LARGE DEVIATION PRINCIPLES OF MUSSEL-ALGAE MODEL WITH STOCHASTIC DIFFUSIONS

by

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In this work, we consider the large deviation principles of the stochastic reaction diffusion mussel-algae model. The weak convergence method is used to prove the large deviation principle by using martingale inequality, shrinkage principle and some special energy estimation.

Key words: *mussel-algae model, large deviation principle, compactness weak convergence method*

Introduction

Since the 1990's, the pioneering work of British biologist Turing has brought signi cant attention biological interaction systems among researchers [1, 2]. Cangelosi [3] investigated the mussel-algae model using weakly non-linear diffusion instability analysis. This model is described by the partial differential equations:

$$\begin{aligned}\frac{\partial M}{\partial s} &= ecAM - d_M \frac{k_M}{k_M + M} + D_M \Delta M \\ \frac{\partial A}{\partial s} &= (A_{UP} - A)\rho - \frac{c}{H} AM - V \frac{\partial A}{\partial X} + D_A \Delta A\end{aligned} \tag{1}$$

where M and A are the population densities of mussels and algae, respectively, and all constants in the model are non-negative.

However, due to uncertainties and unknown factors, real ecological systems are often subjected to various noise perturbations from the environment. Liu *et al*. [4, 5], employed stochastic control and weak convergence methods to demonstrate the Laplace principle, which is equivalent to the large deviation principle. Additionally, Liu [4, 5] established the large deviation principle for stochastic evolution equations with multiplicative noise. Budhiraja [6], as referenced in, utilized the weak convergence approach to derive the large deviation properties of weakly interacting particle systems. Therefore, through the transformation of parameters, the stochastic reaction dispersal mussel-algae model was considered:

$$\begin{gathered}\frac{\partial m}{\partial t} = r_1 \Delta m + m\left(a\delta - \frac{\gamma}{1+m}\right) + \sigma_1 m W, \quad \frac{\partial a}{\partial t} = r_2 \Delta a - a(m+\alpha) + \alpha - \beta \frac{\partial a}{\partial x} + \sigma_2 a W \\ \frac{\partial m}{\partial t} = \frac{\partial a}{\partial t} = 0, \ m(x,0) = m_0, \ a(x,0) = a_0\end{gathered} \tag{2}$$

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where W_t is the white noise of independent space-time, diffusion coefficient r_1 and r_2 – the normal numbers, and σ_1 and σ_2 – the disturbance intensity of white noise.

Main results

In the model (2), take $r_1 = \sigma_2 = \beta = 0$, $r_2 = 1$, $\sigma_1 = \varepsilon^{1/2}$, and the following will prove the results of the large deviations principle in the time interval [0, 1] of the stochastic reaction diffusion planar space system. The mussel-algae model (2) is further rephrased for random reaction diffusion:

$$\begin{gathered}\frac{\partial m}{\partial t} = m\left(a\delta - \frac{\gamma}{1+m}\right) + \sqrt{\varepsilon} mW \\ \frac{\partial a}{\partial t} = \Delta a - a(m+\alpha) - \alpha \\ \frac{\partial m}{\partial t} = \frac{\partial a}{\partial t} = 0,\ m(x,0) = m_0,\ a(x,0) = a_0\end{gathered} \tag{3}$$

Therefore, it is easy to calculate the boundary homeostasis of model (3) as: (0,1) and the normal homeostasis of the coexistence of two populations:

$$m^* = \frac{\alpha(r-1)}{1-\alpha r},\ a^* = \frac{1-\alpha r}{r(1-\alpha)}$$

Let:

$$\psi \in \mathcal{H},\ \varepsilon \in [0,1] \text{ and } \{(\psi_\varepsilon) : 0 \le \varepsilon \le 1\} \subset \mathcal{A}^s$$

consider the following stochastic partial differential equations in order to study the large deviations of the stochastic reaction diffusion mussel-algae model (3):

$$\begin{gathered}\frac{\partial m_{\varepsilon,\psi_\varepsilon}}{\partial t} = m_{\varepsilon,\psi_\varepsilon}\left(a_{\varepsilon,\psi_\varepsilon}\delta - \frac{\gamma}{1+m_{\varepsilon,\psi_\varepsilon}}\right) + \sqrt{\varepsilon} m_{\varepsilon,\psi_\varepsilon} W \\ \frac{\partial a_{\varepsilon,\psi_\varepsilon}}{\partial t} = \Delta a_{\varepsilon,\psi_\varepsilon} - a_{\varepsilon,\psi_\varepsilon}(m_{\varepsilon,\psi_\varepsilon} + \alpha) - \alpha\end{gathered} \tag{4}$$

