EXPLORING MULTIPLE AND SINGULAR SOLITON SOLUTIONS FOR NEGATIVE-ORDER SPACE-TIME FRACTIONAL mKdV EQUATIONS

by

Sulima ZUBAIR [a*], ***Hala AHMED*** [b], ***Batul MAHMOUD*** [b], ***Safa ELAMIN*** [c], ***Hind ATTI*** [c], ***and Bakri YOUNIS*** [d]

[a] Department of Mathematics, College of Sciences, Qassim University, Buraydah, Saudi Arabia
[b] Department of Mathematics, Applied College in Sarat Ubaida, King Khalid University, Abha, Saudi Arabia
[c] Department of Mathematics, Shaqra University, Shaqra, Saudi Arabia
[d] Department of Mathematics and Statistics, College of Science, Taif University, Taif, Saudi Arabia

Original scientific paper
https://doi.org/10.2298/TSCI2501359Z

This study on the negative-order fractional space-time modified KdV (nfmKdV) equation provides a comprehensive analysis of how fractional differentials affect the dynamics of solitons in non-linear wave models. We are referring to introduces the nfmKdV equation, a significant extension of the traditional KdV equation, which is commonly used to model wave propagation in non-linear dispersive media. By developing both focusing and defocusing solutions and employing the Hirota technique to construct multisoliton solutions, the study opens new avenues for the exploration of fractional wave equations in diverse physical contexts. The use of fractional calculus, and specifically negative-order derivatives, enhances the model's ability to describe real-world phenomena with long-range interactions and memory effects, offering significant potential for future research in non-linear and fractional dynamics. This newly established result warrants further investigation determine its applicability to other non-linear fractional order models, and other existing methods may be employed to explore this new development. As the fractional order approaches one, the results align with well-established findings in the literature. This study provides a deeper understanding of the dynamics of solitons in fractional media, which could be useful for modelling soliton propagation in systems where traditional integer-order models fail to capture essential behavior.

Key words: *fractional mKdV model, fractional differential models, Hirota technique*

Introduction

The recursion operator is essential in the analysis of integrable equations, especially in the context of (1+1)-D non-linear PDE. It provides a systematic method for generating an infinite sequence of symmetries, which are essential for understanding the integrability of the system. Through iterative application, the recursion operator generates higher-order symmetries, leading to exact solutions and conserved quantities. This infinite hierarchy of symmetries is key to the phenomenon of solitons, whose shape is preserved during interactions. Additionally, the recursion operator serves as a powerful tool for extending the study of integrability to

* Corresponding author, e-mail: s.zebir@qu.edu.sa

higher dimensions, enabling the modelling of more complex natural phenomena in fields like plasma physics, fluid dynamics, and non-linear optics [1-9].

The recursion operator was first introduced by Olver [4], who demonstrated that applying this operator to a known symmetry produces a new, higher-order symmetry, forming an infinite hierarchy of symmetries. Later, Magri [6] expanded this idea by introducing bi-Hamiltonian systems, which possess two compatible Hamiltonian structures crucial for integrability. Verosky [8] further developed Olver's theory by considering negative-order equations, leading to a broader framework for solving generalized integrable systems. The application of recursion operators and their role in generating infinite symmetries have had profound implications in the theory of non-linear PDE, offering new insights into both integrable and non-integrable systems. This framework has greatly advanced research in mathematical physics, particularly in areas such as fluid dynamics and non-linear optics [10-14].

Recently, there has been an increasing interest in functional differential equation (FDE) because of their wide-ranging applications in engineering and physics. Abundant important phenomena in fields such as acoustics, electromagnetics, electrochemistry, viscoelasticity, and materials science are effectively modeled by FDE. However, solving these FDE can be quite challenging. Typically, there is no universal techniques that produces explicit and numerical solution for non-linear FDE [15-26]. In this framework, fractional differentials are considered in the conformable sense. Non-integer calculus encompasses the idea of differentials and integrals of random order, effectively unifying and generalizing the principles of integer-order derivatives and repeated integrals. Numerous books have been published on non-integer calculus, discussing numerous classifications of non-integer differentiation and integration, including those by Riemann-Liouville (RL), Grunwald-Letnikov, Caputo, and the modified RL. For the purposes of this study, we will employ the conformable fractional differential (CFD). Khalil *et al.* [27] presented the CFD in the limit form:

$$D^{\alpha}\psi(s)=\lim_{\varepsilon\to 0}\frac{\psi(s+\varepsilon s^{1-\alpha})-\psi(s)}{\varepsilon},\ \psi^{(\alpha)}(0)=\lim_{s\to 0^{+}}\psi^{(\alpha)}(s)$$

with $s > 0$ and $\alpha \in (0, 1]$, since $\psi^{(\alpha)}(0)$ is not determined. This fractional differential goes back to the famous integer differential at $\alpha = 1$. The corresponding CFD satisfies the will know axioms of differentiation see [20-26].

