ON THE INVISCID LIMIT OF THE INHOMOGENEOUS NAVIER-STOKES EQUATIONS IN THE HALF SPACE

by

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In this paper, we consider the convergence in L^2 norm, uniformly in time of the inhomogeneous Navier-Stokes system and inhomogeneous Euler equations. Upon the assumption of the Oleinick conditions of no back-flow in the trace of the Euler flow, and of a lower bound for the Navier-Stokes vorticity in a Kato-like boundary-layer, we prove that the inviscid limit holds.

Key words: *inhomogeneous Navier-Stokes equations, boundary-layer, inhomogeneous Euler equations, inviscid limit*

Introduction

We are concerned with the incompressible inhomogeneous Navier-Stokes equations:

$$\begin{gathered}\partial_t\rho + \operatorname{div}(\rho u) = 0\\ \rho\partial_t u + \rho u\cdot\nabla u - \varepsilon\Delta u + \nabla P = 0\\ \operatorname{div} u = 0\end{gathered} \tag{1}$$

and incompressible inhomogeneous Euler equations:

$$\begin{gathered}\overline{\rho}_t + \operatorname{div}(\overline{\rho}\overline{u}) = 0\\ \overline{\rho}\partial_t\overline{u} + \overline{\rho}\overline{u}\cdot\nabla\overline{u} + \nabla\overline{P} = 0\\ \operatorname{div}\overline{u} = 0\end{gathered} \tag{2}$$

in the half plane $\Omega = \{x = (x_1, x_2) \in R^2 : x_2 > 0$ with Dirichlet and slip boundary:

$$u\,|_{\partial\Omega} = 0 \tag{3}$$

and

$$\overline{u}_2\,|_{\partial\Omega} = 0 \tag{4}$$

on the Navier-Stokes eq. (1) and the Euler eq. (2), respectively, where $u = (u_1, u_2)$ and $\overline{u} = (\overline{u}_1, \overline{u}_2)$ are velocity, P, $\overline{P}$ are the pressure, and ρ, $\overline{\rho}$ are the density.

The initial conditions for eqs. (1) and (2) are taken to be the same, $u_0 = \overline{u}_0$. We shall also denote the Navier-Stokes vorticity:

$$w = \partial_1 u_2 - \partial_2 u_1 \tag{5}$$

where $\partial_j = \partial/\partial_{xj}$ and the trace of tangential component of the Euler flow:

$$U = \overline{u}_1\big|_{\partial\Omega} \tag{6}$$

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The behavior of viscous incompressible flows in the inviscid limit is a classical issue in the fluid dynamics. When the fluid domain has no boundary, it is well known that the solution of the Navier-Stokes equations converges to the one of the Euler equations, and this problem is closely related to the boundary-layer problem and Prandtl equation. There have abundant literatures on the problem of inviscid limits, see [1-6] and references therein. In particular, Constantin [7] considered the convergence in the L^2 norm, uniformly in time, of the Navier-Stokes equations with Dirichlet boundary conditions to the Euler equations with slip boundary conditions, and proved that if the Oleinik conditions of no back-flow in the trace of the Euler flow, and of a lower bound for the Navier-Stokes vorticity is assumed in a Kato-like boundary-layer, then the inviscid limit holds. Maekawa [8, 9] also considered the Navier-Stokes equations for viscous incompressible flows in the half-plane under the no-slip boundary condition. By the vorticity formulation, he proved that the local-in-time convergence of the Navier-Stokes flows to the Euler flows outside a boundary-layer and to the Prandtl flows in the boundary-layer in the inviscid limit when the initial vorticity is located away from the boundary. Paddick [10] obtained the existence and the conormal Sobolev regularity of strong solutions to the 3-D compressible isentropic Navier-Stokes system on the half-space with a Navier boundary condition, over a time that is uniform with respect to the viscosity parameters when these are small. Then these solutions converge globally in space and strongly in L^2 towards the solution of the compressible isentropic Euler system when the viscosity parameters go to zero. Recently, there also have been extensive efforts on resolving this inviscid limit problem which lead to many results, for example, see [11-15].