Theorem 1. Let ϕ_n, $\psi \in \mathbb{A}_N^d$, and (m_{ϕ_n}, a_{ϕ_n}) be the solution of model (4), then:

$$\lim_{\phi_n \to \psi} \left\| (m_{\phi_n}, a_{\phi_n}) - (m_\psi, a_\psi) \right\|_{\mathbb{C}([0,1];\mathcal{R}\times C)} = 0$$

Proof. First, for al s, $t \in [0, 1]$ and $\phi \in \mathbb{A}_N^d$, by model (4), there is $\mathcal{P}_i$ ($i = 1, 2$), such that:

$$\left| m_{\phi_n}(t) - m_{\phi_n}(t') \right| \le \int_{t'}^{t} \left[m_{\phi_n}(s) \left(a_{\phi_n}(s)\delta - \frac{\gamma}{1+m_{\phi_n}(s)} \right) \right] \mathrm{d}s + \int_{t'}^{t} m_{\phi_n}(s) \left| \dot{\eta}(s) \right| \mathrm{d}s \le$$

$$\le \mathcal{P}_1(t-t') + \left(\int_{t'}^{t} \int_{t'}^{t} m_{\phi_n}(s) \mathrm{d}s \right)^{1/2} \left(\int_0^1 \left| \dot{\eta}(s) \right|^2 \mathrm{d}s \right)^{1/2} \le \mathcal{P}_1(t-t') + \mathcal{P}_2(t-t')^{1/2}$$

Obviously, $\{m_{\phi_n}(t)\}_{n \ge 1}$ is equivalently continuous, so according to the Arzela-Ascoli theorem, for some subsequences of $\{m(t)\}_{\mathbb{C}([0,\,1];\mathbb{R}}$) and $\{m_{\phi_n}(t)\}_{n \ge 1}$:

$$\lim_{n\to\infty} \left\| m_{\phi_n}(t) - m(t) \right\|_{\mathbb{C}([0,1];\mathcal{R})} = 0 \tag{5}$$

Secondly, there are $\mathcal{P}_3$ that make it:

$$\left\|a_{\phi_n}(t)-a_{\phi_n}(t')\right\|_{\mathbb{C}} \le \left\|a_{\phi_n}(t)-a_{\phi_n}(t')\right\|+\left\|a_{\phi_n}(t)-a_{\phi_n}(t')\right\|\left\|m_{\phi_n}(t)+\alpha\right\|-\alpha \le$$

$$\le \int_0^t \left\|a_{\phi_n}(s)\right\|\mathrm{d}s+\int_0^t \left(\left\|a_{\phi_n}(s)\right\|\left\|m_{\phi_n}(t)+\alpha\right\|-\alpha\right)\mathrm{d}s \le \mathcal{P}_3\int_0^t \left\|a_{\phi_n}(s)\right\|ds$$

Obviously, the available $\{a_{\phi_n}(t)\}_{n\ge 1}$ is the Cauchy sequence on $\mathbb{C}([0, 1]; \mathbb{C})$, so for some $\{a(t)\}_{\mathbb{C}([0,\,1];\mathbb{R})}$, there is $\lim\|_{n\to\infty} a_{\phi_n}(t) - a(t)_{\mathbb{C}([0,\,1];\,\mathbb{C})} = 0$. Thus, by the formulas $\phi_n \to \psi$ and eq. (5), when $n \to \infty$ has:

$$\int_0^t m_{\phi_n}(s)\dot{\phi}_n(s)\mathrm{d}s-\int_0^t m(s)\dot{\psi}(s)\mathrm{d}s=\int_0^t \left[m_{\phi_n}(s)-m(s)\right]\dot{\phi}(s)\mathrm{d}s+\int_0^t \left[\dot{\phi}_n(s)-\dot{\psi}(s)\right]m(s)\mathrm{d}s \to 0$$

