ON THE PERIODIC WAVE SOLUTIONS OF THE (2+1)-D KONOPELCHENKO-DUBROVSKY EQUATION IN FLUID MECHANICS

by

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The central orientation of this work is to plumb the (2+1)-D Konopelchenko-Dubrovsky equation that is utilized widely to describe certain non-linear phenomena in the field of the fluid mechanics. Two effective methods namely the variational method and the energy balance theory are employed to construct the periodic wave solutions. As predicted, the results extracted by these two approaches are almost identical, which is anticipated to offer some new viewpoints to the exploration of the periodic wave theory in physics.

Key words: *periodic wave solution, variational principle, energy balance theory, variational method, Hamiltonian*

Introduction

A good many kinds of complex problems in nature can be described and modeled by the non-linear partial differential equations (NPDE) [1-5]. However, because some basic properties of linear differential equations are no longer tenable in non-linear differential equations, it is difficult to use a unified method to solve the NPDE [6-8]. Therefore, for a long time, solving the exact solutions of NPDE has been a hot topic for researchers [9-12]. In this study, we will look into the (2+1)-D Konopelchenko-Dubrovsky equation (KDE) [13,14]:

$$\begin{gathered}\phi_t-\phi_{xxx}-6b\phi\phi_x+\frac{3}{2}a\phi^2\phi_x-3\phi_y+3a\varphi\phi_x=0\\ \phi_y=\varphi_x\end{gathered} \tag{1}$$

where $\phi=\phi(x,y,t)$, $\varphi=\varphi(x,y,t)$, and a and b are the non-zero real parameters. Equation (1) was proposed by Konopelchenko and Dubrovsky [15] in 1984 and was used extensively to model complex phenomenon in fluid mechanics. Some different efficacious approaches have been adopted to deal with eq. (1) such as the extended F-expansion method [16], modified auxiliary equation approach [17], modified simplest equation method [18], tanh-sech method [19], improved tanh function approach [20], mapping approach [21] and others [22-25]. In this paper, we aim to construct the periodic wave solutions (PWS) of eq. (1) by taking advantage of the variational method (VM) and energy balance theory (EBT).

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The variational principle and the Hamiltonian

For obtaining the periodic solutions, we first apply the traveling waves transformations for eq. (1):

$$\phi(x,y,t)=\phi(\eta),\ \varphi(x,y,t)=\varphi(\eta),\ \eta=x+y+cy+\eta_0 \tag{2}$$

Then, we obtain:

$$\begin{gathered} c\phi'-\phi'''-6b\phi\phi'+\frac{3}{2}a\phi^2\phi'-3\varphi'+3a\varphi\phi'=0 \\ \phi'=\varphi' \end{gathered} \tag{3}$$

Taking a integration of the second equation in eq. (3) yields:

$$\phi=\varphi \tag{4}$$

Taking it into the first equation of eq. (3), then integrating the obtained equation one and taking the integration constant as zero, we get:

$$(c-3)\phi-\left(3b+\frac{3}{2}a\right)\phi^2+\frac{a^2}{2}\phi^3-\phi''=0 \tag{5}$$

Then the variational principle can be established by manipulating the semi-inverse approach [26-29]:

$$J(\phi)=\int\left\{\frac{1}{2}(c-3)\phi^2-\left(b+\frac{1}{2}a\right)\phi^3+\frac{a^2}{8}\phi^4+\frac{1}{2}(\phi')^2\right\}\mathrm{d}\eta \tag{6}$$

For simplicity, we re-write eq. (6) as:

$$J(\phi)=\int\left\{m\phi^2-n\phi^3+k\phi^4+\frac{1}{2}(\phi')^2\right\}\mathrm{d}\eta \tag{7}$$

where

$$m=\frac{1}{2}(c-3),\ n=\left(b+\frac{1}{2}a\right),\ k=\frac{a^2}{8}$$

eq. (7) can be written

$$J(\phi)=\int\left\{m\phi^2-n\phi^3+k\phi^4+\frac{1}{2}(\phi')^2\right\}\mathrm{d}\eta=\int(\Xi-\Theta)\mathrm{d}\eta \tag{8}$$

where Ξ is the kinetic energy and Θ – the stands for the potential energy:

$$\Omega=\frac{1}{2}(\phi')^2 \tag{9}$$

$$\Theta=-m\phi^2+n\phi^3-k\phi^4 \tag{10}$$

Thus we can get the system's Hamiltonian:

$$\mathrm{H}=\Omega+\Theta=\frac{1}{2}(\phi')^2-m\phi^2+n\phi^3-k\phi^4 \tag{11}$$

