AN EFFICIENT MATHEMATICAL METHOD FOR NON-LINEAR BOUSSINESQ-LIKE EQUATION

by

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In this paper, we mainly investigate the non-linear Boussinesq-like equation by using an efficient and simple mathematical method, which is called functional variable method. A large number of new soliton solutions and periodic solutions are successfully obtained. These new solutions are very useful for elucidating corresponding physical and natural phenomena.

Key words: *functional variable method, soliton solution, Boussinesq-like equation*

Introduction

The non-linear Boussinesq-like equation [1]:

$$\frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u}{\partial x^2}-\frac{\partial}{\partial x}\left(6u^2\frac{\partial u}{\partial x}+\frac{\partial^3 u}{\partial x^3}\right)=0 \tag{1}$$

Previous equation plays an important role in many different fields of physics, such as plasma physics, fluid dynamics and ocean engineering. In [1], the non-linear Boussinesq-like equation has been studied by using the sine-Gordon expansion method and the hyperbolic function method. In recent decades, there are many effective methods for solving evolutionary equations, such as symmetry group scheme [2], variational method [3],unified solver method [4], local wave method [5], ϕ^6-model expansion method [6], first integral method [7], Bernoulli sub-equation function method [8], sub-equation method [9], $(1/G')$ method [10], and so on [11-15].

In this work, we provide a concise and efficient mathematical method to study the non-linear Boussinesq-like equation, which is called functional variable method. A large number of different types of soliton solutions and periodic solutions have been successfully derived. Some 3-D and 2-D figures are sketched to illustrate the physical properties of the obtained new solutions.

Functional variable method

In this section, the functional variable method [16] is described in detail.

Consider the non-linear differential equation:

$$N\left(\frac{\partial u}{\partial x^2},\frac{\partial u}{\partial t^2},u,u^3\right)=0 \tag{2}$$

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Employ the transformation:

$$u(x,t) = U(\varphi) \tag{3}$$

$$\varphi = ax + bt \tag{4}$$

Put eqs. (3) and (4) into eq.(2), get:

$$N\left[\left(a^2, b^2\right)\frac{\mathrm{d}U}{\mathrm{d}\varphi^2}, U, U^3\right] = 0 \tag{5}$$

Assume that unknown function K is:

$$\frac{\mathrm{d}U}{\mathrm{d}\varphi} = K(U) \tag{6}$$

Therefore, have:

$$\frac{\mathrm{d}^2U}{\mathrm{d}\varphi^2} = \frac{1}{2}\frac{\mathrm{d}\left[K(U)\right]^2}{\mathrm{d}U} \tag{7}$$

$$\frac{\mathrm{d}^3U}{\mathrm{d}\varphi^3} = \frac{1}{2}\frac{\mathrm{d}^2\left[K(U)\right]^2}{\mathrm{d}U^2}\sqrt{K^2} \tag{8}$$

Hence, we obtain:

$$N\left\{\frac{\mathrm{d}^2K(U)}{\mathrm{d}U^2}, U, K(U), \left[K(U)\right]^3\right\} = 0 \tag{9}$$

The K can be successfully obtained by solving eq. (9).

Application

Consider the non-linear Boussinesq-like equation:

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial}{\partial x}\left(6u^2\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3}\right) = 0 \tag{10}$$

Employ the transformation:

$$\begin{gathered} u(x,t) = U(\varphi) \\ \varphi = ax + bt \end{gathered} \tag{11}$$

Substitute eq. (11) into eq. (10), and get:

$$a^4\frac{\partial^4 U}{\partial\varphi^4} + \left(a^2 - b^2\right)\frac{\partial^2 U}{\partial\varphi^2} + 12a^2U\left(\frac{\partial U}{\partial\varphi}\right)^2 + 6a^2U^2\frac{\partial^2 U}{\partial\varphi^2} = 0 \tag{12}$$

Integral equation twice, so that the integral constant is zero, we have:

$$\frac{\partial^2 U}{\partial\varphi^2} + \frac{a^2 - b^2}{a^4}U + \frac{2}{a^2}U^3 = 0 \tag{13}$$

Equation (13) can be converted into:

$$\frac{\partial^2 U}{\partial\varphi^2} + \gamma U + \chi U^3 = 0 \tag{14}$$

where

$$\gamma = \frac{a^2 - b^2}{a^4}$$
$$\chi = \frac{2}{a^2} \tag{15}$$

Next, we apply the transformation:

$$\frac{\mathrm{d}U}{\mathrm{d}\varphi} = K(U) \tag{16}$$

Hence, eq. (14) can be re-written into:

$$\frac{1}{2}\frac{\mathrm{d}\left[K(U)\right]^2}{\mathrm{d}U} + \gamma U + \chi U^3 = 0 \tag{17}$$

By calculating eq. (17),we have:

$$K(U) = \sqrt{-\gamma}U\sqrt{1 + \frac{\chi}{2\gamma}U^2} \tag{18}$$

Solve eq. (18), and get:

$$U(\varphi) = \pm\sqrt{\frac{2\gamma}{\chi}}\csc\left(\sqrt{-\gamma}\varphi\right) \tag{19}$$