Further, there is:

$$m_{\phi_n}(t)=m_0+\int_0^t \left[m_{\phi_n}(s)\left(a_{\phi_n}(s)\delta-\frac{\gamma}{1+m_{\phi_n}(s)}\right)\right]\mathrm{d}s+\int_0^t m_{\phi_n}(s)\dot{\psi}(s)\mathrm{d}s$$

then we can get:

$$m(t)=m_0+\int_0^t \left[m(s)\left(a(s)\delta-\frac{\gamma}{1+m(s)}\right)\right]\mathrm{d}s+\int_0^t m(s)\dot{\psi}(s)\mathrm{d}s$$

According to the existential uniqueness of the solution, it can be seen that $(m, a) = (m_{\phi_n}, a_{\phi_n})$, and therefore, there is:

$$\lim_{\phi_n\to\psi}\left\|(m_{\phi_n},a_{\phi_n})-(m_\psi,a_\psi)\right\|_{\mathbb{C}([0,1];\mathbb{R}\times C)}=0$$

Theorem 2. For solution $(m_{\varepsilon,\psi_\varepsilon}, a_{\varepsilon,\psi_\varepsilon})$ and family $\{\psi_\varepsilon\} \subset \mathbb{A}_N^d$ of model (4), the distribution under topology $\mathcal{H}$ is:

$$\lim_{n\to\infty}\psi_{\varepsilon_n}=\psi \ \ and \ \ \lim_{\varepsilon\to 0}(m_{\varepsilon,\psi_\varepsilon},a_{\varepsilon,\psi_\varepsilon})=(m_\psi,a_\psi)$$

That is, there are all bounded continuous functions $f: \mathbb{C}([0, 1]; \mathbb{R}\times C)$ such that:

$$\lim_{\varepsilon\to 0} Ef(m_{\varepsilon,\psi_\varepsilon},a_{\varepsilon,\psi_\varepsilon})=Ef(m_\psi,a_\psi)$$

Proof. According to the embedding Skorohod theorem, for the probability space $(\Omega, \mathcal{F}, \mathcal{P})$ it is easy to obtain that the random variables $\{\tilde{m}_{\varepsilon_n}\}$ and $\hat{m}$ have the same distribution as $\{m_{\varepsilon_n}\}$ and m, respectively:

$$\lim_{n\to\infty}\left\|\tilde{m}_{\varepsilon_n}-\tilde{m}\right\|_{\mathbb{C}([0,1];\mathbb{R})}=0 \ \ a.s$$

according to the differential equations of reaction diffusion:

$$\frac{\partial\tilde{a}_{\varepsilon_n}}{\partial t}=\Delta\tilde{a}_{\varepsilon_n}-\tilde{a}_{\varepsilon_n}(\tilde{m}_{\varepsilon_n}+\alpha)-\alpha,\ \frac{\partial\tilde{a}}{\partial t}=\Delta\tilde{a}-\tilde{a}(\tilde{m}+\alpha)-\alpha \tag{6}$$

initial: $\tilde{a}_{\varepsilon_n}(0) = \tilde{a}_0$, $\tilde{a}(0) = \tilde{a}_0$, with *Theorem* 1, it is easy to know:

$$\lim_{n\to\infty}\left\|\tilde{a}_{\varepsilon_n}-\tilde{a}\right\|_{\mathbb{C}([0,1];\mathbb{R})}=0 \ \ a.s$$

Therefore, $(\tilde{m}_{\varepsilon_n}, \tilde{a}_{\varepsilon_n})$ and $(m_{\varepsilon_n}, a_{\varepsilon_n})$ have the same distribution. According to Martingale inequality, there is:

$$\lim_{\varepsilon\to 0^+} \mathbb{E}\sqrt{\varepsilon} \sup_{0\le t\le 1}\left|\int_0^t a_\varepsilon \mathrm{d}W_s\right| = 0$$

Pushable:

$$\lim_{\varepsilon\to 0^+} \mathbb{E}\sqrt{\varepsilon_n} \sup_{0\le t\le 1}\left|\int_0^t a_\varepsilon \mathrm{d}W_s\right| = 0$$

