BELL SHAPE SOLITARY, ANTI-KINK SOLITARY AND PERIODIC WAVE SOLUTIONS OF THE BENJAMIN ONO EQUATION FOR SHALLOW WATER WAVES

by

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In this study, the Benjamin Ono equation which acts a key role for the shallow water waves is explored. Two effective approaches namely the Bernoulli sub-equation function method and simple frequency formulation method are adopted to extract some different wave solutions, which include the bell shape solitary, anti-kink solitary and periodic wave solutions. Correspondingly, the outlines of the diverse wave solutions are unveiled graphically through MAPLE.

Key words: *Bernoulli sub-equation function method, Benjamin Ono equation, simple frequency formulation, solitary wave solutions*

Introduction

The exact solutions of non-linear evolution equations can provide more physical information in the field of non-linear research and offer deeper insights into the study of problems at the physical level [1-3]. Therefore, solving non-linear evolution equations has always been a research hotspot [4-6]. Up to now, a lot of the effective and powerful methods have been developed, for example the Darboux transformation technique [7, 8], variational method [9, 10], Kudryashov's approach [11, 12], Backlund transformations [13, 14], Sardar subequation method [15, 16], Hirota bilinear method [17, 18], generalized auxiliary equation [19, 20], generalized (G'/G)-expansion method [21, 22], and so on [23-26]. In this study, we aim to study the Benjamin Ono equation as [27]:

$$\phi_{tt} + m\left(\phi^2\right)_{xx} + n\phi_{xxxx} = 0 \tag{1}$$

where $\phi = \phi(x, t)$, and m and n are non-zero real numbers. Here, eq. (1) can model the internal waves in deep water. The aim of this work is to develop the diverse wave solutions for eq. (1).

The solitary wave solutions

To find the solitary wave solutions, we consider the transformation:

$$\phi(x,t) = \Phi(\lambda),\ \lambda = \alpha x + \beta t \tag{2}$$

where α and β are non-zero constants. Substituting it into eq. (1) yields:

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$$\beta^2\Phi'' + m^2\alpha^2\left(\Phi^2\right)'' + \alpha^4 n\Phi^{(4)} = 0 \tag{3}$$

where

$$\Phi'' = \frac{\mathrm{d}^2\Phi}{\mathrm{d}\lambda^2},\ \Phi''' = \frac{\mathrm{d}^4\Phi}{\mathrm{d}\lambda^4}$$

Integrate eq. (3) twice and set the integration constant as zero yields:

$$\beta^2\Phi + m^2\alpha^2\Phi^2 + \alpha^4 n\Phi'' = 0 \tag{4}$$

By the Bernoulli sub-equation function method [28], it is assumed eq. (4) has the solution:

$$\Phi = \sum_{i=0}^{s}\mu_i R^i = \mu_0 + \mu_1 R + \mu_2 R^2 + ... + \mu_s R^s \tag{5}$$

there is:

$$R' = bR + dR^K \tag{6}$$

which admits the solutions:

$$R(\lambda) = \left[-\frac{d}{b} + \frac{c}{\mathrm{e}^{b(K-1)\lambda}}\right]^{\frac{1}{1-K}},\ b \neq d \tag{7}$$

$$R(\lambda) = \left[\frac{(c-1)+(c+1)\tanh\left(\dfrac{b(1-K)\lambda}{2}\right)}{1-\tanh\left(\dfrac{b(1-K)\lambda}{2}\right)}\right]^{\frac{1}{1-K}} \tag{8}$$

where $c \in R$, $b \neq 0$, $d \neq 0$, $K \in R - \{0, 1, 2\}$. The $P = P(\lambda)$ is the Bernoulli differential polynomial. The relation of K and s can be found by putting eqs. (5) and (6) into eq. (4) via balancing Φ'' and Φ^2:

$$2K = s + 2 \tag{9}$$

When $K = 3$, $s = 4$, eq. (5) becomes:

$$\Phi = \mu_0 + \mu_1 R + \mu_2 R^2 + \mu_3 R^3 + \mu_4 R^4 \tag{10}$$

Substituting it into eq. (4) and setting the coefficients of R^j ($j = 0, 1,.., 8$) to be zero, we have:

$$\mu_1 = 0,\ \mu_2 = \frac{6d\mu_0}{b},\ \mu_3 = 0,\ \mu_4 = \frac{6d^2\mu_0}{b^2},\ \beta = -m\beta\sqrt{-\mu_0},\ n = -\frac{\mu_0 m^2}{4b^2\alpha^2}$$

