APPLICATION OF ADOMIAN DECOMPOSITION METHOD TO FRACTIONAL ORDER PARTIAL DIFFERENTIAL EQUATIONS

by

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In the paper, we first use G transform decomposition method to solve the non-linear fractional Fisher equation, and verify the effectiveness through concrete examples. Then we use Natural transform decomposition method to solve non-linear fractional KdV equation and prove the solution surface of fractional order converges to integer order solution surface when exponent $\sigma \to 1$.

Key words: *Caputo fractional derivative, G transform, natural transform, Adomian decomposition method*

Introduction

The Adomian decomposition method (ADM) is proposed by American mathematical physicist George Adomian in the 1980's, it can decompose the original equation into linear part and non-linear part, and express the solution in infinite series. The research indicates that the ADM offers advantages of reduced error and enhanced precision compared to traditional methods in solving differential equations. Daftardar-Gejji *et al.* [1] solved a class of fractional order differential equations with ADM and obtained its convergence. On the basis of traditional ADM, Luo [2] proposed the two-step ADM, which has the characteristics of less error and higher precision. Duan *et al.* [3] and Saelao *et al*. [4] solved separately non-linear fractional ODE and linear and non-linear Klein-Gordon equations by using an improved ADM. Optimized and improved by many experts and scholars, Shah *et al.* [5] obtained analytical solutions of fractional order dispersive partial differrential equations by Laplace-Adomian decomposition method (LADM) and Bushnaq *et al.* [6] obtained the solution of the non-linear Fisher differential equation by using natural transform decomposition method (NTDM).

Fisher equation [7] is of great significance in the fields of heat conduction, biology and ecology. In this paper, the non-linear fractional order Fisher equation is studied:

$$\frac{\partial^{\sigma} w(x,t)}{\partial t^{\sigma}} = \frac{\partial^{2\eta} w(x,t)}{\partial x^{2\eta}} + \alpha w(x,t)\left(1 - w^{\beta}(x,t)\right),\ 0 < \sigma,\ \eta \le 1$$

where x, t is the time variable and space variable, σ – the fractional Caputo derivative, $w(x, t)$ – the population density, and $w(x, t) - w^{\beta+1}(x, t)$ represents the population growth rate. When $\beta > 1$, $w^{\beta}(x, t)$ is a non-linear term, when population density is too large, its growth rate will decrease. Therefore, the Fisher equation can better reflect the trend of population change.

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The KdV equation describes the non-linear wave behavior propagating in the medium. We consider non-homogeneous non-linear fractional order KdV equation:

$$\frac{\partial^\sigma w(x,t)}{\partial t^\sigma} = \frac{\partial w(x,t)}{\partial x} + \frac{\partial^2 w(x,t)}{\partial x^2} + \frac{\partial^3 w(x,t)}{\partial x^3} + Nw(x,t) + g(x,t), \; 0 < \sigma \le 1$$

where N is a non-linear operator and $g(x, t)$ is the source term.

Preliminary knowledge

Definition 1. [8, 9] For $n - 1 < \alpha < n$, the α-order Caputo type fractional derivative of the function $f(t)$ is defined:

$${}_a^C D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}} \mathrm{d}\tau$$

Definition 2. [10] For $t \ge 0$, the G transform of $f(x)$ is defined:

$$G\{f(x)\} = u^\alpha \int_0^\infty \mathrm{e}^{-t/u} f(t) \mathrm{d}t$$

Lemma 1. [10] The G transform of the n-order derivative of $f(x)$:

$$G\{f^{(n)}(x)\} = \frac{1}{u^n} G(f) - \sum_{k=0}^{n-1} \frac{f^{(n-1-k)}(0)}{u^k} u^\alpha$$

