SOME INTEGRAL INEQUALITIES ON PROBABILITY SPACE

by

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Holder's inequality and Minkowski's inequality are important tools in probability theory and mathematical analysis and they have wide applications in many branches of mathematics. This paper studies the reverse probability inequalities of these, as well as their generalizations and improvements.

Key words: *Holder's inequality, Minkowski's inequality, Young inequality, reverse probability inequality*

Introduction

It is assumed that (Ω, A, P), consisting of all A-Borel measurable functions, is a probability space. Then we have the following Holder inequality.

If T, H $\in (\Omega, A, P)$, $\alpha > 1$, $1/\alpha + 1/\beta = 1$, then the integral inequality on probability space holds [1]:

$$\int_\Omega |\mathrm{TH}| \mathrm{d}P \le \left(\int_\Omega |\mathrm{T}|^\alpha \mathrm{d}P \right)^{1/\alpha} \left(\int_\Omega |\mathrm{H}|^\beta \mathrm{d}P \right)^{1/\beta} \tag{1}$$

It is well-known that the celebrated inequality was established by the distinguished mathematician Holder [2-5]. Reverse Holder's inequality and some related results can also be found in [6-8].

When T, H $\in (\Omega, A, P)$, $\alpha > 1$ then the Minkowski's inequality holds:

$$\left(\int_\Omega |\mathrm{T}+\mathrm{H}|^\alpha \mathrm{d}P \right)^{1/\alpha} \le \left(\int_\Omega |\mathrm{T}|^\alpha \mathrm{d}P \right)^{1/\alpha} + \left(\int_\Omega |\mathrm{H}|^\alpha \mathrm{d}P \right)^{1/\alpha} \tag{2}$$

Bourin *et al*. [9] have advanced novel refinements of the Minkowski inequality, encompassing a broad spectrum of operator means. In [10], a Pexider-type extension of Minkowski's inequality was successfully presented. Within the realm of pseudo-analysis, Agahi *et al.* [11] have established several refinements of Holder's and Minkowski's inequalities related to the pseudo-integral. Some authors also considered integral Holder's inequality using the definition of integral on time scales. For more results about Holder's inequality and related results, the readers can be referred to [12-15].

This work aims to establish a reverse Holder's type integral inequalities on probability space, some related conclusions are also considered. The arrangement of this is planned as

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follows. We state and prove our main results in the second section. We give a conclusion in the final section.

Main results

We consider three A-Borel measurable functions T, H, and Z, a probability space is denoted by (Ω, A, P).

Theorem 1. Let T, H, Z $\in (\Omega, A, P)$, $\alpha > 1$, and $1/\alpha + 1/\beta = 1$. Then one has:

$$\int_{\Omega} |\mathrm{Z}||\mathrm{TH}| \mathrm{d}P \le \left(\int_{\Omega} |\mathrm{Z}||\mathrm{T}|^{\alpha} \, \mathrm{d}P \right)^{1/\alpha} \left(\int_{\Omega} |\mathrm{Z}||\mathrm{H}|^{\beta} \, \mathrm{d}P \right)^{1/\beta} \tag{3}$$

Remark 1. For $|\mathrm{Z}| = 1$ in *Theorem 1*, inequality (3) can be transformed into inequality (1).

Theorem 2. If T, H, Z $\in (\Omega, A, P)$, $0 < \alpha < 1$, with $\beta = \alpha/(\alpha - 1)$, then one has:

$$\int_{\Omega} |\mathrm{Z}||\mathrm{TH}| \mathrm{d}P \ge \left(\int_{\Omega} |\mathrm{Z}||\mathrm{T}|^{\alpha} \, \mathrm{d}P \right)^{1/\alpha} \left(\int_{\Omega} |\mathrm{Z}||\mathrm{H}|^{\beta} \, \mathrm{d}P \right)^{1/\beta} \tag{4}$$

