NEW GENERALIZATION AND REFINEMENT OF THE LOCAL FRACTIONAL INTEGRAL CAUCHY-SCHWARTZ INEQUALITY ON FRACTAL SPACE

by

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In this paper we investigate a local fractional integral Cauchy-Schwartz inequality on fractal spaces. We first obtain a new generalization of local fractional integral Cauchy-Schwartz inequality and then study some refinements of the obtained result.

Key words: *local fractional integral, Cauchy-Schwartz inequality, refinement, fractal space*

Introduction

It is assumed that $\{\Phi_n\}_{n=1}^m$ and $\{\Psi_n\}_{n=1}^m$ are two non-negative real sequences, then the known Cauchy-Schwartz inequality is the result [1]:

$$\sum_{l=1}^{r}\Phi_l\Psi_l \le \left(\sum_{l=1}^{r}\Phi_l^2\right)^{1/2}\left(\sum_{l=1}^{r}\Psi_l^2\right)^{1/2} \tag{1}$$

Let $E = [a, b]$, $\hbar_1(x)$ and $\hbar_2(x)$ are continuous functions on E. Then the integral form of inequality (1) is called Cauchy-Bunyakovsky inequality [1]:

$$\left(\int_F \hbar_1(x)\hbar_2(x)\mathrm{d}x\right)^2 \le \left(\int_E \hbar_1^2(x)\mathrm{d}x\right)\left(\int_E \hbar_2^2(x)\mathrm{d}x\right) \tag{2}$$

The inequalities (1) and (2) have extensive applications not only in pure mathematics but also in science and engineering. To date, many generalizations and refinements of inequalities (1) and (2) have been studied. For example, Yang [2] established new generalizations by using local fractional calculus as follows.

It is assumed that $\Phi(v)$, $\Psi(v) \in C_\lambda(a, b)$, then:

$$\frac{1}{\Gamma(1+\lambda)}\int_b^a\left|\Phi(v)\Psi(v)\right|(\mathrm{d}v)^\lambda \le \left[\frac{1}{\Gamma(1+\lambda)}\int_b^a\left|\Phi(v)\right|^2(\mathrm{d}v)^\lambda\right]\left[\frac{1}{\Gamma(1+\lambda)}\int_b^a\left|\Psi(v)\right|^2(\mathrm{d}v)^\lambda\right] \tag{3}$$

where $C_\lambda(a, b)$ is the fractal space which consist of local fractional continuous functions defined on the interval $[a, b]$, $0 < \lambda \le 1$.

For more generalizations and refinements about inequalities (1) and (2), the reader can be referred to [2-8] and the cited references therein.

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It is well-known that local fractional calculus [2, 5-7, 9-13], established by Yang, has become a very useful tool to discuss the continuously non-differentiable functions, as well as fractals.

Recently, a new generalization of inequality (2) was established by Montazeri [3]:

$$\sum_{l=1}^{m}\frac{q_l}{Q}\left(\int_b^a \phi_l(z)\varphi(z)\mathrm{d}z\right)^2 \le \sum_{l=1}^{m}\frac{q_l}{Q}\left(\int_a^b \phi_l^2(z)\mathrm{d}z\right)\left(\int_a^b \varphi^2(z)\mathrm{d}z\right) \tag{4}$$

where

$$\{q_l\}_{l=1}^{m} > 0,\ Q=\sum_{l=1}^{m} q_l\ \ \{\phi_l(x)\}_{l=1}^{m}$$

are non-negative and continuous real function sequences on interval [a, b]. Then, in [3], the refinement of inequality (4) was derived:

$$\sum_{l=1}^{m}\frac{q_l}{Q}\left(\int_a^b \phi_l(z)\varphi(z)\mathrm{d}z\right)^2 + \begin{Bmatrix} p_1^2 \\ p_2^2 \end{Bmatrix} \le \sum_{l=1}^{m}\frac{q_l}{Q}\left(\int_a^b \phi_l^2(z)\mathrm{d}z\right)\left(\int_a^b \varphi^2 z\mathrm{d}z\right) \tag{5}$$

where p_1^2 and p_2^2 are the functions associated with $\phi_l(z)$, pi, and $\phi(z)$.