The nfmKdV equation is a generalized form of the standard KdV equation, where the traditional time and spatial derivatives are replaced by fractional derivatives of negative order. This concept arises from the theory of fractional calculus, which allows the order of differentiation be non-integer, providing a more flexible and powerful framework to model complex systems, particularly those exhibiting memory effects, anomalous diffusion, or long-range interactions. In classical models like the KdV equation, the dynamics are governed by local interactions, with integer-order derivatives capturing the change of quantities at specific points in space and time. However, fractional derivatives can model situations where the effects at a given point depend not only on the immediate local surroundings but also on distant past states or long-range interactions. The negative-order aspect of the fractional derivatives is particularly important. In fractional calculus, negative orders can model processes that *retrace* or involve *backward* influences, such as a response that decays or grows in an unusual, non-exponential manner. This is key to understanding the complex dynamics of wave propagation, particularly when dealing with phenomena like inverse dispersion or non-locality.

Linking recursion operators to fractional derivatives

In the classical theory of integrable systems, recursion operators serve as tools for generating higher-order symmetries of evolution equations, often leading to the construction of soliton solutions. These operators are typically integer-order differential operators, but when fractional derivatives are introduced into the framework, the concept of recursion operators must be generalized to handle these non-integer orders. Fractional derivatives extend the conventional concept of differentials to fractional orders, offering a more nuanced description of physical processes characterized anomalous diffusion, memory effects, or non-local behavior. By applying fractional calculus to non-linear evolution equations, we can derive fractional evolution equations, which generalize classical models (such as the KdV equation) to account for more complex dynamics.

Fractional recursion operators

To incorporate fractional derivatives into the theory of recursion operators, we consider fractional recursion operators-operators that, when applied to a symmetry of a fractional evolution equation, produce a new symmetry. These operators are typically defined in terms of fractional derivatives (of the form D^{α}, where α is a fractional order) rather than integer derivatives. The key idea is that, like their integer-order counterparts, fractional recursion operators generate an infinite sequence of symmetries, which is crucial for the complete integrability of fractional evolution systems. For example, the fractional KdV (fKdV) equation, which includes fractional derivatives, the recursion operator $\Omega(\psi)$ might be modified to accommodate these fractional orders. The operator would take a symmetry $\psi(t, x)$ and generate a new symmetry of the fractional equation, incorporating fractional derivatives throughout the process. This is a direct extension of the classical recursion operator theory, where the differential operators are replaced with fractional operators.

Negative-order fractional equations

Similar to the extension of recursion operators to negative orders as introduced by Verosky [8], we can extend the concept to negative-order fractional equations. These equations would involve fractional derivatives of negative order $D^{-\alpha}$, such as, and could be generated by applying the fractional recursion operator to lower-order symmetries. These negative-order fractional equations could describe phenomena such as fractional diffusion or wave propagation with memory effects. Just as recursion operators in the classical sense allow the generation of higher-order symmetries, fractional recursion operators would generate an infinite sequence of symmetries for fractional evolution equations. These symmetries, in turn, can be used to construct exact solutions, including fractional solitons or singular fractional soliton solutions, analogous to the classical case but with more complex, fractional behaviors.

Fractional Hamiltonian structures and multi-Hamiltonian systems

The introduction of fractional derivatives into the framework of recursion operators may also affect the Hamiltonian structure of the system. As we saw with bi-Hamiltonian systems in the classical case [6], where two compatible Hamiltonian structures lead to the integrability of the system, fractional derivatives might result in fractional bi-Hamiltonian systems. In such systems, the fractional recursion operator could be related to two different but compatible fractional Hamiltonian formulations of the equation. The introduction of fractional derivatives into the framework of recursion operators may also affect the Hamiltonian structure of the system. As we saw with bi-Hamiltonian systems in the classical case [6], where two compatible Hamiltonian structures lead to the integrability of the system, fractional derivatives might result in fractional bi-Hamiltonian systems. In such systems, the fractional recursion operator could be related to two different but compatible fractional Hamiltonian formulations of the equation.

Generalizing the Painleve test and soliton solutions

In the context of fractional evolution equations, we can also extend the Painleve test and soliton solutions analysis to fractional models. The Painleve test, which is used to check the integrability of equations by determining the singularity structure of their solutions, can be adapted to handle fractional derivatives. Applying the fractional recursion operator could reveal new types of fractional soliton solutions (such as multi-solitons and singular fractional solitons) which are the natural analogs of the classical solitons but exhibit fractional characteristics such as non-local interactions or long-tail behaviors. The incorporation of fractional derivatives into the framework of integrable systems allows us to explore more general solution types, and the fractional recursion operator becomes an essential tool in deriving these solutions.

Applications in physics and engineering

The introduction of fractional derivatives into integrable models, through fractional recursion operators, opens up new avenues for modelling physical phenomena that exhibit non-local effects, anomalous diffusion, or memory. Examples include, fractional diffusion models, these models describe processes in materials with complex, non-local properties, such as those found in viscoelasticity, porous media, or disordered systems. Plasma physics and fluid dynamics: fractional differential systems have been used to model waves and turbulence in plasmas, where non-local interactions are present. Non-linear optics, the study of light propagation in non-linear media with memory effects can be modeled using fractional evolution equations, with fractional recursion operators providing a means to generate symmetries and exact solutions.