Especially, Masnoudi [16] have shown that, without using the Prandtl equation, and in some particular domains such as the half-space, if the ratio of vertical viscosity to horizontal velocity also goes to zero, then all the weak solutions of the Navier-Stokes equations converge to the solution of the Euler system. In this paper, inspired by this work, we prove the following results.

Main results

*Theorem 1***.** Fix $T > 0$ and $s > 0$, and consider classical solutions (ρ, u, P), $(\bar{\rho}, \bar{u}, \bar{P}) \in L^\infty(0, T; H^s)$ of (1) and (2), with boundary conditions (3) and (4), respectively. Assume that the trace of Euler tangential velocity satisfies $U(x_1, t) \geq 0$, and that for all $\varepsilon > 0$ sufficiently small, the trace of Navier-Stokes vorticity satisfies $w|_{\partial\Omega} \geq 0$, for all $x_1 \in R$ and $t \in [0, T]$. Then:

$$\|\rho-\bar{\rho}\|^2_{L^\infty(0,T;L^2(\Omega))} + \|u-\bar{u}\|^2_{L^\infty(0,T;L^2(\Omega))} \to 0 \tag{7}$$

holds as $\varepsilon \to 0$.

Theorem 2. Fix $T > 0$ and s > 0, and consider classical solutions (ρ, u, P), $(\bar{\rho}, \bar{u}, \bar{P}) \in L^\infty(0, T; H^s)$ of (1) and (2), with boundary conditions (3) and (4), respectively. Let $\sigma(t) = \min\{t, 1\}$ and let M_ε be a positive function which satisfies:

$$\int_0^T M_\varepsilon(t)\mathrm{d}t \to 0 \ \text{ as } \ \varepsilon \to 0 \tag{8}$$

Define the boundary-layer Γ_ε:

$$\Gamma_\varepsilon = \left\{(x_1, x_2) \in \Omega : 0 < x_2 \leq \frac{\varepsilon\sigma(t)}{C}\ln\left(\frac{C}{M_\varepsilon(t)\sigma(t)}\right)\right\} \tag{9}$$

where

$$C = C(\|\rho, \nabla\rho, u\|_{L^\infty(0,T;L^\infty)}, \|\bar{u}\|_{L^\infty(0,T;H^s)}) > 0$$

is a sufficiently large fixed positive constant. Assume that there is no back-flow in the trace of the Euler tangential velocity, *i.e.*:

$$U(x_1,t)\geq 0 \tag{10}$$

for all $x_1 \in$ R and $t \in [0, T]$, and that for all ε sufficiently small, the *very negative part* of the Navier-Stokes vorticity satisfies:

$$\varepsilon^{(r-1)/r}\left\| w(x_1,x_2,t)+\frac{M_\varepsilon(t)}{\varepsilon}\right\|_{L^r(\Gamma_\varepsilon(t))}\leq \sigma(t)^{1/r}M_\varepsilon(t) \tag{11}$$

for some $1\leq r\leq\infty$ and all $t\in[0, T]$, where $f_-=\min[f, 0]$. Then the inviscid limit:

$$\|\rho-\overline{\rho}\|_{L^\infty(0,T;L^2(\Omega))}+\|u-\overline{u}\|_{L^\infty(0,T;L^2(\Omega))}\to 0 \tag{12}$$

holds, with the rate of convergence:

$$\|\rho-\overline{\rho}\|_{L^\infty(0,T;L^2(\Omega))}+\|u-\overline{u}\|^2_{L^\infty(0,T;L^2(\Omega))}=O\left(N(\varepsilon)T+\int_0^T M_\varepsilon(t)\mathrm{d}t\right) \tag{13}$$

as $\varepsilon\longrightarrow 0$, for $N(\varepsilon)=O(\varepsilon)$ when $r>1$ in eq. (11) and $N(\varepsilon)=O(\varepsilon^\delta)$, $\forall\delta\in(0, 1)$ when $r=1$ in eq. (11).

Remark 1. From the results in [10], the uniform estimates of $\|\rho,\nabla\rho, u\|_{L^\infty(O,T;\,L^\infty)}$ are reasonable.

Remark 2. Theorems 1 and *2* are also holds for a bounded domain with a smooth boundary. The only difference between the half space and the bounded domain with a smooth boundary is that we need to choose a local compactly supported boundary-layer corrector.