Lemma 2. [10] The convolution of the G transform is defined:

$$u^\alpha G\{fg\} = G\{f\} G\{g\}$$

Lemma 3. [10] The G transform of the p-order fractional derivative of $f(t)$:

$$G\{{}_0 D_t^p f(t)\} = u^{-p} G\{f\} - u^\alpha \sum_{k=0}^{n-1} \frac{f^k(t)}{u^{p-k-1}} \bigg|_{t=0}$$

Definition 3. [11] $u > 0$, $s > 0$, the Natural transform of $f(x)$ is defined:

$$\mathcal{N}\{f(x)\} = \int_0^\infty f(ux) \mathrm{e}^{-sx} \mathrm{d}x = \frac{1}{u} \int_0^\infty f(x) \mathrm{e}^{-\frac{sx}{u}} \mathrm{d}x$$

Lemma 4. [11] The convolution of Natural transform is defined:

$$\mathcal{N}\{(f * g)(x)\} = uF(s,u) G(s,u)$$

Lemma 5. [6] For $q \in N$, $\alpha > 0$, $q - 1 < \alpha \le q$ and $\mathcal{N}\{w(x, t)\} = R(x, s, r)$, the natural transform of the α-order fractional derivative of $w(x$, t) is defined:

$$\mathcal{N}\{D^\alpha w(x,t)\} = \frac{s^\alpha}{r^\alpha} R(x,s,r) - \sum_{k=0}^{m-1} \frac{s^{\alpha-(k+1)}}{r^{\alpha-k}} \left[D^k w(x,t) \right] \bigg|_{t=0}$$

Using GTDM to solve non-linear fractional order Fisher equation

Theorem 1. For non-linear fractional order Fisher equation:

$$\begin{gathered}\frac{\partial^{\sigma} w(x,t)}{\partial t^{\sigma}}=\frac{\partial^{2\eta} w(x,t)}{\partial x^{2\eta}}+\alpha w(x,t)\left[1-w^{\beta}(x,t)\right]\\ w(x,0)=y(x)\end{gathered} \tag{1}$$

where $0 < \sigma, \eta \le 1$, α and β are non-zero constants, its solution can be expressed:

$$w(x,t)=w_0(x,t)+w_1(x,t)+\cdots+w_n(x,t)+\cdots$$

where

$$w_0(x,t)= y(x), w_n(x,t)=G^{-1}\left\{u^{\sigma} G\left\{\frac{\partial^{2\eta}}{\partial x^{2\eta}} w_{n-1}(x,t)+\alpha w_{n-1}(x,t)-\alpha Q_{n-1}\right\}\right\}$$

Proof. Take G transform of eq. (1), we get:

$$u^{-\sigma} G\{w(x,t)\}-u^{\alpha} \frac{y(x)}{u^{\sigma-1}}=G\left\{\frac{\partial^{2\eta} w(x,t)}{\partial x^{2\eta}}+\alpha w(x,t)\left(1-w^{\beta}(x,t)\right)\right\} \tag{2}$$

continue to inverse G transform of eq. (2), we obtain:

$$w(x,t)=y(x)+G^{-1}\left\{u^{\sigma} G\left\{\frac{\partial^{2\eta} w(x,t)}{\partial x^{2\eta}}+\alpha w(x,t)\left(1-w^{\beta}(x,t)\right)\right\}\right\} \tag{3}$$

Let:

$$w(x,t) w^{\beta}(x,t)=\sum_{n=0}^{\infty} Q_n$$

where

$$Q_n=\frac{1}{\Gamma(n+1)} \frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}\left[\sum_{i=0}^{n} \lambda^i w_i \sum_{i=0}^{n} \lambda^i w_i{}^{\beta}\right]_{\lambda=0}$$

and we obtain

$$Q_0=w_0 w_0{}^{\beta},\ Q_1=w_1 w_0{}^{\beta}+w_0 w_1{}^{\beta}$$

Let:

$$w(x,t)=\sum_{i=0}^{\infty} w_i(x,t)$$

then we have:

$$\sum_{i=0}^{\infty} w_i(x,t)= y(x)+G^{-1}\left\{u^{\sigma} G\left\{\frac{\partial^{2\eta}}{\partial x^{2\eta}} \sum_{i=0}^{\infty} w_i(x,t)+\alpha \sum_{i=0}^{\infty} w_i(x,t)-\alpha \sum_{n=0}^{\infty} Q_n\right\}\right\} \tag{4}$$

Therefore, we get the solution w_i of eq. (1) as shown in *Theorem* 1.