Very recently, Tang *et al.* [8] generalized the inequalities (4) and (5) to calculus theory on time scales.

The motivation of this paper is to study some generalizations and refinements of eqs. (4) and (5) by using the local fractional calculus.

The arrangement of this paper is as follows. In section *Main results*, we establish and prove the main results. At last, a conclusion is given.

Main results

Let $C_\eta(a, b)$ denote the fractal space which consist of local fractional continuous functions defined on the interval [a, b], where $0 < \eta \le 1$. It is assumed that $\{\Omega_l\}_{l=1} > 0$ and $\Omega = \sum_{l=1}^{n} \Omega_l$. Next, we state the main results of this paper.

Theorem 1. It is assumed that $\Omega := [a, b]$. Let φ_i and $\hbar$ be local fractional integrable on $\hat{\Omega}$. Then:

$$\sum_{l=1}^{n}\frac{\Omega_l}{\Omega}\left[\frac{1}{\Gamma(1+\eta)}\int_{\Omega_l}\varphi_l(\mu)\hbar(\mu)(\mathrm{d}\mu)^\eta\right]^2 \le \sum_{l=1}^{n}\frac{\Omega_l}{\Omega}\left[\frac{1}{\Gamma(1+\eta)}\int_{\Omega_l}\varphi_l^2(\mu)(\mathrm{d}\mu)^\eta\right]\left[\frac{1}{\Gamma(1+\eta)}\int_{\Omega_l}\hbar^2(\mu)(\mathrm{d}\mu)^\eta\right] \tag{6}$$

Proof. It is clear that the desired results can be obtained by inequality (6) and a direct computation.

For establishing the refinement of inequality (6), we need the following lemma.

Lemma 2. Assume that $\hat{\Omega} := [a, b]$. Let φ_l be local fractional integrable on $\hat{\Omega}$,

$$\hat{\Theta}(\mu) = \frac{1}{\Omega}\sum_{l=1}^{n}\Omega_l\varphi_l(\mu)\ \text{ and }\ r_l(\mu) = \varphi_l(\mu) - \hat{\Theta}(\mu),\ l = 1,2,...,n$$

Then $r_l(\mu)$ is local fractional integrable on $\hat{\Omega}$ and $\sum_{l=1}^{n} \Omega r_l(\mu) = 0$.

Proof. By the assumption and a direct computation, one can directly obtain the conclusion.

Theorem 3. Under the assumptions of *Lemma 2*, if $\hbar$ is local fractional integrable on $\hat{\Omega}$:

$$\left(\frac{1}{\Gamma(1+\eta)}\int_{\hat{\Omega}}\varphi_l(\mu)\hbar(\mu)(\mathrm{d}\mu)^\eta\right)^2 \le \left(\frac{1}{\Gamma(1+\eta)}\int_{\hat{\Omega}}\varphi_l^2(\mu)(\mathrm{d}\mu)^\eta\right)\left(\frac{1}{\Gamma(1+\eta)}\int_{\hat{\Omega}}\hbar^2(\mu)(\mathrm{d}\mu)^\eta\right)$$

then

$$\sum_{l=1}^{n}\frac{\Omega_l}{\Omega}\left(\frac{1}{\Gamma(1+\eta)}\int_{\hat{\Omega}}\varphi_l(\mu)\hbar(\mu)(\mathrm{d}\mu)^\eta\right)^2+\max\left\{\Pi_1^2,\Pi_2^2\right\}\le\sum_{l=1}^{n}\frac{\Omega_l}{\Omega}\left(\frac{1}{\Gamma(1+\eta)}\int_{\hat{\Omega}}\varphi_l^2(\mu)(\mathrm{d}\mu)^\eta\right)\left(\frac{1}{\Gamma(1+\eta)}\int_{\hat{\Omega}}\hbar^2(\mu)(\mathrm{d}\mu)^\eta\right)$$

where

$$\Pi_1^2=\left(\frac{1}{\Gamma(1+\eta)}\int_{\hat{\Omega}}\hat{\Theta}^2(\mu)(\mathrm{d}\mu)^\eta\right)\left(\frac{1}{\Gamma(1+\eta)}\int_{\hat{\Omega}}\hbar^2(\mu)(\mathrm{d}\mu)^\eta\right)-\left(\frac{1}{\Gamma(1+\eta)}\int_{\hat{\Omega}}\hat{\Theta}(\mu)\hbar(\mu)(\mathrm{d}\mu)^\eta\right)^2$$