The periodic wave solutions

The variational method

To seek the PWS of eq. (1) via the VM [30, 31], we postulate the periodic solution of eq. (5) is:

$$\phi(\eta)=\Xi\cos(\omega\eta) \tag{12}$$

Substituting it into eq. (7), we have:

$$J(\Xi)=\int_0^{T/4}\left\{m\Xi^2\cos^2(\omega\eta)-n\Xi^3\cos^3(\omega\eta)+k\Xi^4\cos^4(\omega\eta)+\frac{1}{2}\Xi^2\omega^2\sin^2(\omega\eta)\right\}\mathrm{d}\eta \tag{13}$$

By the VM, there is:

$$\frac{\mathrm{d}J}{\mathrm{d}\Xi}=0 \tag{14}$$

That is:

$$\begin{aligned}\frac{\mathrm{d}J(\Xi)}{\mathrm{d}\Xi}&=\frac{\mathrm{d}}{\mathrm{d}\Xi}\int_0^{T/4}\left\{m\Xi^2\cos^2(\omega\eta)-n\Xi^3\cos^3(\omega\eta)+k\Xi^4\cos^4(\omega\eta)+\frac{1}{2}\Xi^2\omega^2\sin^2(\omega\eta)\right\}\mathrm{d}\eta=\\&=\frac{1}{\omega}\frac{\mathrm{d}}{\mathrm{d}\Xi}\int_0^{\pi/2}\left\{m\Xi^2\cos^2(\psi)-n\Xi^3\cos^3(\psi)+k\Xi^4\cos^4(\psi)+\frac{1}{2}\Xi^2\omega^2\sin^2(\psi)\right\}\mathrm{d}\psi=\\&=\frac{1}{\omega}\int_0^{\pi/2}\left\{2m\Xi\cos^2(\psi)-3n\Xi^2\cos^3(\psi)+4k\Xi^3\cos^4(\psi)+\Xi\omega^2\sin^2(\psi)\right\}\mathrm{d}\psi=0\end{aligned} \tag{15}$$

The expression can be obtained:

$$\omega^2=\frac{\displaystyle\int_0^{\pi/2}\left\{-2m\Xi\cos^2(\psi)+3n\Xi^2\cos^3(\psi)-4k\Xi^3\cos^4(\psi)\right\}\mathrm{d}\psi}{\displaystyle\int_0^{\pi/2}\Xi\sin^2(\psi)\mathrm{d}\psi} \tag{16}$$

which leads to:

$$\omega=\sqrt{\frac{8n\Xi-3k\pi\Xi^2-2m\pi}{\pi}} \tag{17}$$

Thus, the periodic solution of eq. (5) can be:

$$\phi(\eta)=\Xi\cos\left(\sqrt{\frac{8n\Xi-3k\pi\Xi^2-2m\pi}{\pi}}\eta\right) \tag{18}$$

Finally, we can attain the PWS of eq. (1):

$$\phi(x,y,t)=\varphi(x,y,t)=\Xi\cos\left[\sqrt{\frac{8n\Xi-3k\pi\Xi^2-2m\pi}{\pi}}(x+y+ct+\eta_0)\right] \tag{19}$$

where

$$m=\frac{1}{2}(c-3),\ n=\left(b+\frac{1}{2}a\right),\ k=\frac{a^2}{8}$$

The energy balance teory

To apply the EBT [32], the periodic solution of eq. (5) is assumed:

$$\phi(\eta) = \Xi\cos(\omega\eta), \ \omega > 0 \tag{20}$$

Based on the Hamiltonian eq. (11), we can get the Hamiltonian constant :

$$\mathrm{H}_0 = \Omega + \Theta = -m\Xi^2 + n\Xi^3 - k\Xi^4 \tag{21}$$

The EBT tells that the energy of the system keep unchanged with the η changes:

$$H = \Omega + \Theta = \frac{1}{2}\left[-\Xi\omega\sin(\omega\eta)\right]^2 - m\left[\Xi\cos(\omega\eta)\right]^2 + n\left[\Xi\cos(\omega\eta)\right]^3 - k\left[\Xi\cos(\omega\eta)\right]^4 = H_0 \tag{22}$$

which is:

$$\frac{1}{2}\left[-\Xi\omega\sin(\omega\eta)\right]^2 - m\left[\Xi\cos(\omega\eta)\right]^2 + n\left[\Xi\cos(\omega\eta)\right]^3 - \\ -k\left[\Xi\cos(\omega\eta)\right]^4 = -m\Xi^2 + n\Xi^3 - k\Xi^4 \tag{23}$$