Linking the previous results on recursion operators and integrable systems to fractional derivatives involves extending the classical theory of recursion operators to accommodate fractional calculus. Fractional recursion operators generate infinite sequences of symmetries for fractional evolution equations, mirroring the classical case but with the added complexity of fractional derivatives. This extension not only broadens the scope of integrable systems but also opens new areas of research in mathematical physics, where fractional derivatives are increasingly used to model complex, non-local phenomena. By incorporating fractional recursion operators, we can derive new classes of soliton solutions, explore bi-Hamiltonian structures, and apply these ideas to a variety of physical models involving fractional dynamics. A fractional hereditary symmetry $\Omega(\psi)$ is a fractional operator function $\psi(t, x)$ that produces a hierarchical set of fractional evolution systems, with $\Omega(\psi)$ acting as a recursion operator within this hierarchy. Specifically, $\Omega(\psi)$ serves as the recursion operator for the corresponding series of fractional evolution systems:

$$D_t^\alpha \psi = \Omega^n D_x^\alpha \psi,\ \ n = 0,1,2,.... \tag{1}$$

It is evident that this equation generates a family of (1+1)-D fractional sytems, with the particular form dependent on the significance of n. The fKdV equation, is obtained:

$$D_t^\alpha \varphi + 6\varphi D_x^\alpha \varphi + D_x^{\alpha\alpha\alpha} \varphi = 0 \tag{2}$$

with the recursion operator $\Omega(\varphi)$, defined:

$$\Omega(\varphi) = -D_x^{\alpha\alpha} - 4\varphi - 2\varphi D_x^{-\alpha} \tag{3}$$

where $\ell_{\alpha x}$ is the whole fractional differentials with respect to x, and $\ell_{\alpha x}^{-1}$ – the associated fractional integration operator.

It is widely recognized that the mfKdV models manifest in two distinct formulas: the focusing and the defocusing branches given:

$$D_t^\alpha \varphi + 6\varphi^2 D_x^\alpha \varphi + D_x^{\alpha\alpha\alpha}\varphi = 0 \tag{4}$$

$$D_t^\alpha \varphi - 6\varphi^2 D_x^\alpha \varphi + D_x^{\alpha\alpha\alpha}\varphi = 0 \tag{5}$$

with the corresponding recursion operators given:

$$\Omega_1 = D_x^{\alpha\alpha} - 4\varphi - 4\varphi D_x^{-\alpha} \tag{6}$$

$$\Omega_2 = -D_x^{\alpha\alpha} + 4\varphi + 4\varphi D_x^{-\alpha} \tag{7}$$

Correspondingly, the last terms in eqs. (6) and (7) represents an operator that takes a polynomial $P \in R(\varphi)$, multiplies it by φ, applies the converse operator $D_x^{-\alpha}$, and then multiplies the outcome by 4 $D_x^\alpha\varphi$. The recursion operators in eqs. (6) and (7) have been employed in the literature, especially for developing new equations in higher dimensions. Recall that eq. (1) represents:

$$D_t^\alpha \varphi = \Omega D_x^\alpha \varphi \tag{8}$$

Through the minas-order fractional hierarchy, we mention the sequence of equations characterized by progressively lower (more negative) orders of fractional derivatives:

$$D_t^\alpha \varphi = \Omega^{-1} D_x^\alpha \varphi \tag{9}$$

That is, the powers of Ω move in the reverse direction. In other words, the minas-order equation can be expressed:

$$\Omega D_t^\alpha \varphi = D_x^\alpha \varphi \tag{10}$$

Provided the fractional recursion operator is hereditary and supports an invertible implectic-symplectic separation, the negative-order fractional systems also reveal multi-Hamiltonian constructions [7]. It is worth noting that, for systems in (2+1)-D, such as the fractional KP equation, no standard recursion operator exists.

The Hirota technique is a well-known perturbative technique used to derive multi-soliton solutions for integrable equations. In this study, the Hirota technique proves to be an effective mathematical tool for constructing these multi-soliton solutions for the nfmKdV equation, both for the focusing and defocusing branches. The advantage of using the Hirota technique lies in its ability to handle complex non-linear equations by converting the problem into a series of simple algebraic equations. This method has proven highly efficient in generating exact multi-soliton solutions for a variety of non-linear PDE, especially in systems involving fractional derivatives.

The study distinguishes between two branches of the nfmKdV equation, corresponding to focusing and defocusing behaviors. These branches refer to the two types of wave interactions that can occur depending on the non-linear nature of the equation. This study utilizes the fractional recursion operator to derive the two negative-order forms of the nfmKdV equation: one for the focusing branch and another for the defocusing branch. These two branches correspond to different behaviors of the soliton solutions (focusing leading to possible singularities and defocusing leading to stable solitons).