For simplicity, we denote:

$$\int_\Omega f dx=\int f \ \text{ and } \int_{\partial\Omega} f\mathrm{d}x=\int_{\partial\Omega} f$$

and L^2 norm $\|u\|_{L^2(\Omega)}=\|u\|$.

Proof of *Theorem 1*

Let $\phi=\rho-\overline{\rho}$, $q=P-\overline{P}$ and $v=u-\overline{u}$ be the differences of density, pressure and velocity, respectively. Then we can get the equation about ϕ, q, v:

$$\begin{gathered}\phi_t+v\cdot\nabla\rho+\overline{u}\cdot\nabla\phi=0\\ \rho\partial_t v+\rho v\cdot\nabla v+\rho v\cdot\nabla\overline{u}+\rho\overline{u}\cdot\nabla v+\nabla q-\varepsilon\Delta u+\phi(\partial_t\overline{u}+\overline{u}\cdot\nabla\overline{u})=0,\ \ x\in\Omega\\ \mathrm{div}v=0\end{gathered} \tag{14}$$

with boundary conditions:

$$v_1\big|_{\partial\Omega}=-U,\ \ v_2\big|_{\partial\Omega}=0 \tag{15}$$

and the initial condition:

$$v(t=0,x)=0 \tag{16}$$

Next, we will prove *Theorem 1* by estimating the solution of the system (14)-(16). Computing $\int(14)_1\phi\mathrm{d}x$, we have:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\phi\|^2+\int v\nabla\rho\phi+\int\overline{u}\frac{1}{2}\nabla\phi^2=0$$

Then by the Young's inequality and properties of solutions (ρ, u, P) and $(\overline{\rho}, \overline{u}, \overline{P})$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\phi\|^2\leq C\left(\|v\|^2+\|\phi\|^2\right) \tag{17}$$

Computing $\int(14)_2 v\mathrm{d}x$, we have:

$$\int\left[\rho\partial_t v+\rho v\cdot\nabla v+\rho v\cdot\nabla\overline{u}+\rho\overline{u}\cdot\nabla v+\nabla q-\varepsilon\Delta u+\phi(\partial_t\overline{u}+\overline{u}\cdot\nabla\overline{u})\right]\mathrm{d}x=0 \tag{18}$$

Observe that if $U\geq 0$ and $w|_{\partial\Omega}\geq 0$, then we estimate:

$$-\varepsilon\int\Delta uv\mathrm{d}x=\varepsilon\int\nabla u\nabla v-\varepsilon\int_{\partial\Omega}\partial_2 u_1 v_1=\varepsilon\|\nabla v\|^2+\varepsilon\int\nabla\overline{u}\nabla v-\varepsilon\int_{\partial\Omega}wU\leq\frac{\varepsilon}{2}\|\nabla v\|^2+C\varepsilon\|\nabla\overline{u}\|^2 .$$

Furthermore, by direct calculus, one can obtain:

$$\int\rho\partial_t vv+\rho v\cdot\nabla vv\leq\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\rho|v|^2+\frac{1}{2}\int\overline{u}\cdot\nabla\rho|v|^2$$

$$\int\rho v\cdot\nabla\overline{u}v\leq C\|v\|^2,\ \int\rho\overline{u}\cdot\nabla vv\leq C\|v\|^2$$

$$\int\nabla qv=0,\ \int\phi\left(\partial_t\overline{u}+\overline{u}\cdot\nabla\overline{u}\right)v\leq C\left(\|\phi\|^2+\|v\|^2\right)$$

The combination of eq. (18) and the aforementioned inequality implies:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|v\|^2\leq C\varepsilon+C\left(\|\phi\|^2+\|v\|^2\right) \tag{19}$$

From eqs. (17) and (18), immediately we have:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\phi\|^2+\|v\|^2\right)\leq C\varepsilon+C\left(\|\phi\|^2+\|v\|^2\right) \tag{20}$$

Then by the Gronwall inequality and the initial condition, we get:

$$\|\phi\|^2+\|v\|^2\leq C\varepsilon t \tag{21}$$

This completes the proof.