Without loss of generality, here we set:

$$\omega\eta = \frac{\pi}{4} \tag{24}$$

such that

$$\frac{1}{2}\left(-\frac{\sqrt{2}}{2}\Xi\omega\right)^2 - m\left(\frac{\sqrt{2}}{2}\Xi\right)^2 + n\left(\frac{\sqrt{2}}{2}\Xi\right)^3 - k\left(\frac{\sqrt{2}}{2}\Xi\right)^4 = -m\Xi^2 + n\Xi^3 - k\Xi^4 \tag{25}$$

which leads to:

$$\omega = \sqrt{\left(4-\sqrt{2}\right)n\Xi - 3k\Xi^2 - 2m} \tag{26}$$

Obviously, we can discover that eq. (26) has a well agreement with eq. (18), this strongly confirms the two method are correct and reliable. By eq. (26), the PWS to eq. (1) is found as:

$$\varphi(x,y,t) = \phi(x,y,t) = \Xi\cos\left[\sqrt{\left(4-\sqrt{2}\right)n\Xi - 3k\Xi^2 - 2m}\left(x+y+ct+\eta_0\right)\right] \tag{27}$$

where

$$m = \frac{1}{2}(c-3), \ n = \left(b+\frac{1}{2}a\right), \ k = \frac{a^2}{8}$$

Conclusion

The (2+1)-D KDE is explored in this paper. By applying the VM and EBT, the periodic wave solutions are obtained. And it finds that the two wave solutions extracted by the two methods have a well agreement. The outcomes reveal that the proposed methods are correct and efficacious, and are anticipated to provide some new ideas to the research of the periodic wave theory.

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References

[1] Hosseini, K., *et al.*, New Exact Solutions of the Coupled Sine-Gordon Equations in Non-Linear Optics Using the Modified Kudryashov Method, *Journal of Modern Optics*, *65* (2018), 3, pp. 361-364

