INVESTIGATING ANALYTICAL SOLUTIONS FOR (2+1)-DIMENSIONAL M-TRUNCATED BURGERS MODEL

by

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In this study, we employed the M-truncated fractional singular manifold method to analytically address the (2+1)-dimensional M-truncated fractional Burgers equation. This approach involves reformulating the original fractional differential equation into a more tractable form through the introduction of a singular manifold. This transformation simplifies the problem and often leads to analytical solutions. We derive a general solution expressed in terms of arbitrary functions, which enables us to accommodate variations in system parameters or initial conditions. This results in a versatile expression that captures a broad spectrum of possible solutions, providing a framework for analyzing the dynamics of kink waves in the relevant fractional differential models. We also construct multiple kink wave solutions, offering analytical representations of kink wave behavior within these models. Notably, our findings revert to well-established results when the fractional order is set to one, thereby affirming the consistency of this method with existing theories and validating our approach.

Key words: *M-truncated fractional derivative, fractional calculus, fractional models, Burgers equation*

Introduction

The concept of fractional derivatives dates back to the notable correspondence between Leibniz and L'Hospital in 1695. Over the last sixty years, fractional calculus has significantly influenced a wide range of fields, including physics, chemistry, electrical engineering, biology, economics, image processing, and aerodynamics [1-6]. In the past decade, fractional calculus has emerged as a crucial tool for modelling long-memory processes, attracting the attention of engineers, physicists, and mathematicians alike [7-12]. Understanding the solutions to fractional differential equations is vital for enhancing our grasp of physical processes characterized by fractional orders, with substantial implications for practical applications and real-world impacts. Partial and ordinary differential equations are widely used in disciplines such as fluid dynamics [13-16], system identification [17-19], control theory [20, 21], and image processing [22, 23], among others, to model complex phenomena [10-27].

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The fractional calculus is a field that expands traditional calculus, which typically focuses on integer-order derivatives, to encompass fractional orders [1-12]. This extension leads to various formulations of fractional derivatives, including the Riemann-Liouville (RL) [17], Caputo [19], He's [18], conformable [20], and local fractional derivatives [21, 22]. The RL fractional derivative is a foundational approach based on integrals, while He's fractional derivative utilizes He's polynomials for its definition. Caputo's fractional derivative combines integer-order differentials with the RL framework, making it particularly effective for analyzing initial value problems. The more recent conformable fractional derivative adheres to ordinary product rules and is well-suited for functions with singularities. Each of these definitions offers unique advantages and is employed across various fields, including physics, engineering, and signal processing, to address challenges involving fractional-order models and natural phenomena.

The fractional Burgers equation, a simplified version of the fractional model, effectively captures the interplay between dissipative effects and non-linear propagation. This model has applications across various fields, including hydrodynamics, fluid dynamics, wave propagation in thermo elastic media, acoustic transmission, plasma physics, traffic flow, MHD, shock waves, supersonic flow around airfoils, diffusion-affected waves, liquid dynamics, and information sciences. In this manuscript, we will investigate the general form of the analytical solution, as well as multiple soliton and singular soliton solutions for the model under consideration.

Overview and characteristics of the M-truncated fractional derivative

The truncated Mittag-Leffler function (MLF) can be defined:

$$ {}_{l}E_{\beta}(\varepsilon s^{\alpha}) = \sum_{j=0}^{l} \frac{(\varepsilon s^{\alpha})^{j}}{\Gamma(\beta_{j}+1)}, \ \beta > 0, \ s \in \mathbb{C} \tag{1} $$

Definition 1: Let ψ: $[0, \infty] \to \Re$ be a function, the local truncated M-fractional differential (MFD) of ψ with respect to y is given [25]:

$$ {}_{l}D_{M,t}^{\alpha,\beta}\psi(s) = \lim_{\delta\to 0} \frac{\psi\left[s\, {}_{l}\mathrm{E}_{\beta}(\delta s^{-\alpha})\right] - \psi(s)}{\delta}, \ \forall \beta, \ s > 0, \ \alpha \in (0,1) \tag{2} $$

The MFD adheres to the axioms:

$$ {}_{l}D_{M,s}^{\alpha,\beta} s^{n} = \frac{n}{\Gamma(\beta+1)} s^{n-\alpha}, \ n \in R, \ {}_{l}D_{M,s}^{\alpha,\beta} c = 0, \ \forall \psi(s) = c \tag{3} $$

$$ {}_{l}D_{M,s}^{\alpha,\beta}(c_1\psi + c_2\varphi) = c_1\, {}_{l}D_{M,s}^{\alpha,\beta}\psi + c_2\, {}_{l}D_{M,s}^{\alpha,\beta}\varphi, \ \forall c_1, c_2 \in \Re \tag{4} $$

$$ {}_{l}D_{M,s}^{\alpha,\beta}(\psi\varphi) = \psi\, {}_{l}D_{M,s}^{\alpha,\beta}\varphi + \varphi\, {}_{l}D_{M,s}^{\alpha,\beta}\psi \tag{5} $$

$$ {}_{l}D_{M,s}^{\alpha,\beta}\left(\frac{\varphi}{\psi}\right) = \frac{\psi\, {}_{l}D_{M,s}^{\alpha,\beta}\varphi - \varphi\, {}_{l}D_{M,s}^{\alpha,\beta}\psi}{\psi^{2}} \tag{6} $$