$$\Pi_2^2=\sum_{l=1}^{n}\frac{\Omega_l}{\Omega}\left[\left(\frac{1}{\Gamma(1+\eta)}\int_{\hat{\Omega}}r_l^2(\mu)(\mathrm{d}\mu)^\eta\right)\left(\frac{1}{\Gamma(1+\eta)}\int_{\hat{\Omega}}\hbar^2(\mu)(\mathrm{d}\mu)^\eta\right)-\left(\frac{1}{\Gamma(1+\eta)}\int_{\hat{\Omega}}r_l(\mu)\hbar(\mu)(\mathrm{d}\mu)^\eta\right)^2\right]$$

Proof. It is assumed that Ψ_1, Ψ_2 can be given:

$$\Psi_1=\sum_{l=1}^{n}\frac{\Omega_l}{\Omega}\left(\frac{1}{\Gamma(1+\eta)}\int_{\hat{\Omega}}\varphi_l^2(\mu)(\mathrm{d}\mu)^\eta\right)\left(\frac{1}{\Gamma(1+\eta)}\int_{\hat{\Omega}}\hbar^2(\mu)(\mathrm{d}\mu)^\eta\right)$$

$$\Psi_2=\sum_{l=1}^{n}\frac{\Omega_l}{\Omega}\left(\frac{1}{\Gamma(1+\eta)}\int_{\hat{\Omega}}\varphi_l(\mu)\hbar(\mu)(\mathrm{d}\mu)^\eta\right)^2$$

From $r_l(\mu)$ we have $\varphi_l(\mu) = r_1(\mu) + \hat{\Theta}(\mu)$. So Ψ_1 can be transformed:

$$\begin{aligned}\Psi_1=&\sum_{l=1}^{n}\frac{\Omega_l}{\Omega}\left(\frac{1}{\Gamma(1+\eta)}\int_{\hat{\Omega}}\hat{\Theta}^2(\mu)(\mathrm{d}\mu)^\eta\right)\left(\frac{1}{\Gamma(1+\eta)}\int_{\hat{\Omega}}\hbar^2(\mu)(\mathrm{d}\mu)^\eta\right)+\\&+2\sum_{l=1}^{n}\frac{\Omega_l}{\Omega}\left(\frac{1}{\Gamma(1+\eta)}\int_{\hat{\Omega}}r_l(\mu)\hat{\Theta}(\mu)(\mathrm{d}\mu)^\eta\right)\left(\frac{1}{\Gamma(1+\eta)}\int_{\hat{\Omega}}\hbar^2(\mu)(\mathrm{d}\mu)^\eta\right)+\\&+\sum_{l=1}^{n}\frac{\Omega_l}{\Omega}\left(\frac{1}{\Gamma(1+\eta)}\int_{\hat{\Omega}}r_l^2(\mu)(\mathrm{d}\mu)^\eta\right)\left(\frac{1}{\Gamma(1+\eta)}\int_{\hat{\Omega}}\hbar^2(\mu)(\mathrm{d}\mu)^\eta\right)\end{aligned}\tag{7}$$

Thanks to the first summation in eq. (7):

$$\begin{aligned}&\sum_{l=1}^{n}\frac{\Omega_l}{\Omega}\left(\frac{1}{\Gamma(1+\eta)}\int_{\hat{\Omega}}\hat{\Theta}^2(\mu)(\mathrm{d}\mu)^\eta\right)\left(\frac{1}{\Gamma(1+\eta)}\int_{\hat{\Omega}}\hbar^2(\mu)(\mathrm{d}\mu)^\eta\right)=\\&=\left(\frac{1}{\Gamma(1+\eta)}\int_{\hat{\Omega}}\hat{\Theta}^2(\mu)(\mathrm{d}\mu)^\eta\right)\left(\frac{1}{\Gamma(1+\eta)}\int_{\hat{\Omega}}\hbar^2(\mu)(\mathrm{d}\mu)^\eta\right)\end{aligned}\tag{8}$$