Proof. We may assume, without any loss of generality, that:

$$\left(\int_{\Omega} |\mathrm{Z}||\mathrm{T}|^{\alpha} \, \mathrm{d}P \right)^{1/\alpha} \left(\int_{\Omega} |\mathrm{Z}||\mathrm{H}|^{\beta} \, \mathrm{d}P \right)^{1/\beta} \ne 0$$

and let

$$\zeta = \frac{|\mathrm{Z}||\mathrm{T}|^{\alpha}}{\int_{\Omega} |\mathrm{Z}||\mathrm{T}|^{\alpha} \, \mathrm{d}P}, \quad \upsilon = \frac{|\mathrm{Z}||\mathrm{H}|^{\beta}}{\int_{\Omega} |\mathrm{Z}||\mathrm{H}|^{\beta} \, \mathrm{d}P}$$

Applying the reverse Young inequality in [12], it is natural to obtain:

$$\begin{aligned}
&\int_{\grave{U}} \frac{|\mathrm{Z}|^{1/\alpha} |\mathrm{T}|}{\left(\int_{\grave{U}} |\mathrm{Z}||\mathrm{T}|^{\alpha} \, \mathrm{d}P \right)^{1/\alpha}} \frac{|\mathrm{Z}|^{1/\beta} |\mathrm{H}|}{\left(\int_{\grave{U}} |\mathrm{Z}||\mathrm{H}|^{\beta} \, \mathrm{d}P \right)^{1/\beta}} \mathrm{d}P = \\
&= \int_{\grave{U}} \zeta^{1/\alpha} \upsilon^{1/\beta} \mathrm{d}P \ge \\
&\ge \int_{\grave{U}} \left(\frac{\zeta}{\alpha} + \frac{\upsilon}{\beta} \right) \mathrm{d}P = \\
&= \frac{1}{\alpha} \int_{\grave{U}} \left(\frac{|\mathrm{Z}||\mathrm{T}|^{\alpha}}{\int_{\grave{U}} |\mathrm{Z}||\mathrm{T}|^{\alpha} \, \mathrm{d}P} \right) \mathrm{d}P + \frac{1}{\beta} \int_{\grave{U}} \left(\frac{|\mathrm{Z}||\mathrm{H}|^{\beta}}{\int_{\grave{U}} |\mathrm{Z}||\mathrm{H}|^{\beta} \, \mathrm{d}P} \right) \mathrm{d}P = 1
\end{aligned}$$

Remark 2. For $|\mathrm{Z}| = 1$ in *Theorem 2*, a reverse version of inequality (1) is obtained.

According to *Theorem 1* and *Theorem 2*, the following generalization concerning multiple functions can be presented.

Corollary 1. Let Z, $\mathrm{T}_i \in (\Omega, A, P)$, $\alpha_i \in R$, $i = 1, 2, ..., k$, $1/\alpha_1 + 1/\alpha_2 + ... + 1/\alpha_k = 1$:

– For $\alpha_i > 1$ one has

$$\int_\Omega |Z|\left|\prod_{i=1}^k T_i\right| dP \le \prod_{i=1}^k \left(\int_\Omega |Z||T_i|^{\alpha_i}\, dP\right)^{1/\alpha_i} \tag{5}$$

– For $\alpha_1 \in (0, 1)$ and $\alpha_i < 0$, $i = 2, ..., k$, one has:

$$\int_\Omega |Z|\left|\prod_{i=1}^k T_i\right| dP \ge \prod_{i=1}^k \left(\int_\Omega |Z||T_i|^{\alpha_i}\, dP\right)^{1/\alpha_i} \tag{6}$$

*Theorem 3***.** If T, H, Z ∈ (Ω, A, P), $\alpha > 1$, then one has:

$$\left(\int_\Omega |Z||T+H|^\alpha\, dP\right)^{1/\alpha} \le \left(\int_\Omega |Z||T|^\alpha\, dP\right)^{1/\alpha} + \left(\int |Z||H|^\alpha\, dP\right)^{1/\alpha} \tag{7}$$

Remark 3. Under the condition |Z| = 1 in *Theorem 3*, inequality (7) is diminished to inequality (2).