Through *Theorem 1*, get:

$$\tilde{m}(t) = m_0 + \int_0^t \left[\tilde{m}(s)\left(\tilde{a}(s)\delta - \frac{\gamma}{1+\tilde{m}(s)}\right)\right]\mathrm{d}s + \int_0^t \tilde{m}(s)\dot{\eta}(s)\mathrm{d}s \tag{7}$$

We can be seen from eqs. (6) and (7), $(\tilde{m}, \tilde{a})$ also satisfies the model (4), and according to the uniqueness of the solution, $(\tilde{m}, \tilde{a})$ and (m_η, a_η) have the same distribution, so that there is $m \in \mathbb{C}([0, 1]; \mathbb{R})$ and a subsequence $\{m_{\varepsilon_n}\}$ satisfied: $\lim_{n\to\infty}\varepsilon_n = 0$.

Under the distribution of topological structure $\mathbb{C}$, there is: $\lim_{n\to\infty} m\varepsilon_n = m$. Then prove compactness. for any $\kappa > 0$, there is:

$$\lim_{\xi\to 0}\lim_{\varepsilon\to 0} \mathbb{P}(\lambda > \kappa) = 0;\ \lambda = \sup_{\|t-t'\|\le\xi} \|m_\varepsilon(t) - m_\varepsilon(t')\|$$

for any $t \in [0, 1]$, there is:

$$0 \le m(t)_\varepsilon^2 \le \left[\Theta_m(t)m_0\right]\mathrm{e}^{\left(1-\frac{\sigma_1^2}{2}\right)t+\sigma_1 W_1}$$

where Θ_m is a semigroup generated on, and Θ_m is a contraction semigroup, this can be obtained:

$$\mathcal{P}_4 \doteq \mathbb{E}\sup_{0\le t\le 1} m_\varepsilon^2(t) \le \mathcal{P}_5 \tag{8}$$

Suppose that κ_2, $\lambda_2 = (\mathcal{P}_5/\kappa_2)$, using Chebyshev inequality, for some $\lambda_2 > 0$, have therefore:

$$\mathbb{P}\left[\sup_{0\le t\le 1} m_\varepsilon^2(t) \le \lambda_2\right] \ge 1 - \kappa_2$$

for any $\{0 \le t_1 < t_2 < ...< t_n \le 1\}_{n\in N_+}$, there is $\mathbb{P}((m_\varepsilon^2(t_i)) \le \lambda_2,\ i = 1, 2,..., n) \ge 1 - \kappa_2$, since $\kappa_2 > 0$ is arbitrary, it is easy to obtain that the distribution of $\{m_\varepsilon^2(t_i),\ i = 1, 2,..., n\}$ is compact. Therefore, there is:

$$\sup_{|t-t'|\le\xi} |m_\varepsilon(t) - m_\varepsilon(t')| \le \sup_{|t-t'|\le\xi}\left\{\int_0^t\left[m_\varepsilon(s)\left(a_\varepsilon(s)\delta + \frac{\gamma}{1+m_\varepsilon(s)}\right)\right]\mathrm{d}s + \int_0^t m_\varepsilon(s)\dot{\eta}_\varepsilon(s)\mathrm{d}s + \sqrt{\varepsilon}\int_0^t m_\varepsilon \mathrm{d}W_s\right\} \le$$

$$\le \sup_{|t-t'|\le\xi}\left|\int_0^t m_\varepsilon(s)\dot{\eta}_\varepsilon(s)\mathrm{d}s\right| + \sqrt{\varepsilon}\sup_{|t-t'|\le\delta}\left|\int_0^t m_\varepsilon \mathrm{d}W_s\right|$$

and there is

$$0 \le \sup_{|t-t'|\le\xi} a(x,t) \le \max_{x\in\Omega} a_0(x)$$

Using the Holder inequality, there is:

$$\sup_{|t-t'|\le\xi}\left|\int_{t'}^{t} m_\varepsilon(s)\dot{\psi}_\varepsilon(s)\mathrm{d}s\right| \le \sup_{|t-t'|\le\xi}\int_{t'}^{t}\left(m_\varepsilon^2(s)\mathrm{d}s\right)^{1/2}\|\psi_\varepsilon\|_{\mathcal{H}} \le \sqrt{\xi}\sup_{0\le t\le 1} m_\varepsilon(s)$$