By simplifying the second summation of the eq. (7):

$$\begin{aligned}&2\sum_{l=1}^{n}\frac{\Omega_l}{\Omega}\left(\frac{1}{\Gamma(1+\eta)}\int_{\hat{\Omega}}r_l(\mu)\hat{\Theta}(\mu)(\mathrm{d}\mu)^\eta\right)\left(\frac{1}{\Gamma(1+\eta)}\int_{\hat{\Omega}}\hbar^2(\mu)(\mathrm{d}\mu)^\eta\right)=\\&=2\left(\frac{1}{\Gamma(1+\eta)}\int_{\hat{\Omega}}\hbar^2(\mu)(\mathrm{d}\mu)^\eta\right)\sum_{l=1}^{n}\frac{\Omega_l}{\Omega}\left(\frac{1}{\Gamma(1+\eta)}\int_{\hat{\Omega}}r_l(\mu)\hat{\Theta}(\mu)(\mathrm{d}\mu)^\eta\right)=\\&=2\left(\frac{1}{\Gamma(1+\eta)}\int_{\hat{\Omega}}\hbar^2(\mu)(\mathrm{d}\mu)^\eta\right)\left(\frac{1}{\Gamma(1+\eta)}\int_{\hat{\Omega}}\sum_{l=1}^{n}\frac{\Omega_l}{\Omega}r_l(\mu)\hat{\Theta}(\mu)(\mathrm{d}\mu)^\eta\right)=\\&=2\left(\frac{1}{\Gamma(1+\eta)}\int_{\hat{\Omega}}\hbar^2(\mu)(\mathrm{d}\mu)^\eta\right)\left(\frac{\hat{\Theta}(\mu)}{\Omega}\frac{1}{\Gamma(1+\eta)}\int_{\hat{\Omega}}\sum_{l=1}^{n}\Omega_l r_l(\mu)(\mathrm{d}\mu)^\eta\right)\end{aligned}\tag{9}$$

According to *Lemma 2* and the eq. (9), we can infer:

$$2\sum_{l=1}^{n}\frac{\Omega_l}{\Omega}\left(\frac{1}{\Gamma(1+\eta)}\int_{\hat{\Omega}} r_l(\mu)\hat{\Theta}(\mu)(\mathrm{d}\mu)^{\eta}\right)\left(\frac{1}{\Gamma(1+\eta)}\int_{\hat{\Omega}}\hbar^2(\mu)(\mathrm{d}\mu)^{\eta}\right)=0 \tag{10}$$

Consequently, based on the eqs. (8) and (10), the eq. (7) can be transformed:

$$\begin{aligned}\Psi_1 &= \left(\frac{1}{\Gamma(1+\eta)}\int_{\hat{\Omega}}\hat{\Theta}^2(\mu)(\mathrm{d}\mu)^{\eta}\right)+\left(\frac{1}{\Gamma(1+\eta)}\int_{\hat{\Omega}}\hbar^2(\mu)\hbar(\mathrm{d}\mu)^{\eta}\right)+\\ &+\sum_{l=1}^{n}\frac{\Omega_l}{\Omega}\left(\frac{1}{\Gamma(1+\eta)}\int_{\hat{\Omega}} r_i^2(\mu)(\mathrm{d}\mu)^{\eta}\right)+\left(\frac{1}{\Gamma(1+\eta)}\int_{\hat{\Omega}}\hbar^2(\mu)\hbar(\mathrm{d}\mu)^{\eta}\right)\end{aligned} \tag{11}$$

Based on the definition of Ψ_2, we have:

$$\begin{aligned}\Psi_2 &= \sum_{l=1}^{n}\frac{\Omega_l}{\Omega}\left(\frac{1}{\Gamma(1+\eta)}\int_{\hat{\Omega}}\hat{\Theta}(\mu)\hbar(\mu)(\mathrm{d}\mu)^{\eta}\right)^2+2\left(\frac{1}{\Gamma(1+\eta)}\int_{\hat{\Omega}}\hat{\Theta}(\mu)\hbar(\mu)(\mathrm{d}\mu)^{\eta}\right)\cdot\\ &\cdot\sum_{l=1}^{n}\frac{\Omega_l}{\Omega}\left(\frac{1}{\Gamma(1+\eta)}\int_{\hat{\Omega}} r_i(\mu)\hbar(\mu)(\mathrm{d}\mu)^{\eta}\right)+\sum_{l=1}^{n}\frac{\Omega_l}{\Omega}\left(\frac{1}{\Gamma(1+\eta)}\int_{\hat{\Omega}} r_i(\mu)\hbar(\mu)(\mathrm{d}\mu)^{\eta}\right)^2\end{aligned}$$