Proof of *Theorem 2*

Constructing a suitable boundary-layer corrector, φ, to account for the mismatch between the Euler and Navier-Stokes boundary conditions is a very important method for the proof of the zero viscosity limit [3]. For instance, Gie and Kelliher [6] proved the convergence, as ε tends to zero, of the Navier-Stokes solutions to the Euler solution both in the natural energy norm and uniformly in time and space by constructing an explicit corrector. Kelliher [17] gave a brief comparison of various correctors and established necessary and sufficient conditions for solutions to the Navier-Stokes equations with Dirichlet boundary conditions to converge in a strong sense to a solution the Euler equations if the viscosity is taken to be zero.

Now firstly, we begin with the construction of the corrector as follows.

The boundary-layer corrector. Choose $\psi:[0,\infty)\to[0,\infty)$ to be a C^∞ function, supported in [1/2, 4], which is non-negative and has mass $\int\psi(z)\mathrm{d}z=1$. Recall that $\sigma(t)=\min\{t,1\}$.

For $\alpha\in(0,1]$, to be chosen later, we introduce:

$$\varphi(x_1,x_2,t)=(\varphi_1,\varphi_2)(x_1,x_2,t)$$

where

$$\varphi_1=-U(x_1,t)\left[\mathrm{e}^{-x_2/\alpha\sigma(t)}-\alpha\sigma(t)\psi(x_2)\right] \tag{22}$$

$$\varphi_2(x_1,x_2,t)=\alpha\sigma(t)\partial_1 U(x_1,t)\left[\left(1-\int_0^{x_2}\psi(y)\mathrm{d}y\right)-\mathrm{e}^{-x_2/\alpha\sigma(t)}\right] \tag{23}$$

and

$$\varphi(x_1, x_2, 0) = \varphi_0(x_1, x_2)$$

The corrector φ has following properties.

Lemma 1.

- $\varphi_1(x_1, 0, t) = -u(x_1, t)$, $\varphi_2(x_1, 0, t) = 0$, so $\overline{u} + \varphi = 0$ on $\partial\Omega$ and $\varphi_1 \longrightarrow 0$ as $x_2 \longrightarrow \infty$ exponentially,
- $\mathrm{div}\varphi = 0$, and
- following estimates hold for $1 \le p \le \infty$:

$$\|\varphi_1\|_{L^p} + \|\partial_1\varphi_1\|_{L^p} \le C(\alpha\sigma)^{1/p},\ \|\varphi_2\|_{L^p} + \|\partial_1\varphi_2\|_{L^p} \le C\alpha\sigma$$

Energy equation. We can rewrite the second equation in [14] as:

$$\rho\partial_t(v-\varphi) + \rho v\cdot\nabla v + \rho v\cdot\nabla\overline{u} + \rho\overline{u}\cdot\nabla v + \nabla q - \varepsilon\Delta u + \phi(\partial_t\overline{u} + \overline{u}\cdot\nabla\overline{u}) + \rho\partial_t\varphi = 0 \tag{24}$$

Since $v - \varphi = 0$ on $\partial\Omega$, we may multiply eq. (24) by $v - \varphi$ and integrate by parts to obtain:

$$\int\left(\rho\partial_t(v-\varphi) + \rho v\cdot\nabla u + \rho\overline{u}\cdot\nabla v + \nabla q - \varepsilon\Delta u + \phi(\partial_t\overline{u} + \overline{u}\cdot\nabla\overline{u}) + \rho\partial_t\varphi\right)(v-\varphi) = 0$$

Next, we will estimate all previous terms:

$$\int\rho\partial_t(v-\varphi)(v-\varphi) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\rho(v-\varphi)^2 + \frac{1}{2}\int u\cdot\nabla\rho(v-\varphi)^2 \tag{25}$$

$$\begin{aligned}\int\rho v\cdot\nabla u(v-\varphi) &= \int\rho(v-\varphi)\cdot\nabla u(v-\varphi) + \int\rho\varphi\cdot\nabla u(v-\varphi) = \\ &= -\frac{1}{2}\int v\cdot\nabla\rho|v-\varphi|^2 + \int\rho(v-\varphi)\cdot\nabla\overline{u}(v-\varphi) + \int\rho\varphi\cdot\nabla\overline{u}(v-\varphi)\end{aligned} \tag{26}$$

