac{c}{e^{2b\left(mx+ny-\frac{\alpha m^2+4bm^3n+n^2\beta+mn\gamma}{n}t\right)}}\right]} \tag{19}$$

Case 2: For $b = d$, we get the exact wave solution as:

$$u(x,y,t) = \varepsilon_0 + \frac{4mb\left[1-\tanh\left(-b\left(mx+ny-\dfrac{\alpha m^2+4bm^3n+n^2\beta+mn\gamma}{n}t\right)\right)\right]}{\left[(c-1)+(c+1)\tanh\left(-b\left(mx+ny-\dfrac{\alpha m^2+4bm^3n+n^2\beta+mn\gamma}{n}t\right)\right)\right]} \tag{20}$$

Results and discussion

In this section, we will unveil the performances of the attained wave solutions with the help of the MAPLE.

When we take $\varepsilon_0 = 1$, $m = 1$, $n = 1$, $\alpha = 1$, $\beta = 1$, and $\gamma = 1$, the outlines of eq. (12) in the interval $x, y \in [-10, 10]$ are unfolded in fig. 1. It is obvious that the shape is the anti kink solitary wave.

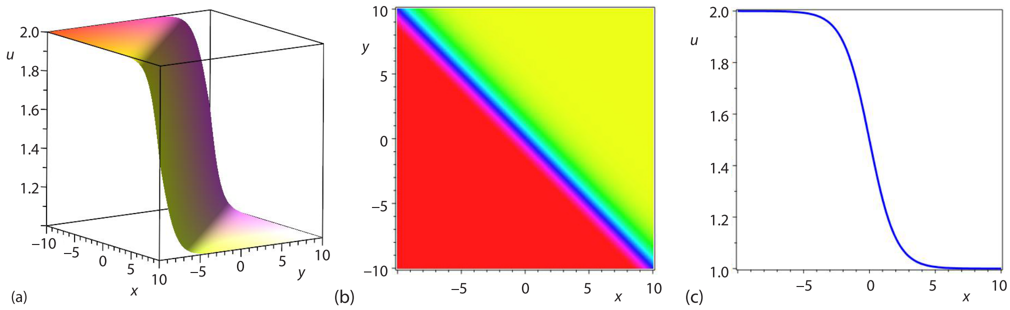

Figure 1. The outlines of eq. (12)

Figure 2 displays the behaviors of eq. (19) for $\varepsilon_0 = 1$, $m = 1$, $n = 1$, $\alpha = 1$, $\beta = 1$, $\gamma = 1$, $b = 1$, $d = 2$, $c = -1$. Here we can discover that the waveform is the anti-solitary wave. For $\varepsilon_0 = 1$, $m = 1$, $n = 1$, $\alpha = 1$, $\beta = 1$, $\gamma = 1$, $b = 1$, and $c = -1$, we depict the behaviors of eq. (20) in fig. 3, which shows that the wave is the singular wave.

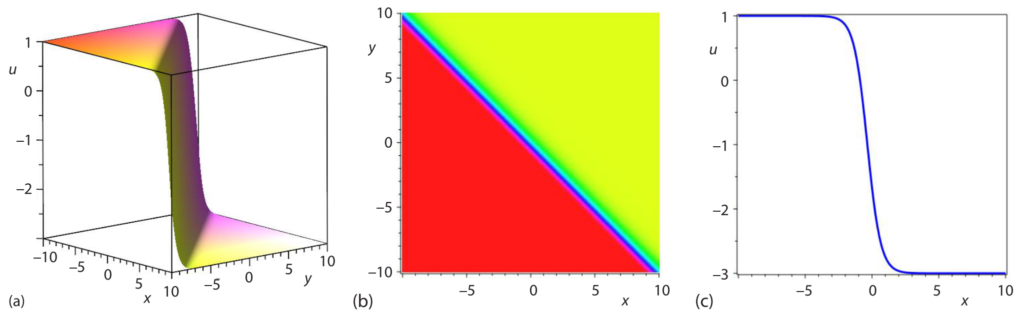

Figure 2. The outlines of eq. (19)

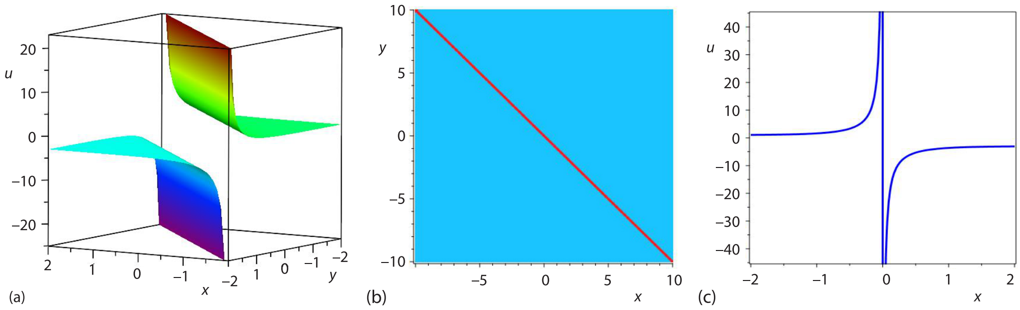

Figure 3. The outlines of eq. (20)

Conclusion

This work has probed some new wave exact solutions to the new (2+1)-D shallow water wave equation by taking two powerful tools, namely Kudryashov method and the Bernoulli sub-equation function method. The solitary wave and singular wave solutions were obtained. MAKING use of the MAPLE, the outlines of the extracted wave solutions are presented graphically.

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