Case 1: For $b \neq d$, we get the solitary wave solution:

$$\varphi(x,t) = \mu_0 + \frac{6d\mu_0}{b\left[-\dfrac{d}{b} + \dfrac{c}{e^{2b\left(\alpha x - m\alpha\sqrt{-\mu_0}t\right)}}\right]} + \frac{6d^2\mu_0}{b^2\left[-\dfrac{d}{b} + \dfrac{c}{e^{2b\left(\alpha x - m\alpha\sqrt{-\mu_0}t\right)}}\right]^2} \tag{11}$$

Case 2. For $b = d$, we the solitary wave solution:

$$\phi(x,t) = \mu_0 + \frac{6\mu_0\left[1-\tanh\left[-b\left(\alpha x - m\alpha\sqrt{-\mu_0}t\right)\right]\right]}{\left[(c-1)+(c+1)\tanh\left[-b\left(\alpha x - m\alpha\sqrt{-\mu_0}t\right)\right]\right]} + \\ + \frac{6\mu_0\left[1-\tanh\left[-b\left(\alpha x - m\alpha\sqrt{-\mu_0}t\right)\right]\right]^2}{\left[(c-1)+(c+1)\tanh\left[-b\left(\alpha x - m\alpha\sqrt{-\mu_0}t\right)\right]\right]^2} \tag{12}$$

The periodic wave solutions

To apply the simple frequency formulation, we can re-write eq. (4):

$$\Phi'' + \frac{\beta^2}{\alpha^4 n}\Phi + \frac{m^2}{\alpha^2 n}\Phi^2 = 0 \tag{13}$$

According to the simple frequency formulation, there is the expression:

$$\Phi'' + f(\Phi) = 0 \tag{14}$$

with

$$f(\Phi) = \frac{\beta^2}{\alpha^4 n}\Phi + \frac{m^2}{\alpha^2 n}\Phi^2 \tag{15}$$

It is assumed that the solution of eq. (13) as:

$$\Phi = \Lambda\cos(\varpi\lambda),\ \varpi > 0 \tag{16}$$

Then the frequency amplitude relationship can be obtained as [29]:

$$\varpi = \sqrt{\frac{\mathrm{d}f(\Phi)}{\mathrm{d}\Phi}}\Bigg|_{\Phi=\frac{\Lambda}{2}} = \sqrt{\frac{\beta^2}{\alpha^4 n} + \frac{m^2\Lambda}{\alpha^2 n}} \tag{17}$$

Then the periodic wave solution of eq. (1) can be obtained as:

$$\phi(x,t) = \Lambda\cos\left[\sqrt{\frac{\beta^2}{\alpha^4 n} + \frac{m^2\Lambda}{\alpha^2 n}}(\alpha x + \beta t)\right] \tag{18}$$

Results and discussion

The purpose of this part is to display the outlines of the attained wave solutions and give the discussion.

If we use the parameters as $\mu_0 = -1$, $b = 1$, $d = -1$, $m = 1$, $\alpha = 1$, $c = 1$, the shapes of eq. (10) are revealed in the form of the 3-D plot, density plot and the 2-D curve in fig. 1, which indicates the wave is the anti kink solitary wave. Figure 2 depicts the dynamic behaviors of the eq. (11) with $\mu_0 = -1$, $b = -1$, $c = -1$, $m = 1$, $\alpha = 1$. Obviously, the wave structure is the bell shape solitary wave.

When the parameters are choosen as $\Lambda = 0.5$, $m = 1$, $n = 1$, $\alpha = 1$, and $\beta = 1$, the behaviors of eq. (6) are unfolded in fig. 3, here we can find the wave is a perfect periodic wave.

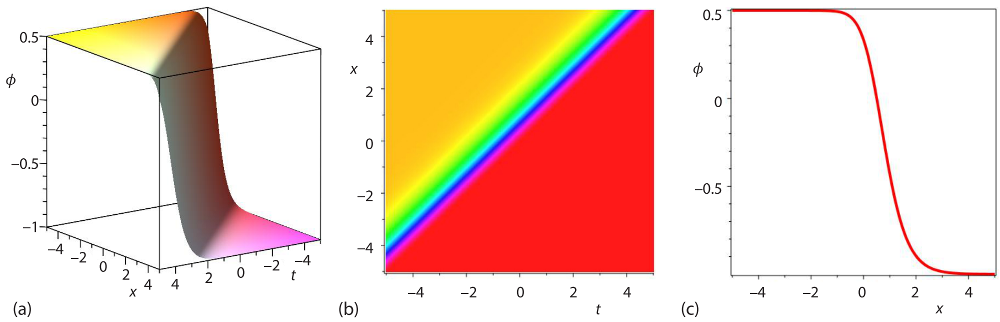

Figure 1. The dynamic behavior of eq. (11)