From eqs. (25) and (26), we have:

$$\rho\partial_t(v-\varphi)(v-\varphi) + \int\rho v\cdot\nabla u(v-\varphi) \ge \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\rho(v-\varphi)^2 - \|v-\varphi\|^2 + \int\rho\varphi\cdot\nabla\overline{u}(v-\varphi)$$

$$\int\rho\overline{u}\cdot\nabla v(v-\varphi) = \int\rho\overline{u}\cdot\nabla(v-\varphi)(v-\varphi) + \int\rho\overline{u}\cdot\nabla\varphi(v-\varphi),\ \int\nabla q(v-\varphi) = 0 \tag{27}$$

$$-\int\varepsilon\Delta u(v-\varphi) = \varepsilon\int\nabla u\nabla(v-\varphi) = \varepsilon\int|\nabla u|^2 - \varepsilon\int\nabla u\nabla\overline{u} - \varepsilon\int\nabla u\nabla\varphi \le \frac{3}{4}\varepsilon\int|\nabla u|^2 - C\varepsilon - \varepsilon\int\nabla u\nabla\varphi$$

Then we have:

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\rho|v-\varphi|^2 + \frac{3}{4}\varepsilon\|\nabla u\|^2 \le \\ &\le C\|v-\varphi\|^2 + \varepsilon\int\nabla u\nabla\varphi + \int\left(\rho\varphi\cdot\nabla\overline{u} - \rho\overline{u}\cdot\nabla\varphi - \phi(\partial_t\overline{u} + \overline{u}\cdot\nabla\overline{u}) - \rho\partial_t\varphi\right)(v-\varphi) = \\ &= C\|v-\varphi\|^2 + I_1 + I_2\end{aligned} \tag{28}$$

Firstly, we deal with $I_1 = \varepsilon\int\nabla u\nabla\varphi$.

$$I_1 = \varepsilon\int\nabla u\nabla\varphi = \varepsilon\int\left(\partial_1 u_1\partial_1\varphi_1 + \partial_1 u_2\partial_1\varphi_2 + \partial_2 u_1\partial_2\varphi_1 + \partial_2 u_2\partial_2\varphi_2\right) \tag{29}$$

From *Lemma 1*, we see:

$$\varepsilon\int\partial_1 u_1\partial_1\varphi| \le \frac{\varepsilon}{8}\|\partial_1 u_1\|^2 + C\varepsilon\alpha\sigma \tag{30}$$

$$\varepsilon\int\partial_1 u_2\partial_1\varphi_2 \le \frac{\varepsilon}{8}\|\partial_1 u_2\|^2 + C\varepsilon\alpha\sigma \tag{31}$$

$$\varepsilon\int\partial_2 u_2\partial_2\varphi_2 \le \frac{\varepsilon}{8}\left\|\partial_2 u_2\right\|^2 + C\varepsilon\alpha\sigma \tag{32}$$

For the term $\varepsilon\int\partial_2 u_1\partial_2\varphi_1$, observe that $w = \partial_1 u_2 - \partial_2 u_1$. Thus, for some $\beta \in (\alpha, 1/4)$ and $M > 0$, we use the bound:

$$w(x_1,x_2,t) \ge -\frac{M}{\varepsilon} + \tilde{w}(x_1,x_2,t),(x_1,x_2) \in \varGamma_\beta = R\cdot(0,\beta), t\in[0,T] \tag{33}$$

where

$$\tilde{w}(x_1,x_2,t) = \min\left\{w(x_1,x_2,t) + \frac{M}{\varepsilon}, 0\right\} \le 0$$

Then we can decompose:

$$\varepsilon\int\partial_2 u\partial_2\varphi = -\varepsilon\int_{\Gamma_\beta} w\partial_2\varphi - \varepsilon\int_{\Gamma_\beta^C} w\partial_2\varphi + \varepsilon\int\partial_1 u_2\partial_2\varphi = J_1 + J_2 + J_3$$