Theorem 4. If, H, Z ∈ (Ω, A, P), $0 < \alpha < 1$, then one has:

$$\left(\int_\Omega |Z||T+H|^\alpha\, dP\right)^{1/\alpha} \ge \left(\int_\Omega |Z||T|^\alpha\, dP\right)^{1/\alpha} + \left(\int_\Omega |Z||H|^\alpha\, dP\right)^{1/\alpha} \tag{8}$$

Proof. Assume that $0 < \alpha < 1$

$$\frac{1}{\gamma} = \alpha,\ \frac{1}{\lambda} = 1-\alpha,\ a_k = \mu_k^\alpha,\ b_k = \nu_k^{1/\alpha-1}$$

based on Holder's inequality

$$\sum_{k=1}^n a_k b_k \le \left(\sum_{k=1}^n a_k^\gamma\right)^{1/\gamma} \left(\sum_{k=1}^n b_k^\lambda\right)^{1/\lambda},\ \gamma > 1,\ \frac{1}{\gamma}+\frac{1}{\lambda} = 1$$

it can be concluded that:

$$\sum_{k=1}^n \mu_k^\alpha \nu_k^{1/\alpha-1} \le \left(\sum_{k=1}^n \mu_k\right)^\alpha \left(\sum_{k=1}^n \nu_k^{1/\alpha}\right)^{1-\alpha} \tag{9}$$

Let

$$\theta = \int_\Omega |Z||T|^\alpha\, dP,\ \eta = \int_\Omega |Z||H|^\alpha\, dP$$

$$M = \left(\int_\Omega |Z||T|^\alpha\, dP\right)^{1/\alpha} + \left(\int_\Omega |Z||H|^\alpha\, dP\right)^{1/\alpha} = \theta^{1/\alpha} + \eta^{1/\alpha}$$

from eq. (9), that

$$M = \theta^{1/\alpha} + \eta^{1/\alpha} = \theta^{1/\alpha-1}\int_\Omega |Z||T|^\alpha\, dP + \eta^{1/\alpha-1}\int_\Omega \int_\Omega |Z||H|^\alpha\, dP =$$

$$= \int_\Omega |Z|\left(|T|^\alpha \theta^{1/\alpha-1} + |H|^\alpha \eta^{1/\alpha-1}\right) dP \le \int_\Omega |Z||T+H|^\alpha \left(\theta^{1/\alpha} + \eta^{1/\alpha}\right)^{1-\alpha} dP =$$

$$= \int_\Omega |Z||T+H|^\alpha M^{1-\alpha} dP = M^{1-\alpha}\int_\Omega |Z||T+H|^\alpha\, dP$$

Building on the aforementioned outcome, we promptly derive Minkowski's inequality, thereby fully establishing the theorem. Drawing upon *Theorems 3 and 4*, the subsequent generalization is validated.

Corollary 2. Let Z, $T_i \in (\Omega, A, P)$. Then:

– For $\alpha > 1$, one has

$$\left(\int_\Omega |Z|\left|\sum_{i=1}^m T_i\right|^\alpha dP\right)^{1/\alpha} \le \sum_{i=1}^m \left(\int_\Omega |Z||T_i|^\alpha dP\right)^{1/\alpha} \tag{10}$$

For $0 < \alpha < 1$, Z, $T_i \in (\Omega, A, P)$, one has

$$\left(\int_\Omega |Z|\left|\sum_{i=1}^m T_i\right|^\alpha dP\right)^{1/\alpha} \ge \sum_{i=1}^m \left(\int_\Omega |Z||T_i|^\alpha dP\right)^{1/\alpha} \tag{11}$$

Subsequently, we present a counterpart to *Corollary 2*.