According to eqs. (7) and (8), there are:

$$\mathbb{E}\sup_{|t-t'|\le\xi}\left|\int_{t'}^{t} m_{\varepsilon(s)}\dot{\psi}_\varepsilon(s)\mathrm{d}s\right| \le \mathcal{P}_6\xi^{1/2}$$

Using the properties of martingale, there is:

$$\mathbb{E}\sqrt{\varepsilon}\sup_{0\le t\le 1}\left|\int_0^t m_\varepsilon \mathrm{d}W_s\right| \le \sqrt{\varepsilon}\left(\mathbb{E}\sup_{0\le t\le 1}\left|\int_0^t m_\varepsilon \mathrm{d}W_s\right|^2\right)^{1/2} \le \sqrt{2\varepsilon}\left(\int_0^t \mathbb{E}|m_\varepsilon|^2\,\mathrm{d}s\right)^{1/2} \le \mathcal{P}_7\sqrt{\varepsilon}$$

Combined with the previous analysis, it can be obtained:

$$\mathbb{E}\sup_{|t-t'|\le\xi}|m(t)-m(t')| \le \max\{\mathcal{P}_6,\mathcal{P}_7\}\left(3\xi+\xi^{1/2}+\sqrt{\varepsilon}\right)$$

Then take:

$$\lambda_2=\sup_{|t-t'|\le\xi}|m(t)-m(t')|,\ \lambda_3=\sup_{|t-t'|\le\xi}|m_\varepsilon(t)-m_\varepsilon(t')|$$

According to Chebyshev inequality, it can be obtained:

$$\mathbb{P}(\lambda_3>\kappa)\le\frac{\mathbb{E}(\lambda_2)}{\lambda}\le 3\xi+\xi^{1/2}+\sqrt{\varepsilon}$$

Assumption 3. There exists a measurable mapping F^0:$C(0, T)$:$H) \to \theta$ such that the following conditions hold:

– For each $M<\infty$, set

$$\mathrm{Y}_m \doteq \left\{F^0\left(\int_0^t m(s)\mathrm{d}s\right): m\in\mathcal{K}_M\right\}$$

is a compact set of θ.

– Considering $M<\infty$ and family $\{m^\theta\}\subset\mathcal{H}_M$, let m^θ converge to m in distribution (as a $\mathcal{K}_M$-valued random element), then:

$$F^\theta\left(W(\cdot)+\frac{1}{\sqrt{\theta}}\int_0^{\cdot} m(s)\mathrm{d}s\right) \text{ converges to } F^0\left(\int_0^{\cdot} m(s)\mathrm{d}s\right)$$

in distribution. For each $m\in\theta$, first define:

$$I(m)\doteq\inf_{\left\{m\in L^2([0,T]:H_0):m=F_0\left(0\int^{\cdot} m(s)\mathrm{d}s\right)\right\}}\left\{\frac{1}{2}\int_0^T\|m(s)\|_0^2\mathrm{d}s\right\} \tag{9}$$

where the integer value on the empty set is taken as ∞.

Theorem 4. Suppose $X^\phi = F^\phi(W(\cdot))$, and $\{F^\phi\}$ satisfies *Assumption 3*, then there is a family $\{X^\phi\}_{\phi>0}$ satisfying the Laplace principle in θ, and the rate function $I(x)$ is defined in eq. (9).

Proof. In order to prove the theorem, we must first prove:

$$\lim_{\phi\to 0}\phi\log E\left\{\exp\left[-\frac{1}{\phi}\hbar(X^\phi)\right]\right\} = -\inf_{x\in\theta}\{\hbar(x)+I(X)\}$$

holds for all real-valued, bounded and continuous functions $\hbar$ on θ.