Since the analytical solution obtained by using GTDM to solve Fisher equation is the same as that obtained in [6], and [6] cites the conclusion of [12] to prove the convergence of this analytical solution, so the analytical solution obtained here converges.

Example 1. Solving Fisher's equation for $\alpha = 1$, $\beta = 1$, $\eta = 0.5$:

$$\frac{\partial^\sigma w(x,t)}{\partial t^\sigma} = \frac{\partial w(x,t)}{\partial x} + w(x,t)\big(1 - w(x,t)\big) \qquad w(x,0) = k \tag{5}$$

where $0 < \sigma \le 1$.

Take G transform and inverse G transform of eq. (5), we have:

$$w(x,t) = k + G^{-1}\left\{u^\sigma G\left\{\frac{\partial w(x,t)}{\partial x} + w(x,t)\big(1 - w(x,t)\big)\right\}\right\} \tag{6}$$

Let:

$$w(x,t)w(x,t) = \sum_{i=0}^{\infty} Q_i, \quad w(x,t) = \sum_{i=0}^{\infty} w_i(x,t)$$

and plug into eq. (6), we get

$$\sum_{i=0}^{\infty} w_i(x,t) = k + G^{-1}\left\{u^\sigma G\left\{\frac{\partial}{\partial x}\sum_{i=0}^{\infty} w_i(x,t) + \sum_{i=0}^{\infty} w_i(x,t) - \sum_{i=0}^{\infty} Q_i\right\}\right\}$$

where

$$w_0(x,t) = k, \quad Q_0 = k^2, \quad w_1(x,t) = \left(k - k^2\right)\frac{t^\sigma}{\Gamma(\sigma+1)}$$

$$Q_1 = 2k\left(k - k^2\right)\frac{t^\sigma}{\Gamma(\sigma+1)}, \quad w_2(x,t) = \left(k - 3k^2 + 2k^3\right)\frac{t^{2\sigma}}{\Gamma(2\sigma+1)}$$

Therefore, we get the solution shown in eq. (7) and figs. 1 and 2:

$$w(x,t) = k + \left(k - k^2\right)\frac{t^\sigma}{\Gamma(\sigma+1)} + \left(k - 3k^2 + 2k^3\right)\frac{t^{2\sigma}}{\Gamma(2\sigma+1)} + \cdots \tag{7}$$

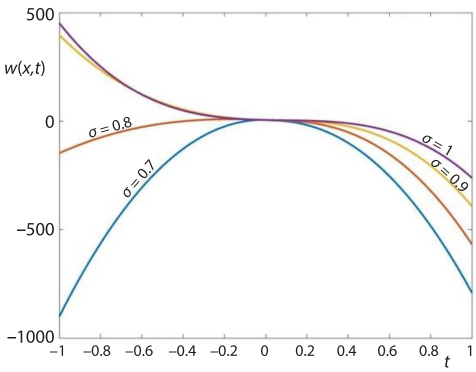

Figure 1. When $k = 5$, the solution curve of eq. (7) corresponding to the different σ

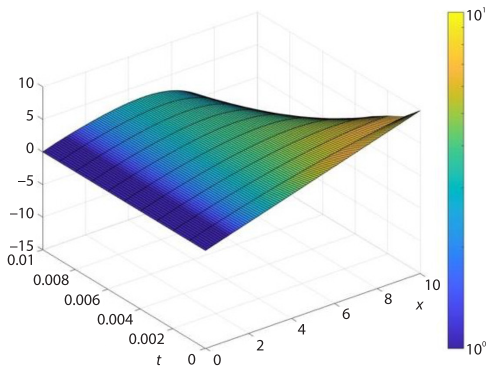

Figure 2. When $\sigma = 0.5$, the solution surface of eq. (7)