Case 1. When $a^2 - b^2/a^4 < 0$, we obtain the soliton solutions:

$$u_1\left(x,t\right) = i\sqrt{\frac{b^2 - a^2}{a^2}}\operatorname{csch}\left(\sqrt{-\frac{a^2 - b^2}{a^4}}(ax + bt)\right) \tag{20}$$

$$u_2\left(x,t\right) = -i\sqrt{\frac{b^2 - a^2}{a^2}}\operatorname{csch}\left(\sqrt{-\frac{a^2 - b^2}{a^4}}(ax + bt)\right) \tag{21}$$

Case 2. When $a^2 - b^2/a^4 > 0$, we obtain the periodic solutions:

$$u_3\left(x,t\right) = i\sqrt{\frac{b^2 - a^2}{a^2}}\csc\left(\sqrt{\frac{a^2 - b^2}{a^4}}(ax + bt)\right) \tag{22}$$

$$u_4\left(x,t\right) = -i\sqrt{\frac{b^2 - a^2}{a^2}}\csc\left(\sqrt{\frac{a^2 - b^2}{a^4}}(ax + bt)\right) \tag{23}$$

In fig. 1(a), the 3-D graph of $|u_{1,2}(x, t)|$ is plotted with parameters $a = 2$, $b = 3$. In fig. 1(b), 2-D graph of $|u_{1,2}(x, t)|$ is plotted with parameters $a = 6$, $b = 7$ at different times $t = 2$ and $t = 5$. In fig. 2(a), the 3-D graph of $|u_{1,2}(x, t)|$is plotted with parameters $a = 5$, $b = 4$. In fig. 2(b), 2-D graph of $|u_{1,2}(x, t)|$ is plotted with parameters $a = 9$, $b = 6$ at different times $t = 3$ and $t = 6$.

Conclusion

In this paper, the non-linear Boussinesq-like equation is studied by adopting the functional variable method, and some soliton solutions and periodic solutions are successfully derived. These obtained solutions are completely new and have not yet appeared in the literature that currently exists. In the process of solving this paper, we can find that the method used in this paper is very simple and effective.

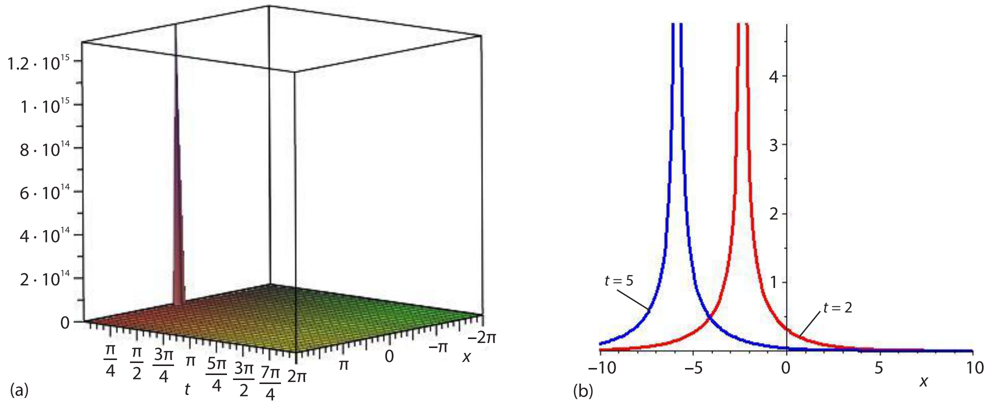

Figure.1. The graphs of $|u_{1,2}(x, t)|$

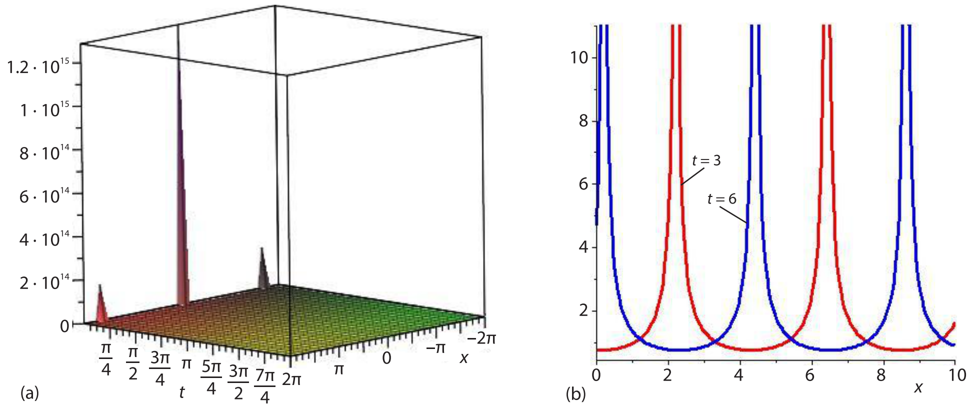

Figure.2. The graphs of $|u_{3,4}(x, t)|$

Nomenclature

t – time co-ordinate, [s]

x – space co-ordinate, [m]

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