Corollary 3. Assume that Z, $T_i \in (\Omega, A, P)$, then

– For $\alpha > 1$, one has

$$\int_\Omega |Z|\left(\sum_{j=1}^m |T_i|\right)^\alpha dP \ge \sum_{j=1}^m \int_\Omega |Z||T_i|^\alpha dP \tag{12}$$

– For $0 < \alpha < 1$, Z, $T_i \in (\Omega, A, P)$, one has

$$\int_\Omega |Z|\left(\sum_{j=1}^m |T_i|\right)^\alpha dP \le \sum_{j=1}^m \int_\Omega |Z||T_i|^\alpha dP \tag{13}$$

Proof.

– When $\alpha > 1$, by setting $s = \alpha$, $r = 1$ within Jensen's inequality, one obtains:

$$|T_1| + |T_2| + \cdots + |T_m| \ge \left(|T_1|^\alpha + |T_2|^\alpha + \cdots + |T_m|^\alpha\right)^{1/\alpha}$$

it is easy to get

$$|Z|\left(|T_1| + |T_2| + \cdots + |T_m|\right)^\alpha \ge |Z|\left(|T_1|^\alpha + |T_2|^\alpha + \cdots + |T_m|^\alpha\right)$$

Integrating the aforementioned inequality, it can be derived that inequality (12) is true.

– For $0 < \alpha < 1$, by setting $s = \alpha$, $r = 1$ within Jensen's inequality, one obtains:

$$|T_1| + |T_2| + \cdots + |T_m| \le \left(|T_1|^\alpha + |T_2|^\alpha + \cdots + |T_m|^\alpha\right)^{1/\alpha}$$

from the aforementioned inequality, it is natural to get

$$|Z|\left(|T_1| + |T_2| + \cdots + |T_m|\right)^\alpha \le |Z|\left(|T_1|^\alpha + |T_2|^\alpha + \cdots + |T_m|^\alpha\right)$$

By the integration of the previously mentioned inequality, the desired result is achieved.

In the subsequent theorem, we present several improvements to Minkowski's integral inequality on probability space.

Theorem 5. Let Z, $T_i \in (\Omega, A, P)$, $\alpha > 0$, $\gamma, \lambda \in R\backslash\{0\}$, and $\gamma \ne \lambda$:

– Assume that $\alpha, \gamma, \lambda \in R$ are different such that $\gamma, \lambda > 1$ and $(\gamma - \lambda)/(\alpha - \lambda) > 1$. Then:

$$\int_{\Omega}|Z||T+H|^{\alpha}\,dP \le \left[\left(\int_{\Omega}|Z||T|^{\gamma}\,dP\right)^{1/\gamma}+\left(\int_{\Omega}|Z||H|^{\gamma}\,dP\right)^{1/\gamma}\right]^{\gamma(\alpha-\lambda)/(\gamma-\lambda)}\cdot$$
$$\cdot\left[\left(\int_{\Omega}|Z||T|^{\lambda}\,dP\right)^{1/\lambda}+\left(\int_{\Omega}|Z||H|^{\lambda}\,dP\right)^{1/\lambda}\right]^{\lambda(\gamma-\alpha)/(\gamma-\lambda)} \tag{14}$$

– Assume tha $\alpha, \gamma, \lambda \in R$ are different such that $0 < \gamma < 1$ and $(\gamma-\lambda)/(\alpha-\lambda) < 1$. Then:

$$\int_{\Omega}|Z||T+H|^{\alpha}\,dP \ge \left[\left(\int_{\Omega}|Z||T|^{\gamma}\,dP\right)^{1/\gamma}+\left(\int_{\Omega}|Z||H|^{\gamma}\,dP\right)^{1/\gamma}\right]^{\gamma(\alpha-\lambda)/(\gamma-\lambda)}\cdot$$
$$\cdot\left[\left(\int_{\Omega}|Z||T|^{\lambda}\,dP\right)^{1/\gamma}+\left(\int_{\Omega}|Z||H|^{\lambda}\,dP\right)^{1/\gamma}\right]^{\lambda(\gamma-\alpha)/(\gamma-\lambda)} \tag{15}$$

Proof.