Using NTDM to solve fractional partial differential equation

Theorem 2. For non-homogeneous non-linear fractional order KdV equation:

$$\frac{\partial^\sigma w(x,t)}{\partial t^\sigma}=\frac{\partial w(x,t)}{\partial x}+\frac{\partial^2 w(x,t)}{\partial x^2}+\frac{\partial^3 w(x,t)}{\partial x^3}+Nw(x,t)+g(x,t),\quad w(x,0)=y(x) \tag{8}$$

where $0<\sigma\le 1$, N is the non-linear operator and $g(x, t)$ – the source term, the solution is:

$$w(x,t)=w_0(x,t)+w_1(x,t)+\cdots+w_n(x,t)+\cdots,$$

where $w_0(x, t) = y(x)$

$$w_n(x,t)=\mathcal{N}^{-1}\left\{\frac{r^\sigma}{s^\sigma}\mathcal{N}\left\{\frac{\partial}{\partial x}w_{n-1}(x,t)+\frac{\partial^2}{\partial x^2}w_{n-1}(x,t)+\frac{\partial^3}{\partial x^3}w_{n-1}(x,t)+A_{n-1}\right\}\right\}.$$

Proof. Take natural transform and inverse natural transform of eq. (8), we have:

$$w(x,t)=y(x)+\mathcal{N}^{-1}\left\{\frac{r^\sigma}{s^\sigma}\mathcal{N}\left\{\frac{\partial w(x,t)}{\partial x}+\cdots+\frac{\partial^3 w(x,t)}{\partial x^3}+Nw(x,t)+g(x,t)\right\}\right\} \tag{9}$$

let

$$Nw(x,t)=\sum_{i=0}^{\infty}A_i,\, w(x,t)=\sum_{i=0}^{\infty}w_i(x,t)$$

where A_i is Adomian polynomial:

$$A_i=\frac{1}{\Gamma(n+1)}\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}\left[\sum_{i=0}^{n}\lambda^i w_i\right]_{\lambda=0}$$

and we have

$$A_0=N(w_0),\ A_1=N'(w_0)w_1,\ A_2=N'(w_0)w_2+N''(w_0)\frac{w_1^{\,2}}{2!}$$

then:

$$\sum_{i=0}^{\infty}w_i(x,t)=y(x)+\mathcal{N}^{-1}\left\{\frac{r^\sigma}{s^\sigma}\mathcal{N}\left\{\sum_{j=1}^{3}\frac{\partial^j}{\partial x^j}\sum_{i=0}^{\infty}w_i(x,t)+\sum_{i=0}^{\infty}A_i+g(x,t)\right\}\right\}$$

Therefore, we get the solution w_i of eq. (8) as shown in *Theorem 2*.

Example 2. Solving fractional dispersion KdV equation:

$$\begin{gathered}\frac{\partial^\sigma v(x,t)}{\partial t^\sigma}+\frac{\partial^3 v(x,t)}{\partial x^3}=-\sin\pi x\sin t-\pi^3\cos\pi x\cos t\\ v(x,0)=\sin\pi x\end{gathered} \tag{10}$$

where $0<\sigma\le 1$.