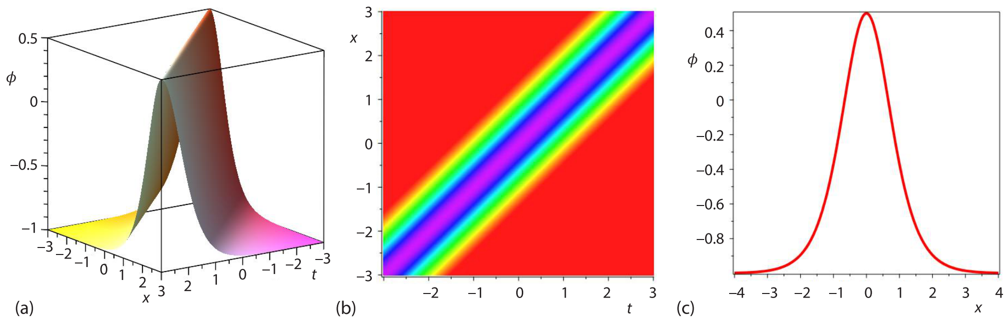

Figure 2. The dynamic behavior of eq. (12)

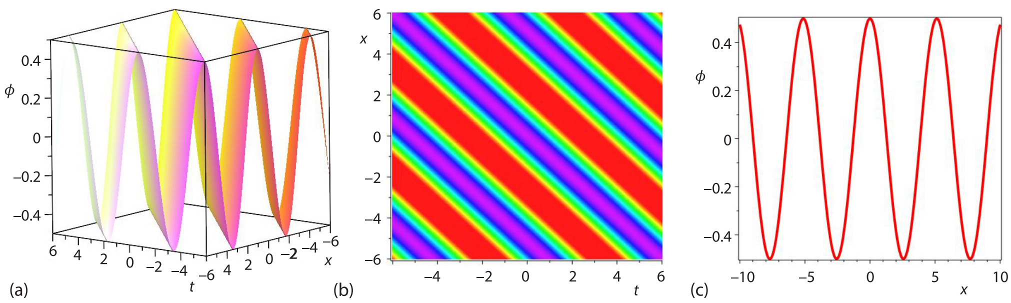

Figure 3. The dynamic behavior of eq. (18)

Conclusion

This work has probed the Benjamin Ono equation equation for the shallow water waves. Applying the Bernoulli sub-equation function method and simple frequency formulation method, different wave solutions like the bell shape solitary, anti-kink solitary and periodic wave solutions were obtained. Additionally, the dynamics of the extracted wave solutions were presented with the help of MAPLE.

References

[1] Duran, S., An Investigation of the Physical Dynamics of a Traveling Wave Solution Called a Bright Soliton, *Physica Scripta*, *96* (2021), 12, ID125251
[2] Duran, S., *et al*., A Study on Solitary Wave Solutions for the Zoomeron Equation Supported by 2-D Dynamics, *Physica Scripta*, *98* (2023), 12, ID125265