The focusing branch of the negative-order modified fractional KdV equation

The nfmKdV equation for the focusing branch eq. (4) corresponds to the case where non-linearities cause the wave to concentrate or collapse in a finite time. In many physical systems, the focusing branch is associated with phenomena like soliton blow-up, where the wave amplitude grows indefinitely at a finite point in time. In the context of the nfmKdV equation, this branch describes the tendency of wave packets to focus and form singularities. First, we formulate the nfmKdV equation for the focusing branch eq. (4) by applying the negative-order fractional hierarchy:

$$\Omega D_t^\alpha \varphi = D_x^\alpha \varphi \tag{11}$$

with the recursion operator Ω for the focusing branch is provided in eq. (6). In essence, we utilize:

$$\left(-D_x^{\alpha\alpha} - 4\varphi - 4\varphi D_x^{-\alpha}\right) D_t^\alpha \varphi = D_x^\alpha \varphi \tag{12}$$

which yields:

$$-D_x^{\alpha\alpha} D_t^\alpha \varphi - 4\varphi^2 D_t^\alpha \varphi - 4D_x^\alpha \varphi D_x^{-\alpha}(\varphi D_t^\alpha \varphi) = D_x^\alpha \varphi \tag{13}$$

Thus, we can apply:

$$D_x^{-\alpha}(\varphi D_t^\alpha \varphi) = -\frac{D_x^{\alpha\alpha} D_t^\alpha \varphi + 4\varphi^2 D_t^\alpha \varphi + D_x^\alpha \varphi}{4D_x^\alpha \varphi} \tag{14}$$

To eliminate $D_x^{-\alpha}(\varphi D_t^\alpha \varphi)$ we take fractional differentiate for the two sides of eq. (13) with respect to x, resulting:

$$-D_x^{\alpha\alpha\alpha} D_t^\alpha \varphi - 4\varphi^2 D_x^\alpha D_t^\alpha \varphi - 12\varphi D_x^\alpha \varphi D_t^\alpha \varphi - 4D_x^{\alpha\alpha} \varphi D_x^{-\alpha}(\varphi D_t^\alpha \varphi) = D_x^{\alpha\alpha} \varphi \tag{15}$$

Replacing (14) into (15) results in the nmfKdV equation:

$$-D_x^\alpha \varphi D_x^{\alpha\alpha\alpha} D_t^\alpha \varphi - 4\varphi^2 D_x^\alpha \varphi D_x^\alpha D_t^\alpha \varphi - 12\varphi (D_x^\alpha \varphi)^2 D_t^\alpha \varphi + D_x^{\alpha\alpha} \varphi D_x^{\alpha\alpha} D_t^\alpha \varphi + 4\varphi^2 D_x^{\alpha\alpha} \varphi D_t^\alpha \varphi = 0 \tag{16}$$

Examining the fractional Painleve property of eq. (16)

The fractional Painleve property is an extension of the Painleve property to FDE. This means that for a given FDE, one looks for conditions under which the general solution has the same desirable behavior as in the classical Painleve property, the solutions should have no singularities other than isolated poles, even in the fractional context. Similar to the classical Painleve property, the solution should only exhibit poles as singularities, and these should be isolated, rather than having branch points, which are more characteristic of non-integrable systems. If a FDE possesses the fractional Painleve property, it suggests that the system might exhibit integrable behavior or solvable dynamics, similar to the classical integrable equations that are known to possess the Painleve property. Equation (16) is called to exhibit the fractional Painleve property if its solutions are 'single-valued' on arbitrary non-characteristic, movable singularity fractional manifolds. In other words, the solutions can be expressed as Laurent series:

$$\varphi = \sum_{s=0}^{\infty} \varphi_s \psi^{s+n}$$

with a sufficient number of arbitrary functions among φ_s in addition ψ. The Painleve property test, as outlined by the Weiss-Tabor-Carnevale technique, involves three key steps. Firstly, the order n and the coefficient φ_0 must be determined. For this purpose, we put $\varphi = \varphi_0 \psi^n$, into eq. (16). By balancing the non-linear and fractional dispersive terms, we obtain:

$$n = -1, \quad \varphi_0 = \pm i D_x^\alpha \psi, \quad i = \sqrt{-1}$$

The next stage is to determine the resonant points. Substituting:

$$\varphi = \varphi_0 \psi^{-1} + \varphi_j \psi^{j-1}$$

into eq. (16), along with eq. (17), and setting the coefficient of ψ^{j-7} equal to zero, we get:

$$D_t^\alpha \psi (D_x^\alpha \psi)^5 \varphi_j (j+1)(j-2)(j-3)(j-4) = 0 \tag{18}$$

from this, we observe that four resonant points happen at $j = -1$, 2, 3, and 4. The final step is to test the resonant conditions. We may suppose:

$$\varphi = \sum_{s=0}^{4} \varphi_s \psi^{s=1}$$

To simplify the calculations, we adopt Kruskal's ansatz:

$$\psi = \frac{x^\alpha}{\alpha} + \psi\left(\frac{t^\alpha}{\alpha}\right)$$

Thus, the coefficients φ_s will be functions of t only. By putting the expression (18) into eq. (16) and collecting the terms based on the different powers of ψ, we arrive:

$$\begin{gathered} \pm 12 i\varphi_1 D_t^\alpha \psi = 0,\ \pm 4iD_t^\alpha \varphi_1 = 0,\ 4\varphi_1(9\varphi_2 D_t^\alpha \psi - D_t^\alpha \varphi_1) = 0 \\ 4(18\varphi_1\varphi_3 D_t^\alpha \psi \pm 2i\varphi_1^2 D_t^\alpha \psi - 3\varphi_1 D_t^\alpha \varphi_2 - 7\varphi_2 D_t^\alpha \varphi_1) = 0 \end{gathered} \tag{19}$$

From the first equation we deduce that $\varphi_1 = 0$, so the reminder three equations are satisfied identically, indicating that φ_2, φ_3, and φ_4 are arbitrary functions of t.