Take natural transform and inverse natural transform of eq. (10), we have:

$$v(x,t)=\sin\pi x-\mathcal{N}^{-1}\left\{\frac{r^\sigma}{s^\sigma}\mathcal{N}\left\{\frac{\partial^3 v(x,t)}{\partial x^3}+\sin\pi x\sin t+\pi^3\cos\pi x\cos t\right\}\right\} \tag{11}$$

let

$$v(x,t)=\sum_{i=0}^{\infty} v_i(x,t)$$

and take the first five expansions of sin and cos, we get:

$$\sum_{i=0}^{\infty} v_i(x,t)=\sin \pi x-\sin \pi x\left(\frac{t^{\sigma+1}}{\Gamma(\sigma+2)}-\cdots+\frac{t^{\sigma+9}}{\Gamma(\sigma+10)}\right)-$$
$$-\pi^3 \cos \pi x\left(\frac{t^{\sigma}}{\Gamma(\sigma+1)}-\cdots+\frac{t^{\sigma+8}}{\Gamma(\sigma+9)}\right)-\mathcal{N}^{-1}\left\{\frac{r^{\sigma}}{s^{\sigma}}\mathcal{N}\left\{\frac{\partial^3}{\partial x^3}\sum_{i=0}^{\infty} v_i(x,t)\right\}\right\}$$

Therefore, v_0 is the first three terms of the previous formula, v_1 can be expressed:

$$v_1(x,t)=\pi^3 \cos \pi x\frac{t^{\sigma}}{\Gamma(\sigma+1)}-\pi^3 \cos \pi x\left(\frac{t^{2\sigma+1}}{\Gamma(2\sigma+2)}-\cdots+\frac{t^{2\sigma+9}}{\Gamma(2\sigma+10)}\right)+$$
$$+\pi^6 \sin \pi x\left(\frac{t^{2\sigma}}{\Gamma(2\sigma+1)}-\cdots+\frac{t^{2\sigma+8}}{\Gamma(2\sigma+9)}\right)$$

Therefore, the solution of eq. (10):

$$v(x,t)=v_0(x,t)+v_1(x,t)+v_2(x,t)+\ldots \tag{12}$$

when $\sigma = 0.5$ and $\sigma = 1$, as shown in figs. 3 and 4, respectively. Combined with figs. 3 and 4, we verify the conclusion given in [5]. When the fractional exponent approaches the integer exponent, the solution surface of fractional order converges to the solution surface of integer order.

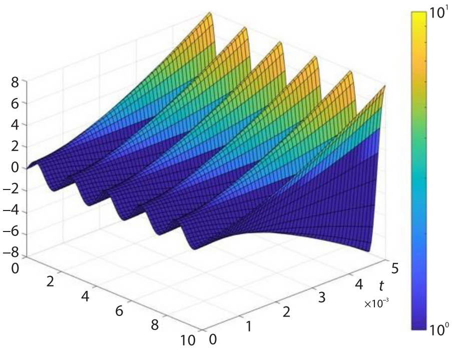

Figure 3. When σ = 0.5, the solution surface of eq. (12)

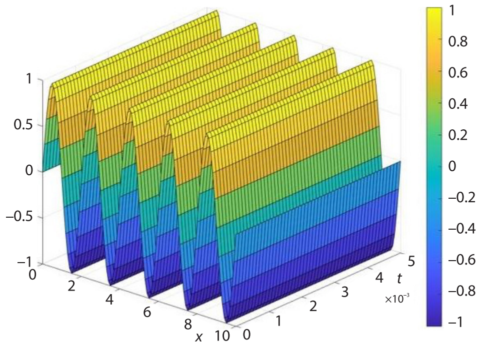

Figure 4. When σ = 1, the solution surface of eq. (12)

Conclusion

In the paper, we used GTDM and NTDM to solve non-linear fractional order Fisher equation and KdV equation. It is also verified that when exponent $\sigma \to 1$, the solution surface of the fractional order KdV equation converges to the solution surface of integer order. We will continue to explore the use of GTDM to solve more general Fisher equation in the future:

$$\frac{\partial^{\sigma} w(x,t)}{\partial t^{\sigma}}=\frac{\partial^{2\eta} w(x,t)}{\partial x^{2\eta}}+w^{\alpha}(x,t)\left(1-w^{\beta}(x,t)\right),\ 0<\sigma,\ \eta\le 1$$

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