By the construction of corrector, φ,we have the explicit formula:

$$\partial_2\varphi = \frac{1}{\alpha\sigma}U(x_1,t)\mathrm{e}^{-x_2/\alpha\sigma} - \alpha\sigma U(x_1,t)\psi'(x_2) \tag{34}$$

for all $(x_1, x_2) \in \varOmega$ and $t \in [0, T]$. Using the no-back-flow condition $U \ge 0$ and the bound eq. (30), for $r \in [1, \infty]$, we have:

$$\begin{aligned} J_1 &= -\frac{\varepsilon}{\alpha\sigma}\int_{\Gamma_R} wU(x_1,t)\mathrm{e}^{-x_2/\alpha\sigma} + \varepsilon\alpha\sigma\int_{\Gamma_R} U\psi' \le \\ &\le \frac{M}{\alpha\sigma}\int_{x_2<\beta} U(x_1,t)\mathrm{e} - x_2/\alpha\sigma + \frac{v\mathrm{e}}{\alpha\sigma}\int_{x_2<\beta}(-\tilde{w}U\mathrm{e}^{-x_2/\alpha\sigma}) + \varepsilon\alpha\sigma\int_{x_2<\beta} wU\psi' \le \\ &\le CM(1-\mathrm{e}^{-\beta/\alpha\sigma}) + \frac{C\varepsilon}{\alpha\sigma}\left\|\tilde{w}\right\|_{L^r(\Gamma_\beta)}\left[\frac{(r-1)\alpha\sigma}{r}(1-\mathrm{e}^{-r\beta/(r-1)\alpha\sigma})\right]^{(r-1)/r} + C\varepsilon\alpha\sigma\left\|\nabla u\right\| + C\varepsilon(\alpha\sigma)^2 \le \\ &\le \frac{\varepsilon}{12}\left\|\nabla u\right\|^2 + CM + C\varepsilon(\alpha\sigma)^{-1/r}\left\|\tilde{w}\right\|_{L^r(\Gamma_\beta)} \end{aligned} \tag{35}$$

For J_2:

$$\begin{aligned} J_2 &= -\varepsilon\int_{x_2>\beta} wU\left(\frac{1}{\alpha\sigma}\mathrm{e}^{-x_2/\alpha\sigma} - \alpha\sigma\psi'(x_2)\right) \le \\ &\le C\frac{\varepsilon\mathrm{e}^{-\beta/\alpha\sigma}}{(\alpha\sigma)^{1/2}}\left\|\nabla u\right\| + \varepsilon\alpha\sigma\left\|\nabla u\right\| \le \frac{\varepsilon}{12}\left\|nau\right\|^2 + \frac{C\varepsilon}{\alpha\sigma}\mathrm{e}^{-2\beta/\alpha\sigma} + C\varepsilon(\alpha\sigma)^2 \end{aligned} \tag{36}$$

Lastly, using the Yang's inequality and *Lemma 1*, one has:

$$J_3 \le \frac{\varepsilon}{12}\left\|\nabla u\right\|^2 + C\varepsilon(\alpha\sigma) \tag{37}$$

Combining eqs. (29), (30)-(32), and eqs. (35)-(37), we have:

$$\varepsilon\int\nabla u\nabla\varphi \le \frac{\varepsilon}{2}\left\|\nabla u\right\|^2 + CM + C\varepsilon(\alpha\sigma)^{-1/r}\left\|\tilde{w}\right\|_{L^r(\Gamma_\beta)} + \frac{C\varepsilon}{\alpha\sigma}\mathrm{e}^{-2C\beta/\alpha\sigma} + C\varepsilon\alpha\sigma \tag{38}$$

Next, we deal with I_2 in eq. (28). By the properties in *Lemma 1*, we have:

$$\int(\rho\varphi\cdot\nabla\bar{u} - \phi(\partial_t\bar{u} + \bar{u}\cdot\nabla\bar{u}) - \rho\partial_t\varphi)(v-\varphi) \le C\left\|v-\varphi\right\|^2 + C\left\|\phi\right\|^2 + C\alpha^2 + C\alpha\sigma \tag{39}$$

For the term $\int\rho\overline{u}\cdot\nabla\varphi(v-\varphi)$, we see that:

$$\begin{aligned}\int\rho\overline{u}\cdot\nabla\varphi(v-\varphi)&=-\int\rho\frac{1}{2}\overline{u}\cdot\nabla|\varphi|^2+\int\rho\overline{u}\cdot\nabla\varphi(u-\overline{u})\\&\le\frac{1}{2}\int|\varphi|^2\operatorname{div}(\rho\overline{u})-\int\overline{u}\otimes\varphi:\nabla[(\rho(u-\overline{u})]\\&\le C\alpha\sigma-\int\overline{u}\otimes\varphi:[\nabla\rho u+\rho\nabla u-\nabla(\rho u)]\\&\le C\alpha\sigma-\int\rho\overline{u}\otimes\varphi:\nabla u\le C\alpha\sigma-\int\rho\overline{u}\cdot\nabla\varphi u\end{aligned}\tag{40}$$