– Firstly, we prove the lower bound. according to *Theorem 2*, we can obtain:

$$-\phi\log E\left\{\exp\left[-\frac{1}{\phi}\hbar(X)^\phi\right]\right\} = \inf_{m\in\mathcal{A}} E\left(\frac{\phi}{2}\int_0^T \|m(s)\|_0^2 \mathrm{d}s + \hbar F^\phi\left(W(\cdot)+\int_0^{\cdot} m(s)\int_0^{\cdot} m(s)\mathrm{d}s\right)\right) =$$

$$= \inf_{m\in\mathcal{A}} E\left(\frac{1}{2}\int_0^T \|m(s)\|_0^2 \mathrm{d}s + \hbar F^\phi\left(W(\cdot)+\frac{1}{\sqrt{\phi}}\int_0^{\cdot} m(s)\mathrm{d}s\right)\right)$$

$\alpha > 0$ is fixed at this time, and for each $\theta > 0$, there exists a $m^\theta \in \mathcal{A}$ such that:

$$\inf_{m\in\mathcal{A}} E\left(\frac{1}{2}\int_0^T \|m(s)\|_0^2 \mathrm{d}s + \hbar F^\phi\left(W(\cdot)+\frac{1}{\sqrt{\phi}}\int_0^{\cdot} m(s)\mathrm{d}s\right)\right) \ge$$

$$\ge E\left(\frac{1}{2}\int_0^T \left\|m^\phi(s)\right\|_0^2 \mathrm{d}s + \hbar F^\phi\left(W(\cdot)+\frac{1}{\sqrt{\phi}}\int_0^{\cdot} m^\phi(s)\mathrm{d}s\right)\right) - \alpha$$

Next, we prove:

$$\lim_{\phi\to 0}\inf E\left(\frac{1}{2}\int_0^T \left\|m^\phi(s)\right\|_0^2 \mathrm{d}s + \hbar F^\phi\left(W(\cdot)+\frac{1}{\sqrt{\phi}}\int_0^{\cdot} m(s)\mathrm{d}s\right)\right) \ge \inf_{x\in\theta}\{I(x)+\hbar(x)\}$$

without losing generality, we first assume that for all $\phi > 0$ and a finite number, $\mathcal{S}$, there are

$$\int_0^T \left\|m^\phi(s)\right\|_0^2 \mathrm{d}s \le \mathcal{S}\ \ a.s$$

If $\mathcal{R} \doteq \|\hbar_\infty$, then there is:

$$\sup_{\phi>0} E\left(\frac{1}{2}\int_0^T \left\|m^\phi(s)\right\|_0^2 \mathrm{d}s\right) \le 2\mathcal{R} + \alpha < \infty$$

Define the stop time:

$$\Gamma_{\mathcal{S}}^\phi \doteq \inf\left\{t\in[0,T]: \int_0^t \left\|m^\phi(s)\right\|_0^2 \mathrm{d}s \ge \mathcal{S}\right\} \wedge T$$

further obtained

$$v\left\{m^\phi \ne m^{\phi,\mathcal{S}}\right\} \le v\left\{\int_0^T \left\|m^\phi(s)\right\|_0^2 \mathrm{d}s \ge \mathcal{S}\right\} \le \frac{2\mathcal{R}+\alpha}{\mathcal{S}}$$

The aforementioned results indicate:

$$E\left(\frac{1}{2}\int_0^T\left\|m^{\phi,\mathcal{S}}(s)\right\|_0^2\mathrm{d}s+\hbar F^\phi\left(W(\cdot)+\frac{1}{\sqrt{\phi}}\int_0^T m^{\phi,\mathcal{S}}(s)\mathrm{d}s\right)\right)-\frac{2\mathcal{R}(2\mathcal{R}+\alpha)}{\mathcal{S}}-\alpha$$

Therefore, for the proof of $m^\phi(s)$, only $m^{\phi,\mathcal{S}}(s)$ substitution is needed. Available from *Assumption 3*:

$$\liminf_{\phi\to0}E\left(\frac{1}{2}\int_0^T\left\|m^\phi(s)\right\|_0^2\mathrm{d}s+\hbar F^\phi\left(W(\cdot)+\frac{1}{\sqrt{\phi}}\int_0^T m^\phi(s)\mathrm{d}s\right)\right)\geq E\left(\frac{1}{2}\int_0^T\left\|m(s)\right\|_0^2\mathrm{d}s+\hbar\left(F^\phi\left(\int_0^{\cdot} m^\phi(s)\mathrm{d}s\right)\right)\right)\geq$$

$$\geq\inf_{\left\{m\in\theta\times L^2([0,T]:H_0):m=F_0\left(\int_0^{\cdot}m(s)\mathrm{d}s\right)\right\}}\left\{\frac{1}{2}\int_0^T\left\|m(s)\right\|_0^2\mathrm{d}s+\hbar(x)\right\}\geq\inf_{x\in\theta}\left\{I(x)+\hbar(x)\right\}$$

Therefore, the proof of the lower bound is completed.