By similar to the computation process of Ψ_1, we have:

$$\Psi_2=\left(\frac{1}{\Gamma(1+\eta)}\int_{\hat{\Omega}}\hat{\Theta}(\mu)\hbar(\mu)(\mathrm{d}\mu)^{\eta}\right)^2+\sum_{l=1}^{n}\frac{\Omega_l}{\Omega}\left(\frac{1}{\Gamma(1+\eta)}\int_{\hat{\Omega}} r_l(\mu)\hbar(\mu)(\mathrm{d}\mu)^{\eta}\right)^2 \tag{12}$$

With the help of the eqs. (11) and (12), we can infer:

$$\begin{aligned}\Psi_1-\Psi_2 &= \left[\left(\frac{1}{\Gamma(1+\eta)}\int_{\hat{\Omega}}\hat{\Theta}^2(\mu)(\mathrm{d}\mu)^{\eta}\right)\left(\frac{1}{\Gamma(1+\eta)}\int_{\hat{\Omega}}\hbar^2(\mu)(\mathrm{d}\mu)^{\eta}\right)-\left(\frac{1}{\Gamma(1+\eta)}\int_{\hat{\Omega}}\hat{\Theta}(\mu)\hbar(\mu)(\mathrm{d}\mu)^{\eta}\right)^2\right]+\\ &+\sum_{l=1}^{n}\frac{\Omega_l}{\Omega}\left[\left(\frac{1}{\Gamma(1+\eta)}\int_{\hat{\Omega}} r_l^2(\mu)(\mathrm{d}\mu)^{\eta}\right)\left(\frac{1}{\Gamma(1+\eta)}\int_{\hat{\Omega}}\hbar^2(\mu)(\mathrm{d}\mu)^{\eta}\right)-\left(\frac{1}{\Gamma(1+\eta)}\int_{\hat{\Omega}} r_l(\mu)\hbar(\mu)(\mathrm{d}\mu)^{\eta}\right)^2\right]\end{aligned}$$

Thanks to the inequality (3) and the assumption of *Theorem 3*, we have:

$$\left(\frac{1}{\Gamma(1+\eta)}\int_{\hat{\Omega}}\hat{\Theta}^2(\mu)(\mathrm{d}\mu)^{\eta}\right)\left(\frac{1}{\Gamma(1+\eta)}\int_{\hat{\Omega}}\hbar^2(\mu)(\mathrm{d}\mu)^{\eta}\right)-\left(\frac{1}{\Gamma(1+\eta)}\int_{\hat{\Omega}}\hat{\Theta}(\mu)\hbar(\mu)(\mathrm{d}\mu)^{\eta}\right)^2\ge 0$$

$$\left(\frac{1}{\Gamma(1+\eta)}\int_{\hat{\Omega}} r_l^2(\mu)(\mathrm{d}\mu)^{\eta}\right)\left(\frac{1}{\Gamma(1+\eta)}\int_{\hat{\Omega}}\hbar^2(\mu)(\mathrm{d}\mu)^{\eta}\right)-\left(\frac{1}{\Gamma(1+\eta)}\int_{\hat{\Omega}} r_l(\mu)\hbar(\mu)(\mathrm{d}\mu)^{\eta}\right)^2\ge 0$$