[3] Seadawy, A. R., *et al*., Analytical Wave Solutions of the (2+1)-Dimensional Boiti-Leon-Pempinelli and Boiti-Leon-Manna-Pempinelli Equations by Mathematical Methods, *Mathematical Methods in the Applied Sciences*, *44* (2021), 18, pp. 14292-14315
[4] Seadawy, A. R., *et al*., Bifurcation Solitons, Y-type, Distinct Lumps and Generalized Breather in the Thermophoretic Motion Equation Via Graphene Sheets, *Alexandria Engineering Journal*, *87* (2024), Jan., pp. 374-388
[5] Liu, J. H., *et al*., On the Variational Principles of the Burgers-Korteweg-de Vries Equation in Fluid Mechanics, *EPL*, *149* (2025) 5, ID52001
[6] Hosseini, K., *et al*., New Optical Solitons of Cubic-Quartic Non-Linear Schrodinger Equation, *Optik*, *157* (2018), Mar., pp. 1101-1105
[7] Ma, W. X., A Novel Kind of Reduced Integrable Matrix mKdV Equations and Their Binary Darboux Transformations, *Modern Physics Letters B*, *36* (2022), 20, ID2250094
[8] Yang, D. Y., *et al.*, Lax Pair, Darboux Transformation, Breathers and Rogue Waves of an N-Coupled Non-Autonomous Non-Linear Schrodinger System for an Optical Fiber or a Plasma, *Non-Linear Dynamics*, *107* (2022), 3, pp. 2657-2666
[9] Wang, K. J., *et al*., Phase Portrait, Bifurcation and Chaotic Analysis, Variational Principle, Hamiltonian, Novel Solitary and Periodic Wave Solutions of the New Extended Korteweg-De Vries-Type Equation, *Mathematical Methods in the Applied Sciences*, On-line first, https://doi.org/10.1002/mma.10852, 2025
[10] Liang, Y. H., *et al*., Diverse Wave Solutions to the New Extended (2+1)-Dimensional Non-Linear Evolution Equation: Phase Portrait, Bifurcation and Sensitivity Analysis, Chaotic Pattern, Variational Principle and Hamiltonian, *International Journal of Geometric Methods in Modern Physics*, *22* (2025), ID2550158
[11] Zayed, E. M. E., *et al*., Cubic-Quartic Solitons in Couplers with Optical Metamaterials Having Power Law of Refractive Index, *Journal of Non-linear Optical Physics and Materials*, *29* (2020), 3, ID 20500095
[12] Ozisik, M., *et al*., The Bell-Shaped Perturbed Dispersive Optical Solitons of Biswas-Arshed Equation Using the New Kudryashov's Approach, *Optik*, *267* (2022), ID169650
[13] Ma, Y. X., *et al*., Painleve Analysis, Backlund Transformations and Traveling-Wave Solutions for a (3+1)-Dimensional Generalized Kadomtsev-Petviashvili Equation in a Fluid, *International Journal of Modern Physics B*, *35* (2021), 7, ID2150108
[14] Yin, Y. H., *et al.*, Bäcklund Transformation, Exact Solutions and Diverse Interaction Phenomena to a (3+1)-Dimensional Non-Linear Evolution Equation, *Non-Linear Dynamics*, *108* (2022), 4, pp. 4181-4194
[15] Muhammad, T., *et al*., Traveling Wave Solutions to the Boussinesq Equation Via Sardar Sub-Equation Technique, *AIMS Mathematics*, *7* (2022), 6, pp. 11134-11149
[16] Onder, I., *et al*., On the Optical Soliton Solutions of Kundu-Mukherjee-Naskar Equation via Two Different Analytical Methods, *Optik*, *257* (2022), ID168761
[17] Wang, K. J., *et al.*, Lump Wave, Breather Wave and the other Abundant Wave Solutions to the (2+1)-Dimensional Sawada-Kotera-Kadomtsev Petviashvili Equation for Fluid Mechanic, *Pramana*, *99* (2025), 1, ID40
[18] Wang, K. J., *et al*., Novel Singular and Non-Singular Complexiton, Interaction Wave and the Complex Multi-Soliton Solutions to the Generalized Non-Linear Evolution Equation, *Modern Physics Letters B*, *39* (2025), ID2550135
[19] Zayed, E. M. E., *et al.*, Cubic-Quartic Polarized Optical Solitons and Conservation Laws for Perturbed Fokas-Lenells Model, *Journal of Non-linear Optical Physics and Materials*, *30* (2021), 3, ID21500053
[20] Abdou, M. A., A Generalized Auxiliary Equation Method and Its Applications, *Non-Linear Dynamics*, *52* (2008), 1, pp. 95-102
[21] Alam, M. N., *et al*., Exact Traveling Wave Solutions of the KP-BBM Equation by Using the New Approach of Generalized (G′/G)-Expansion Method, *SpringerPlus*, *2* (2013), 1, pp. 1-7
[22] Teymuri Sindi, C., *et al*., Wave Solutions for Variants of the KdV-Burger and the K(N,N)-Burger Equations by the Generalized G′/G-Expansion Method, *Mathematical Methods in the Applied Sciences*, *40* (2017), 12, pp. 4350-4363
[23] Gkogkou, A., *et al.*, Inverse Scattering Transform for the Complex Coupled Short-Pulse Equation, *Studies in Applied Mathematics*, *148* (2022), 2, pp. 918-963
[24] Wang, K. J, *et al*., Bifurcation Analysis, Chaotic Behaviors, Variational Principle, Hamiltonian and Diverse Optical Solitons of the Fractional Complex Ginzburg-Landau Model, *International Journal of Theoretical Physics*, On-line first: https://doi.org/10.1007/s10773-025-05977-9, 2025
[25] Raza, N., *et al*., Optical Dark and Dark-Singular Soliton Solutions of (1+2)-Dimensional Chiral Non-Linear Schrodinger's Equation, *Waves in Random and Complex Media*, *29* (2019), 3, pp. 496-508

[26] Aderyani, S. R., *et al.*, The Exact Solutions of the Conformable Time-Fractional Modified Non-Linear Schrödinger Equation by the Trial Equation Method and Modified Trial Equation Method, *Advances in Mathematical Physics*, *2022* (2022), ID4318192
[27] Alquran, M., *et al*., New Topological and Non-Topological Unidirectional-Wave Solutions for the Modified-Mixed KdV Equation and Bidirectional-Waves Solutions for the Benjamin Ono Equation Using Recent Techniques, *Journal of Ocean Engineering and Science*, *7* (2022), 2, pp. 163-169
[28] Syam, M. I., The Solution of Cahn-Allen Equation Based on Bernoulli Sub-Equation Method, *Results in Physics*, *14* (2019), ID102413
[29] He, J.-H., The Simplest Approach to Non-Linear Oscillators, *Results in Physics*, *15* (2019), ID102546

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