– Secondly, then prove the upper bound. because the function $\hbar$ is bounded and satisfies:

$$\inf_{x\in\theta}\{I(x)+\hbar(x)\}<\infty$$

Therefore, it is assumed that any $\alpha>0$, $x_0\in\theta$ makes:

$$I(x_0)+\hbar(x_0)\leq\inf_{x\in\theta}\left\{I(x)+\hbar(x)\right\}+\frac{\alpha}{2}$$

Select:

$$m\in L^2([0,T]:H_0)$$

to make

$$\frac{1}{2}\int_0^T\left\|m(s)\right\|_0^2\mathrm{d}s\leq I(x)+\frac{\alpha}{2},\ \ x_0=F^0\left(\int_0^{\cdot}m(s)\mathrm{d}s\right)$$

By the *Theorem 2*, for a bounded continuous function $\hbar(x)$, there is:

$$\limsup_{\phi\to0}\left(-\phi\log E\left(\exp\left\{-\frac{\hbar(X^\phi)}{\phi}\right\}\right)\right)=\limsup_{\phi\to0}E\left(\frac{1}{2}\int_0^T\left\|m(s)\right\|_0^2\mathrm{d}s+\hbar F^\phi\left(W(\cdot)+\frac{1}{\sqrt{\phi}}\int_0^{\cdot}m(s)\mathrm{d}s\right)\right)\leq$$

$$\leq\limsup_{\phi\to0}E\left(\frac{1}{2}\int_0^T\left\|m(s)\right\|_0^2\mathrm{d}s+\hbar F^\phi\left(W(\cdot)+\frac{1}{\sqrt{\phi}}\int_0^{\cdot}m(s)\mathrm{d}s\right)\right)$$

$$=\frac{1}{2}\int_0^T\left\|m(s)\right\|_0^2\mathrm{d}s+\limsup_{\phi\to0}E\left(\hbar F^\phi\left(W(\cdot)+\frac{1}{\sqrt{\phi}}\int_0^{\cdot}m(s)\mathrm{d}s\right)\right)\leq$$

$$\leq\limsup_{\phi\to0}E\left(\hbar F^\phi\left(W(\cdot)+\frac{1}{\sqrt{\phi}}\int_0^{\cdot}m(s)\mathrm{d}s\right)\right)+I(x_0)+\frac{\alpha}{2}$$

Then we can see from *Assumption 3* that when $\phi \longrightarrow 0$ is:

$$E\left(\hbar F^{\phi}\left(W(\cdot)+\frac{1}{\sqrt{\phi}}\int_{0}^{T}m(s)\mathrm{d}s\right)\right)$$

It converges to:

$$\hbar\left(F^{0}\left(\int_{0}^{\cdot}m(s)\mathrm{d}s\right)\right)=h(x_0)$$

it can be expressed as:

$$\inf_{x\in\theta}\left\{I(x)+\hbar(x)\right\}+\alpha \quad (\forall\alpha)$$

Theorem 5. Suppose that $(m_\varepsilon, a_\varepsilon)$ is the solution of the stochastic mussel-algae model (3), then $\{(m_\varepsilon, a_\varepsilon)\}_{\varepsilon>0}$ satisfies the large deviation principle and the rate function at $\mathbb{C}([0, 1];\mathbb{R}\times\mathbb{C})$:

$$I[(m,a)] \doteq \inf_{\{m\in L^2([0,T]:H_0):(m_\psi,a_\psi)=(m,a)\}}\left\{\frac{1}{2}\|\psi\|_{\mathcal{H}}^2\right\}$$

where (m_ψ, a_ψ) is the solution of model (3) and satisfies $(m, a) \in \mathbb{C}([0, 1];\mathbb{R}\times\mathbb{C})$.

Proof. According to *Theorems 1*, *Theorems 2*, and *Theorems 4*, we can easily complete the proof of *Theorem 5*, so *Theorem 5* is completed.

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