By using the previous discussion, we can yield the two inequalities:

$$\Psi_1-\Psi_2\ge\left(\frac{1}{\Gamma(1+\eta)}\int_{\hat{\Omega}}\hat{\Theta}^2\text{ƛ}(\mathrm{d}\text{ƛ})^{\eta}\right)\left(\frac{1}{\Gamma(1+\eta)}\int_{\hat{\Omega}}\hbar^2\text{ƛ}(\mathrm{d}\text{ƛ})^{\eta}\right)-\left(\frac{1}{\Gamma(1+\eta)}\int_{\hat{\Omega}}\hat{\Theta}(\text{ƛ})\hbar(\text{ƛ})(\mathrm{d}\text{ƛ})^{\eta}\right)^2$$

$$\begin{aligned}\Psi_1-\Psi_2 &\ge \sum_{l=1}^{n}\frac{\Omega_l}{\Omega}\left[\left(\frac{1}{\Gamma(1+\eta)}\int_{\hat{\Omega}} r_l^2(\mu)(\mathrm{d}\mu)^{\eta}\right)\left(\frac{1}{\Gamma(1+\eta)}\int_{\hat{\Omega}}\hbar^2(\mu)(\mathrm{d}\mu)^{\eta}\right)-\right.\\ &\left.-\left(\frac{1}{\Gamma(1+\eta)}\int_{\hat{\Omega}} r_l(\mu)\hbar(\mu)(\mathrm{d}\mu)^{\eta}\right)^2\right]\end{aligned}$$

Let Π_1^2 and Π_2^2 can be defined:

$$\Pi_1^2=\left(\frac{1}{\Gamma(1+\eta)}\int_{\hat{\Omega}}\hat{\Theta}^2(\mu)(\mathrm{d}\mu)^\eta\right)\left(\frac{1}{\Gamma(1+\eta)}\int_{\hat{\Omega}}\hbar^2(\mu)(\mathrm{d}\mu)^\eta\right)-\left(\frac{1}{\Gamma(1+\eta)}\int_{\hat{\Omega}}\hat{\Theta}(\mu)\hbar(\mu)(\mathrm{d}\mu)^\eta\right)^2$$

$$\Pi_2^2=\sum_{l=1}^{n}\frac{\Omega_l}{\Omega}\left[\left(\frac{1}{\Gamma(1+\eta)}\int_{\hat{\Omega}}r_l^2(\mu)(\mathrm{d}\mu)^\eta\right)\left(\frac{1}{\Gamma(1+\eta)}\int_{\hat{\Omega}}\hbar^2(\mu)(\mathrm{d}\mu)^\eta\right)-\left(\frac{1}{\Gamma(1+\eta)}\int_{\hat{\Omega}}r_l(\mu)\hbar(\mu)(\mathrm{d}\mu)^\eta\right)^2\right]$$

In terms of the expressions of Π_1^2 and Π_2^2:

$$\Psi_1-\Psi_2=\sum_{l=1}^{n}\frac{\Omega_l}{\Omega}\left(\frac{1}{\Gamma(1+\eta)}\int_{\hat{\Omega}}\varphi_l^2(\mu)(\mathrm{d}\mu)^\eta\right)\left(\frac{1}{\Gamma(1+\eta)}\int_{\hat{\Omega}}\hbar^2(\mu)(d\mu)^\eta\right)-$$
$$-\sum_{i=1}^{n}\frac{\Omega_l}{\Omega}\left(\frac{1}{\Gamma(1+\eta)}\int_{\hat{\Omega}}\varphi_i(\mu)\hbar(\mu)(\mathrm{d}\mu)^\eta\right)^2\geq\Pi_1^2$$

$$\Psi_1-\Psi_2=\sum_{l=1}^{n}\frac{\Omega_l}{\Omega}\left(\frac{1}{\Gamma(1+\eta)}\int_{\hat{\Omega}}\varphi_l^2(\mu)(d\mu)^\eta\right)\left(\frac{1}{\Gamma(1+\eta)}\int_{\hat{\Omega}}\hbar^2(\mu)(\mathrm{d}\mu)^\eta\right)-$$
$$-\sum_{l=1}^{n}\frac{\Omega_l}{\Omega}\left(\frac{1}{\Gamma(1+\eta)}\int_{\hat{\Omega}}\varphi_l(\mu)\hbar(\mu)(d\mu)^\eta\right)^2\geq\Pi_2^2$$