Calculating multi-soliton solutions for eq. (16)

Solitons are wave solutions to non-linear equations that maintain their shape and speed during propagation, often described as *localized waves*. In this study, the researchers use the Hirota technique, a powerful perturbative technique, to construct multi-soliton solutions for the nfmKdV equation. To derive the dispersion relation, we set:

$$\varphi(t^\alpha, x^\alpha) = RD_x^\alpha \left\{ \tan^{-1}\left[f\, \frac{t^\alpha, x^\alpha}{g(t^\alpha, x^\alpha)} \right] \right\} \tag{20}$$

where the auxiliary functions $f(t^\alpha, x^\alpha)$ and $g(t^\alpha, x^\alpha)$, for the single soliton solutions, is given:

$$f(t^\alpha, x^\alpha) = 1 + e^{(kx^\alpha - \omega t^\alpha)/\alpha},\quad g(t^\alpha, x^\alpha) = 1 \tag{21}$$

Replacing eq. (20) into the nmfKdV (16) gives the dispersion relation by $R = 2$, and $\omega = -1/k$. The simplest soliton solutions where a single solitary wave travels without changing shape. This solution represents a single, localized disturbance in the medium, so the single soliton solutions take the formula:

$$\varphi(t^\alpha, x^\alpha) = \frac{2k e^{(kx^\alpha + t^\alpha/k)/\alpha}}{1 + e^{2(kx^\alpha + t^\alpha/k)/\alpha}} \tag{22}$$

For computing the two soliton solutions, we put the auxiliary function:

$$f(t^\alpha, x^\alpha) = e^{\theta_1} + e^{\theta_2},\ g(t^\alpha, x^\alpha) = 1 - a_{12}\, e^{\theta_1+\theta_2},\ \theta_i = \frac{\dfrac{k_i x^\alpha + t^\alpha}{k_i}}{\alpha},\quad i = 1, 2 \tag{23}$$

Using eq. (23) in eq. (20) and substituting the result in eq. (16), we obtain the following phase shift coefficient:

$$a_{12} = \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2} \tag{24}$$

and hence we set the phase shifts:

$$a_{ij} = \frac{\left(k_i - k_j\right)^2}{\left(k_i + k_j\right)^2}, \; 1 \le i \le j \le 3 \tag{25}$$

Combining eqs. (24) and (23) and substituting the outcome into eq. (20), we obtain the two-soliton solutions. In this case, two solitons interact with each other. This solution highlights the ability of solitons to collide without changing their form, a characteristic feature of integrable systems like the KdV equation. For the three-soliton solutions, we set:

$$f(t^\alpha, x^\alpha) = \mathrm{e}^{\theta_1} + \mathrm{e}^{\theta_2} + \mathrm{e}^{\theta_3} - b_{123}\,\mathrm{e}^{\theta_1+\theta_2+\theta_3}, \; g(t^\alpha, x^\alpha) = 1 - a_{12}\,\mathrm{e}^{\theta_1+\theta_2} - a_{23}\,\mathrm{e}^{\theta_2+\theta_3} - a_{13}\,\mathrm{e}^{\theta_1+\theta_3}$$
$$\theta_i = \frac{\dfrac{k_i x^\alpha + t^\alpha}{k_i}}{\alpha}, \; i = 1,2,3 \tag{26}$$

Proceeding as before, we find that:

$$b_{123} = a_{12} a_{23} a_{13} \tag{27}$$

This indicates that three-soliton solutions are attainable. The three-soliton solutions is a more complex interaction, where three solitary waves interact in a similar manner to the two-soliton solutions. This demonstrates the non-linear interaction between multiple waves, where the solitons can pass through one another while retaining their identity. The presence of three-soliton solutions typically suggests the integrability of the equation being studied, and *N*-soliton solutions can be derived for any finite *N,* where *N > 1*. The general *N*-soliton solutions provides a uniform formula that describes the interaction of an arbitrary number of solitons. This is a powerful result, as it allows the exact dynamics of multiple solitons to be calculated efficiently. The uniformity of this formula demonstrates the robustness of the Hirota method in handling complex interactions in fractional equations. However, integrability must be verified using additional methods.

The defocusing branch of the negative-order mfKdV equation

For the defocusing branch, the study also derives multiple singular soliton solutions. These solutions may correspond to *blow-up* or singularities in the system, where the amplitude of the wave tends to infinity under certain conditions. These types of solutions are significant in understanding phenomena like wave collapse or extreme wave events in non-linear media. We derive the nmfKdV equation for the defocusing branch:

$$D_t^\alpha \varphi - 6\varphi^2 D_x^\alpha \varphi + D_x^{\alpha\alpha\alpha} \varphi = 0 \tag{28}$$

this describes the opposite behavior, where the wave tends to spread out or disperse over time. For soliton solutions, this would correspond to waves maintaining their shape and speed without collapsing. The defocusing case is typically associated with stable wave propagation, where solitons retain their form over long periods. By applying the negative-order fractional hierarchy:

$$\Omega D_t^\alpha \varphi = D_x^\alpha \varphi \tag{29}$$

with the recursion operator Ω for the focusing branch is provided in eq. (7). In essence, we utilize:

$$\left(-D_x^{\alpha\alpha} + 4\varphi + 4\varphi D_x^{-\alpha}\right) D_t^\alpha \varphi = D_x^\alpha \varphi \tag{30}$$

which yields:

$$-D_x^{\alpha\alpha} D_t^\alpha \varphi + 4\varphi^2 D_t^\alpha \varphi + 4 D_x^\alpha \varphi D_x^{-\alpha} (\varphi D_t^\alpha \varphi) = D_x^\alpha \varphi \tag{31}$$