– It is clear that $(\gamma-\lambda)/(\alpha-\lambda) > 1$, and

$$\int_{\Omega}|Z||T+H|^{\alpha}\,dP = \int_{\Omega}|Z|\left(|T+H|^{\gamma}\right)^{(\alpha-\lambda)/(\gamma-\lambda)}\left(|T+H|^{\lambda}\right)^{(\gamma-\alpha)/(\gamma-\lambda)}\,dP$$

from eq. (3) with indices $(\gamma-\lambda)/(\alpha-\lambda) < 1$ and $(\gamma-\lambda)/(\gamma-\alpha)$ it is easy to be seen:

$$\int_{\Omega}|Z||T+H|^{\alpha}\,dP \le \left(\int_{\Omega}|Z||T+H|^{\gamma}\,dP\right)^{(\alpha-\lambda)/(\gamma-\lambda)}\left(\int_{\Omega}|Z||T+H|^{\lambda}\,dP\right)^{(\gamma-\alpha)/(\gamma-\lambda)} \tag{16}$$

On the other hand, by using eq. (7) for $\gamma > 1$ and $\lambda > 1$, respectively, one has:

$$\left(\int|Z||T+H|^{\gamma}\,dP\right)^{1/\gamma} \le \left(\int|Z||T|^{\gamma}\,dP\right)^{1/\gamma}+\left(\int|Z||H|^{\gamma}\,dP\right)^{1/\gamma} \tag{17}$$

and

$$\left(\int_{\Omega}|Z||T+H|^{\lambda}\,dP\right)^{1/\lambda} \le \left(\int_{\Omega}|Z||T|^{\lambda}\,dP\right)^{1/\lambda}+\left(\int_{\Omega}|Z||H|^{\lambda}\,dP\right)^{1/\lambda} \tag{18}$$

From eqs. (16)-(18), the desired result is derived.

– It is obvious that $(\gamma-\lambda)/(\alpha-\lambda) < 1$ and

$$\int_{\Omega}|Z||T+H|^{\alpha}\,dP = \int_{\Omega}|Z|\left(|T+H|^{\gamma}\right)^{(\alpha-\lambda)/(\gamma-\lambda)}\left(|T+H|^{\lambda}\right)^{(\gamma-\alpha)/(\gamma-\lambda)}\,dP$$

with the help of eq. (4) with indices $(\gamma-\lambda)/(\alpha-\lambda)$ and $(\gamma-\lambda)/(\gamma-\alpha)$, correspondingly, one has

$$\int_{\Omega}|Z||T+H|^{\alpha}\,dP \ge \left(\int_{\Omega}|Z||T+H|^{\gamma}\,dP\right)^{(\alpha-\lambda)/(\gamma-\lambda)}\left(\int_{\Omega}|Z||T+H|^{\lambda}\,dP\right)^{(\gamma-\alpha)/(\gamma-\lambda)} \tag{19}$$

Alternatively, applying the reverse Minkowski's inequality (2.6) see in [10] to the scenarios involving $0 < \gamma < 1$ and $0 < \lambda < 1$, we get infer that the subsequent propositions are correct:

$$\left(\int_{\Omega}|Z||T+H|^{\gamma}\,dP\right)^{1/\gamma}\geq\left(\int_{\Omega}|Z||T|^{\gamma}\,dP\right)^{1/\gamma}+\left(\int_{\Omega}|Z||H|^{\gamma}\,dP\right)^{1/\gamma}\tag{20}$$

and

$$\left(\int_{\Omega}|Z||T+H|^{\lambda}\,dP\right)^{1/\lambda}\geq\left(\int_{\Omega}|Z||T|^{\lambda}\,dP\right)^{1/\lambda}+\left(\int_{\Omega}|Z||H|^{\lambda}\,dP\right)^{1/\lambda}\tag{21}$$

With the help of eqs. (19), (20), and (2), the desired result can be established.

Conclusion

These studies not only help to deepen our understanding of the classical inequalities but also have significant application value in fields such as optimization theory, probability theory, and statistics. By researching the reverses and generalizations of these inequalities, we can better address practical problems and promote the development of related disciplines.

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