Based on the aforementioned inequalities, it can be obtained:

$$\sum_{l=1}^{n}\frac{\Omega_l}{\Omega}\left(\frac{1}{\Gamma(1+\eta)}\int_{\hat{\Omega}}\varphi_l^2(\mu)(\mathrm{d}\mu)^\eta\right)\left(\frac{1}{\Gamma(1+\eta)}\int_{\hat{\Omega}}\hbar^2(\mu)(\mathrm{d}\mu)^\eta\right)-$$
$$-\sum_{l=1}^{n}\frac{\Omega_l}{\Omega}\left(\frac{1}{\Gamma(1+\eta)}\int_{\hat{\Omega}}\varphi_l(\mu)\hbar(\mu)(\mathrm{d}\mu)^\eta\right)^2\geq\max\{\Pi_1^2,\Pi_2^2\} \tag{13}$$

From eq. (13), we have:

$$\sum_{l=1}^{n}\frac{\Omega_l}{\Omega}\left(\frac{1}{\Gamma(1+\eta)}\int_{\hat{\Omega}}\varphi_l(\mu)\hbar(\mu)(\mathrm{d}\mu)^\eta\right)^2+\max\{\Pi_1^2,\Pi_2^2\}\leq$$
$$\leq\sum_{l=1}^{n}\frac{\Omega_l}{\Omega}\left(\frac{1}{\Gamma(1+\eta)}\int_{\hat{\Omega}}\varphi_l^2(\mu)(\mathrm{d}\mu)^\eta\right)\left(\frac{1}{\Gamma(1+\eta)}\int_{\hat{\Omega}}\hbar^2(\mu)(\mathrm{d}\mu)^\eta\right)$$

which is desired results. We finish the proof of *Theorem 3*.

Conclusion

In this paper, the results established by Montazeri, see [3], is generalized to local fractional calculus theory on fractal spaces. The conclusion obtained in this paper shows two distinct refinements for local fractional integral Cauchy-Bunyakovsky inequality. Some applications of the obtained results of this paper shall be considered in future work.

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References

[1] Mitrinovic, D. S., *et al*., *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, Dordrecht, Boston, London, UK, 1993
[2] Yang, X. J., *Local Fractional Functional Analysis and Its Applications,* Asian Academic Publisher, Hong Kong, China, 2011
[3] Montazeri, S., New Refinements of the Cauchy–Bunyakovsky Inequality, *Journal of Inequalities and Application*s, *2023* (2023), 161, pp. 1-5
[4] Masjed-Jamei, M., A Generic Refinement to the Cauchy-Schwarz Inequality, *Publications de l'Institut Mathematique*, *114* (2023), 128, pp. 103-109
[5] Chen, G. S., Generalizations of Holder's and Some Related Integral Inequalities on Fractal Space, *Journal of Function Spaces and Applications*, *2013* (2013), ID198405
[6] Chen, G. S., *et al*., Some Further Generalizations of Hölder's Inequality and Related Results on Fractal Space, *Abstract and Applied Analysis*, *2014* (2014), ID832802
[7] Chen, G. S., A Subdividing of Local Fractional Integral Holder's Inequality on Fractal Space, *Advanced Materials Research*, *998* (2014), pp. 976-979
[8] Yang, X. J., Local Fractional Integral Transforms, *Progress in Non-linear Science*, *4* (2011), pp. 1-225
[9] Yang, X. J., *Advanced Local Fractional Calculus and Its Applications*, World Science Publisher, New York, NY, USA, 2012
[10] Yang, X. J., *et al.*, Mathematical Aspects of the Heisenberg Uncertainty Principle within Local Fractional Fourier Analysis, *Boundary Value Problems*, *2013* (2013), 131, pp. 1-16
[11] Yang, A. M., *et al*., Local Fractional Series Expansion Method for Solving Wave and Diffusion Equations on Cantor Sets, *Abstract and Applied Analysis*, *2013* (2013), ID351057
[12] Yang, X. J., *et al.*, Local Fractional Variational Iteration Method for Diffusion and Wave Equations on Cantor Sets, *Romanian Journal of Physics*, *59* (2014), 1-2, pp. 36-48
[13] Tang, H. Y., *et al.*, New Generalization and Refinement of the Diamond-α Integral Cauchy Schwartz Inequality on Time Scales, *Fractals*, *23* (2024), 3, ID2440054

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