Thus, we can apply:

$$D_x^{-\alpha}(\varphi D_t^\alpha \varphi) = \frac{D_x^{\alpha\alpha} D_t^\alpha \varphi - 4\varphi^2 D_t^\alpha \varphi + D_x^\alpha \varphi}{4 D_x^\alpha \varphi} \tag{32}$$

To eliminate $D_x^{-\alpha}(\varphi D_t^\alpha \varphi)$ we take fractional differentiate both sides of eq. (31) with respect to x, resulting:

$$-D_x^{\alpha\alpha\alpha} D_t^\alpha \varphi + 4\varphi^2 D_x^\alpha D_t^\alpha \varphi + 12\varphi D_x^\alpha \varphi D_t^\alpha \varphi + 4 D_x^{\alpha\alpha} \varphi D_x^{-\alpha}(\varphi D_t^\alpha \varphi) = D_x^{\alpha\alpha} \varphi \tag{33}$$

Replacing eq. (32) into eq. (33) results in the nmfKdV equation:

$$-D_x^\alpha \varphi D_x^{\alpha\alpha\alpha} D_t^\alpha \varphi + 4\varphi^2 D_x^\alpha \varphi D_x^\alpha D_t^\alpha \varphi + 12\varphi (D_x^\alpha \varphi)^2 D_t^\alpha \varphi + D_x^{\alpha\alpha} \varphi D_x^{\alpha\alpha} D_t^\alpha \varphi - 4\varphi^2 D_x^{\alpha\alpha} \varphi D_t^\alpha \varphi = 0 \tag{34}$$

Examining the Painleve property of eq. (34)

Firstly, the leading order n and the leading coefficient φ_0 must be determined. To do this, we substitute $\varphi = \varphi_0 \psi^n$, into eq. (34). By balancing the non-linear and fractional dispersive terms, we obtain:

$$n = -1, \ \varphi_0 = \pm D_x^\alpha \psi$$

Substituting:

$$\varphi = \varphi_0 \psi^{-1} + \varphi_j \psi^{j-1}$$

into eq. (34), and setting the coefficient of ψ^{j-7} equal to zero, we get:

$$D_t^\alpha \psi (D_x^\alpha \psi)^5 \varphi_j (j+1)(j-2)(j-3)(j-4) = 0$$

from this, we observe that four resonant points happen at $j = -1$, 2, 3, and 4. The final step is to test the resonant conditions. We may suppose:

$$\varphi = \sum_{s=0}^{4} \varphi_s \psi^{s=1}$$

we adopt Kruskal's ansatz:

$$\psi = \frac{x^\alpha}{\alpha} + \psi\left(\frac{t^\alpha}{\alpha}\right)$$

thus, the coefficients φ_s will be functions of t only. By substituting the expansion eq. (35) into eq. (34) and setting the terms based on the different powers of ψ to zero, we can find also $\varphi = 0$, and φ_2, φ_3, φ_4 are arbitrary functions of t. Thus, eq. (34) is integrable in the sense of having the Painleve property.

Calculating multi-soliton solutions for eq. (34)

To derive the dispersion relation, we set:

$$\varphi(t^\alpha, x^\alpha) = R D_x^\alpha \left\{ \ln\left[\frac{f(t^\alpha, x^\alpha)}{g(t^\alpha, x^\alpha)}\right] \right\} \tag{35}$$

where the auxiliary functions $f(t^\alpha, x^\alpha)$ and $g(t^\alpha, x^\alpha)$, for the single soliton solutions, is given:

$$f(t^\alpha, x^\alpha) = 1 + \mathrm{e}^{(kx^\alpha - \omega t^\alpha)/\alpha}, \quad g(t^\alpha, x^\alpha) = 1 - \mathrm{e}^{(kx^\alpha - \omega t^\alpha)/\alpha} \tag{36}$$

Replacing eq. (36) into the nmfKdV eq. (34) gives the dispersion relation by $R = 1$, and $\omega = -1/k$ so the single soliton solutions soliton solutions take the formula:

$$\varphi(t^\alpha, x^\alpha) = 2ke^{(kx^\alpha + t^\alpha/k)/\alpha} \Big/ \left(1 - e^{2(kx^\alpha + t^\alpha/k)/\alpha}\right), \tag{37}$$

For computing the two soliton solutions, we put the auxiliary function:

$$f(t^\alpha, x^\alpha) = 1 + \mathrm{e}^{\theta_1} + \mathrm{e}^{\theta_2} + a_{12}\,\mathrm{e}^{\theta_1+\theta_2},\ g(t^\alpha, x^\alpha) = 1 - \mathrm{e}^{\theta_1} - \mathrm{e}^{\theta_2} + a_{12}\,\mathrm{e}^{\theta_1+\theta_2}$$
$$\theta_i = \frac{k_i x^\alpha - k_i^3 t^\alpha}{\alpha},\ i = 1, 2 \tag{38}$$