Using the basic energy inequality, we can arrive:

$$\|u\|\le|u_0|\le C\tag{41}$$

Hence, combination of this inequality and eq. (40) implies:

$$\int\rho\overline{u}\nabla\varphi(v-\varphi)\le C\alpha\sigma\tag{42}$$

Then it follows from eqs. (39)-(42) that:

$$I_2\le C\|v-\varphi\|^2+C\|\phi\|^2+C\alpha^2+C\alpha\sigma\tag{43}$$

Now from eqs. (28), (38), and (43), we see that:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|v-\varphi\|^2&\le C\|v-\varphi\|^2+C\|\phi\|^2+C\alpha^2+C\alpha\sigma+\\&+CM+C\varepsilon(\alpha\sigma)^{-1/r}\|\tilde{w}\|_{L^r(\Gamma_\beta)}+\frac{C\varepsilon}{\alpha\sigma}\mathrm{e}^{-2C\beta/\alpha\sigma}+C\varepsilon\alpha\sigma\end{aligned}\tag{44}$$

At last, from eq. $(14)_1$, we can rewrite it:

$$\phi_t+(v-\varphi)\cdot\nabla\rho+\overline{u}\cdot\nabla\phi+\varphi\cdot\nabla\rho=0\tag{45}$$

Then, computing $\int(45)\cdot\phi$, we have:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\phi\|^2\le C\|\phi\|^2+\|v-\varphi\|^2+C\alpha\sigma\tag{46}$$

therefore by eqs. (44) and (46), we obtain that:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\phi\|^2+\|v-\varphi\|^2\right)&\le C\left(\|v-\varphi\|^2+\|\phi\|^2\right)+C\alpha^2+C\alpha\sigma+CM+\\&+C\varepsilon(\alpha\sigma)^{-1/r}\|\tilde{w}\|_{L^r(\Gamma_\beta)}+\frac{C\varepsilon}{\alpha\sigma}\mathrm{e}^{-2C\beta/\alpha\sigma}+C\varepsilon\alpha\sigma\end{aligned}\tag{47}$$

Choose $\alpha=\varepsilon^\gamma$, $\gamma\in(0,1)$:

$$\beta=\frac{\varepsilon\sigma}{2C}\ln\left(\frac{1}{M\sigma}\right)\tag{48}$$

Then from eq. (47), we have:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\phi\|^2+\|v-\varphi\|^2\right)\le C\left(\|v-\varphi\|^2+\|\phi\|^2\right)+CN(\varepsilon)+CM\tag{49}$$

where $N(\varepsilon)\to0$ as $\varepsilon\to0$. By the Gronwall inequality, we get:

$$\|\phi\|^2+\|v\|^2\le\|\phi\|^2+\|v-\varphi\|^2+\|\varphi\|^2\le CN(\varepsilon)\sigma+CTe^{CT}\left(N(\varepsilon)T+\int_0^t M(s)\mathrm{d}s\right)\tag{50}$$

This completes the proof.

Conclusion

In this paper, the convergence in L^2 norm, uniformly in time of the inhomogeneous Navier Stokes system and inhomogeneous Euler equations are discussed. More precisely, with the assumption of the Oleynick conditions of no back-flow in the trace of the Euler flow, and of a lower bound for the Navier-Stokes vorticity in a Kato-like boundary-layer, the inviscid limit holds when the viscosity parameters go to zero and the rate of convergence is also given.

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Nomenclature

P, $\bar{P}$ – pressure, [Nm^{-2}]
$u = (u_1 - u_2)$ – velocity, [ms^{-1}]
$\bar{u} = (\bar{u}_1 - \bar{u}_2)$ – velocity, [ms^{-1}]

Greek symbols

Γ – boundary of Ω, [m]
ρ, $\bar{\rho}$ – density, [kgm^{-3}]
Ω – domain, [m^2]

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