[2] Wang, K. L., *et al*., New Perspective to the Coupled Fractional Non-linear Schrodinger Equations in Dual-core Optical Fibers, *Fractals*, *33* (2025), ID2550034
[3] Ahmad, H., *et al*., New Approach on Conventional Solutions to Non-linear Partial Differential Equations Describing Physical Phenomena, *Results in Physics*, *41* (2022), ID105936
[4] Wang, K. J., *et al*., Novel Singular and Non-singular Complexiton, Interaction Wave and the Complex Multi-Soliton Solutions to the Generalized Non-linear Evolution Equation, *Modern Physics Letters B*, *39* (2025), ID2550135
[5] Wang, K. J., *et al*., Lump Wave, Breather Wave and the other Abundant Wave Solutions to the (2+1)-Dimensional Sawada-Kotera-Kadomtsev Petviashvili Equation for Fluid Mechanic, *Pramana*, *99* (2025), 1, ID40
[6] Wang, K. L., New Dynamical Behaviors and Soliton Solutions of the Coupled Non-linear Schrödinger Equation, *International Journal of Geometric Methods in Modern Physics*, *22* (2025), ID2550047
[7] Sheng, P., *et al*., Vibration Properties and Optimized Design of a Non-linear Acoustic Metamaterial Beam, *Journal of Sound and Vibration*, *492* (2021), ID115739
[8] Wang, K. J., *et al*., Bifurcation and Sensitivity Analysis, Chaotic Behaviors, Variational Principle, Hamiltonian and Diverse Wave Solutions of the New Extended Integrable Kadomtsev-Petviashvili Equation, *Physics Letters A*, *534* (2025), ID130246
[9] Attia, R. A. M., et al., Computational and Numerical Simulations for the Deoxyribonucleic Acid (DNA) Model, Discrete & Continuous Dynamical Systems-S, 14 (2021), 10, 3459
[10] Wang, L. L., et al., Stable Soliton Propagation in a Coupled (2+1) Dimensional Ginzburg-Landau System, Chinese Physics B, 29(2020), 7, ID070502
[11] Yan, Y. Y., *et al*., Soliton Rectangular Pulses and Bound States in a Dissipative System Modeled by the Variable-Coefficients Complex Cubic-quintic Ginzburg-Landau Equation, *Chinese Physics Letters*, *38* (2021), 9, ID094201
[12] Al Kalbani, K. K., *et al*., Pure-Cubic Ooptical Solitons by Jacobi's Elliptic Function Approach, *Optik*, *243* (2021), ID167404
[13] Gupta, A. K., On the Exact Solution of Time-Fractional (2+1) Dimensional Konopelchenko-Dubrovsky Equation, *International Journal of Applied and Computational Mathematics*, *5* (2019), 3, ID95
[14] Suleman, H. A., *et al*., On Exact and Approximate Solutions of (2+1)-D Konopelchenko-Dubrovsky Equation via Modified Simplest Equation and Cubic B-spline Schemes, *Thermal Science*, *23* (2019), Suppl. 6, pp. S1889-S1899
[15] Konopelchenko, B. G., *et al*., Some New Integrable Non-Linear Evolution Equations in 2+1 Dimensions, *Physics Letters A*, *102* (1984), 1-2, pp. 15-17
[16] Sheng, Z., The Periodic Wave Solutions for the (2+1)-Dimensional Konopelchenko-Dubrovsky Equations, *Chaos, Solitons & Fractals*, *30* (2006), 5, pp. 1213-1220
[17] Khater, M. M. A., *et al*., Lump Soliton Wave Solutions for the (2+1)-Dimensional Konopelchenko-Dubrovsky Equation and KdV Equation, *Modern Physics Letters B*, *33* (2019), 18, ID1950199
[18] Alfalqi, S. H., *et al*., On Exact and Approximate Solutions of (2+1)-Dimensional Konopelchenko-Dubrovsky Equation Via Modified Simplest Equation and Cubic B-spline Schemes, *Thermal Science*, *23* (2019), 6, pp. 1889-1899
[19] Wazwaz, A. M., New Kinks and Solitons Solutions to the (2+1)-dimensional Konopelchenko-Dubrovsky Equation, *Mathematical and Computer Modelling*, *45* (2007), 3, pp. 473-479
[20] Sheng, Z., Symbolic Computation and New Families of Exact Non-travelling Wave Solutions of (2+1)-Dimensional Konopelchenko-Dubrovsky Equations, Chaos, *Solitons & Fractals*, *31* (2007), 4, pp. 951-959
[21] Feng, W. G., *et al.*, Explicit Exact Solutions for the (2+1)-Dimensional Konopelchenko-Dubrovsky Equation, *Applied Mathematics and Computation*, *210* (2009), 2, pp. 298-302
[22] Barman, H. K., *et al*., Solutions to the Konopelchenko-Dubrovsky Equation and the Landau-Ginzburg-Higgs Equation via the Generalized Kudryashov Technique, *Results in Physics*, *24* (2021), ID104092
[23] Zhi, H., Symmetry Reductions of the Lax Pair for the 2+1-Dimensional Konopelchenko-Dubrovsky Equation, *Applied Mathematics and Computation*, *210* (2009), 2, pp. 530-535
[24] He, T., Bifurcation of Traveling Wave Solutions of (2+1)-Dimensional Konopelchenko-Dubrovsky Equations, *Applied mathematics and computation*, *204* (2008), 2, pp. 773-783
[25] Yu, W. F., *et al.*, Interactions Between Solitons and Cnoidal Periodic Waves of the (2+1)-Dimensional Konopelchenko-Dubrovsky Equation, *Communications in Theoretical Physics*, *62* (2014), 3, ID297
[26] He, J. H., Semi-Inverse Method of Establishing Generalized Variational Principles for Fluid Mechanics With Emphasis on Turbomachinery Aerodynamics, *International Journal of Turbo & Jet Engines*, *14* (1997), 1, pp. 23-28

[27] He, J. H., A Family of Variational Principles for Compressible Rotational Blade-to-Blade Flow Using Semi-Inverse Method, *International Journal of Turbo & Jet Engines*, *15* (1998) 2, pp. 95-100
[28] Liu, J. H., *et al*., On the Variational Principles of the Burgers-Korteweg-de Vries Equation in Fluid Mechanics, *EPL*, *149* (2025), 5, ID52001
[29] Wang, K. J, *et al.*, Bifurcation Analysis, Chaotic Behaviors, Variational Principle, Hamiltonian and Diverse Optical Solitons of the Fractional Complex Ginzburg-Landau Model, *International Journal of Theoretical Physics*, On-line first: https://doi.org/10.1007/s10773-025-05977-9, 2025
[30] He, J. H., Variational Approach for Non-linear Oscillators, *Chaos, Solitons & Fractals*, *34* (2007), 5, pp. 1430-1439
[31] He, J. H., The Simplest Approach to Non-linear Oscillators, *Results Phys*., *15* (2019), ID102546
[32] Liang, Y. H., *et al*., Diverse Wave Solutions to the New Extended (2+1)-Dimensional Non-Linear Evolution Equation: Phase Portrait, Bifurcation and Sensitivity Analysis, Chaotic Pattern, Variational Principle and Hamiltonian, *International Journal of Geometric Methods in Modern Physics*, *22* (2025), ID2550158

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