Using eq. (38) in eq. (35) and substituting the result in eq. (34), we obtain the following phase shift coefficient:

$$a_{12} = \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2} \tag{39}$$

and hence we set the phase shifts:

$$a_{ij} = \frac{(k_i - k_j)^2}{(k_i + k_j)^2},\ 1 \le \mathrm{i} \le j \le 3 \tag{40}$$

For the three-soliton solutions, we set:

$$f(t^\alpha, x^\alpha) = 1 + \mathrm{e}^{\theta_1} + \mathrm{e}^{\theta_2} + \mathrm{e}^{\theta_3} + a_{12}\,\mathrm{e}^{\theta_1+\theta_2} + a_{23}\,\mathrm{e}^{\theta_2+\theta_3} + a_{13}\,\mathrm{e}^{\theta_1+\theta_3} + b_{123}\,\mathrm{e}^{\theta_1+\theta_2+\theta_3}$$
$$g(t^\alpha, x^\alpha) = 1 - \mathrm{e}^{\theta_1} - \mathrm{e}^{\theta_2} - \mathrm{e}^{\theta_3} + a_{12}\,\mathrm{e}^{\theta_1+\theta_2} + a_{23}\,\mathrm{e}^{\theta_2+\theta_3} + a_{13}\,\mathrm{e}^{\theta_1+\theta_3} - b_{123}\,\mathrm{e}^{\theta_1+\theta_2+\theta_3}$$
$$\theta_i = \frac{\dfrac{k_i x^\alpha + t^\alpha}{k_i}}{\alpha},\ i = 1, 2, 3 \tag{41}$$

Proceeding as before, we find that:

$$b_{123} = a_{12} a_{23} a_{13} \tag{42}$$

We note that all the results obtained in [28] are recovered when $\alpha = 1$.

Conclusions

The recursion operator plays a central role in the study of the nfmKdV equation. By generating multi-soliton from simpler one-soliton, it allows for a systematic exploration of soliton dynamics in fractional systems. The combination of the recursion operator with techniques like the Hirota technique enables the derivation of focusing and defocusing soliton solutions, as well as the study of their interactions in the presence of fractional derivatives. This approach is crucial for modelling non-linear fractional systems and provides valuable insights into the behavior of solitons in systems characterized by memory, long-range interactions, and non-local effects. The fractional Painleve property is a natural extension of the classical Painleve property to FDE. It is important because it ensures that the solutions to these equations exhibit analytic behavior (except for isolated poles) and are consistent with the behavior of integrable systems. This property is useful for understanding the integrability of fractional systems, which arise in many modern applications, such as anomalous diffusion, non-locality, and systems with memory effects. The study of the fractional Painleve property opens up new avenues for solving FDE, especially in the context of non-linear dynamics and wave propagation.

While the results presented here are promising, several avenues for future exploration remain: The current work focuses on (1+1)-D fractional system. Extending the negative-order fractional mKdV equations to higher fractional dimensional settings, such as (2+1)-D or

(3+1)-D, could provide a deeper understanding of soliton solutions in more complex physical systems. Although we have derived exact solutions analytically, it would be valuable to investigate the numerical stability and accuracy of these solutions using computational methods. This could include the development of efficient algorithms for solving nmKdV equations and exploring their behavior in various parameter regimes. Investigating the non-integrable versions of the nmKdV equations and understanding how they behave under perturbation could offer new insights into real-world phenomena that are governed by similar equations but may not possess exact soliton solutions. The study of soliton interactions within the context of the nmKdV equations, particularly the focusing and defocusing forms, could provide further insights into the dynamics of multi-soliton solutions. Investigating higher-order interactions and the stability of these solutions in both branches could reveal important physical behaviors.

Several open problems emerge from the work presented here: While we have shown the existence of multiple soliton solutions and singular soliton solutions, an open question is whether more general classes of soliton solutions exist for the nmKdV equations, especially in higher-order cases or in non-integrable settings. The detailed exploration of the symmetries of the nmKdV equations and their corresponding recursion operators could lead to new insights into the underlying structure of these equations. Understanding how these symmetries relate to physical properties of the systems modeled by these equations is still an open problem. While Backlund transformations have been extensively studied for the mKdV equation, developing Backlund transformations specifically for the nmKdV equations could provide a powerful tool for generating new solutions and exploring the interconnections between different integrable systems. Another open problem is the application of the inverse scattering transform to the nmKdV equations. Investigating the solvability of the nmKdV equations through inverse scattering transform would provide a deeper understanding of their integrability and could help in the analysis of their long-time behavior.

Acknowledgment

The authors extend their appreciation the Deanship of Scientific Research at King Khalid University for funding this work through large group Research Project under Grant No. RGP2/461/44

References

[1] Baldwin, D., Hereman, W., A Symbolic Algorithm for Computing Recursion Operators of Non-Linear Partial Differential Equations, *International Journal of Computer Mathematics*, *87* (2010), 5, pp. 1094-1119
[2] Fokas, A. S., Symmetries and Integrability, *Studies in Applied Mathematics*, *77* (1987), 3, pp. 253-299
[3] Sanders, J. A., Wang J. P., Integrable Systems And Their Recursion Operators, Non-Linear Analysis, *47* (2001), 8, pp. 5213-5240
[4] Olver, P. J., Evolution Equations Possessing Infinitely Many Symmetries, *Journal of Mathematical Physics*, *18* (1977), 6, pp. 1212-1215
[5] Lou, S., Higher-Dimensional Integrable Models with a Common Recursion Operator, *Communications in Theoretical Physics*, *28* (1997), 6, pp. 41-50
[6] Magri, F., *Lectures Notes in Physics*, Springer, Berlin, Germany, 1980
[7] Zhang, D., et al., Soliton Scattering with Amplitude Changes of a Negative Order AKNS Equation, *Physica D*, *238* (2009), 23-24, pp. 2361-2367
[8] Verosky, J. M., Negative Powers of Olver Recursion Operators, *Journal of Mathematical Physics*, *32* (1991), 7, pp. 1733-1736
[9] Qiao, Z., Fan, E., Negative-Order Korteweg-de Vries Equation, *Physical Review E*, *86* (2012), 016601
[10] Hirota, R., *The Direct Method in Soliton Theory*, Cambridge University Press, Cambridge, UK, 2004
[11] Xu, G. Q., The Integrability for a Generalized Seventh-Order KdV Equation: Painleve Property, Soliton Solutions, Lax Pairs and Conservation Laws, *Physica Scripta*, *89* (2014), 125201

[12] Xu, G. Q., Painleve Classification of a Generalized Coupled Hirota System, *Physical Review E*, *74* (2006) 027602
[13] Xu, G. Q., Searching for Painleve Integrable Conditions of Non-Linear PDEs with Constant Parameters Using Symbolic Computation, *Computational Physics Communication*, *178* (2008), 7, pp. 505-517
[14] Xu, G. Q., A Note on the Painleve Test for Non-Linear Variable-Coefficient PDE, *Computational Physics Communication*, *180* (2009), 7, pp. 1137-1144
[15] Adem, K. R., Khalique, C. M., Exact Solutions and Conservation Laws of Zakharov-Kuznetsov Modified Equal width Equation with Power Law Non-Linearity, Non-Linear Analysis, *Real World Applications*, *13* (2012), 4, pp. 1692-1702
[16] Ablowitz, M. J., Clarkson, P. A., *Solitons, Non-Linear Evolution Equations and Inverse Scattering*, Cambridge University Press, Cambridge, UK, 1992
[17] Hasegawa, A., Kodama, Y., *Solitons in Optical Communications*, Oxford University Press, Oxford, UK, 1995
[18] Lakshmananm M., Rajasekar, S., *Non-Linear Dynamics: Integrability, Chaos and Patterns*, Springer, Berlin, Germany, 2003
[19] Wazwaz, A. M., *Partial Differential Equations and Solitary Waves Theory*, Springer, Berlin, Germany, 2009
[20] Wazwaz, A. M., Multiple Kink Solutions and Multiple Singular Kink Solutions for (2+1)-Dimensional Non-Linear Models Generated by the Jaulent-Miodek Hierarchy, *Phys. Lett. A*, *373* (2009), 21, pp. 1844-1846
[21] El-Shehawy, S. A., Abdel-Salam, E. A.-B., The q-Deformed Hyperbolic Secant Family, *Int. J. Appl. Math. Stat*., *29* (2012), Sept., pp. 51-62
[22] Abdel-Salam, E. A.-B., Periodic Structures Based on the Symmetrical Lucas Function of the (2+1)-Dimensional Dispersive Long-Wave System, *Zeitschrift für Naturforschung A*, *63* (2008), 10-11, pp. 671-678
[23] Abdel-Salam, E. A.-B., *et al*., Analytical Solution the Conformable Fractional Lane-Emden Type Equations Arising in Astrophysics, *Scientific African*, *8* (2020), e00386
[24] Abdel-Salam, E. A.-B., Mourad, M. F., Fractional Quasi AKNS-Technique for Non-Linear Space-Time Fractional Evolution Equations, *Math. Meth. Appl. Sci*., *42* (2018), 18, pp. 5953-5968
[25] Abdel-Salam, E. A.-B., *et al*., Geometrical Study and Solutions for Family of Burgers-Like Equation with Fractional Order Space Time, *Alexandria Engineering Journal*, *61* (2022), 1, pp. 511-521
[26] Nouh, M. I., *et al*., Modelling Fractional Polytropic Gas Spheres Using Artificial Neural Network, *Neural Comput. and Applic.*, *33* (2021), Aug., pp. 4533-4546
[27] Khalil, R., *et al.*, A New Definition of Fractional Derivative, *J. Comput. Appl. Math*., *264* (2014), 65
[28] Nouh, M. I., Abdel-Salam, E. A.-B., Analytical Solution the Fractional Polytropic Gas Spheres, *Eur. Phys. J. Plus*, *133* (2018), 149

Paper submitted: July 7, 2024
Paper revised: November 15, 2024
Paper accepted: November 21, 2024