$$ {}_{l}D_{M,s}^{\alpha,\beta}\varphi(\psi) = {}_{l}D_{M,s}^{\alpha,\beta}\psi\left(\frac{\mathrm{d}\varphi}{\mathrm{d}\psi}\right), \ {}_{l}D_{M,s}^{\alpha,\beta}\varphi(s) = \frac{s^{1-\alpha}}{\Gamma(\beta+1)}\frac{\mathrm{d}\varphi}{\mathrm{d}s} \tag{7} $$

where φ, ψ are the two α-differentiable functions of a dependent variable, the aforementioned relations are proved in reference [25]. Choosing $\beta = 1$ and $l = 1$ on the two sides of eq. (1), we have:

$$ {}_1D^{\alpha,1}_{M,\mathrm{t}}\psi(s)=\lim_{\delta\to 0}\frac{\psi\left[s_1E_1(\delta s^{-\alpha})\right]-\psi(s)}{\delta},\ \ \forall\, s>0,\ \alpha\in(0,1) $$

But, it is know that:

$$ {}_1E_1(\delta s^{-\alpha})=\sum_{r=0}^{1}\frac{(\delta s^{-\alpha})^r}{\Gamma(2)}=1+\delta s^{-\alpha} $$

Thus, we conclude:

$$ {}_1D^{\alpha,1}_{M,\mathrm{s}}\psi(s)=\lim_{\delta\to 0}\frac{\psi(s+\delta s^{1-\alpha})-\psi(s)}{\delta}=D^{\alpha}_t\psi(s),\ \ \forall\, s>0,\ \alpha\in(0,1] $$

which is exactly the conformable fractional derivative [24]. Simply we write ${}_1D^{\alpha,\beta}_M$ as $D^{\alpha,\beta}_M$. The MFD of some functions [25]:

$$ D^{\alpha,\beta}_{M,\mathrm{s}}\mathrm{e}^{cs}=\frac{cs^{1-\alpha}\mathrm{e}^{cs}}{\Gamma(\beta+1)},\ D^{\alpha,\beta}_{M,\mathrm{s}}\sin(cx)=\frac{cs^{1-\alpha}\cos(cs)}{\Gamma(\beta+1)},\ D^{\alpha,\beta}_{M,\mathrm{s}}\cos(cs)=-\frac{cs^{1-\alpha}\sin(cs)}{\Gamma(\beta+1)} $$
$$ D^{\alpha,\beta}_{M,\mathrm{s}}\mathrm{e}^{cx^{\alpha}}=\frac{c\alpha\mathrm{e}^{cs^{\alpha}}}{\Gamma(\beta+1)},\ D^{\alpha,\beta}_{M,\mathrm{s}}\sin(cs^{\alpha})=\frac{c\alpha\cos(cs^{\alpha})}{\Gamma(\beta+1)},\ D^{\alpha,\beta}_{M,\mathrm{s}}\cos(cs^{\alpha})=-\frac{c\alpha\sin(cs^{\alpha})}{\Gamma(\beta+1)} \quad (8) $$

The MFD is applicable to non-differentiable functions, making it particularly useful for scenarios involving discontinuous media. Currently, fractional calculus extends the concepts of integer-order integration and differentiation incorporate fractional orders. Recently, non-linear fractional models have emerged as a significant area of research, attracting attention from physicists, mathematicians, astronomers, and engineers alike. These models find widespread applications across various scientific disciplines, including plasma physics, condensed matter physics, biomathematics, chemistry, biology, communication, and astronomy. Fractional calculus is essential in engineering and physics, with applications in areas such as fractal wave propagation, particle physics, electrical systems, and wave mechanics.

Fractional Burgers equation

The fractional (2+1)-Burgers equation is a fractional PDE that emerges in the study of fluid dynamics, especially concerning turbulence and shock wave propagation. This equation extends the classical fractional Burgers equation encompass two spatial dimensions and one temporal dimension. It models the evolution of velocity fields in 2-D flows, incorporating dissipative effects due to viscosity. Notably, it displays fascinating phenomena such as shock wave formation and turbulence, making it a key focus of research in fluid dynamics and related disciplines.

The (2+1) Burgers equation with fractional space and time derivatives:

$$ D^{\alpha,\beta}_{M,\mathrm{t}}M=D^{\alpha\alpha,\beta}_{M,\mathrm{x}}M+2ND^{\alpha,\beta}_{M,\mathrm{x}}M,\ \ D^{\alpha,\beta}_{M,\mathrm{x}}M=D^{\alpha,\beta}_{M,\mathrm{y}}N \quad (9) $$

where $D^{\alpha,\beta}_{M,t}$ is the MFD for t, D^{α}_{x} – the MFD for x, $D^{\alpha\alpha,\beta}_{M,x}$ – the twice α fractional derivative which means that:

$$ D^{\alpha\alpha,\beta}_{M,\mathrm{x}}u=D^{\alpha,\beta}_{M,\mathrm{x}}\left(D^{\alpha,\beta}_{M,\mathrm{x}}u\right) $$

Equation (9) is the generalization of the (2+1)-Burgers equation [28]:

$$ M_t=M_{xx}+2NM_x,\ \ M_x=N_y $$

The fractional (2+1) Burgers equation generalizes the classical fractional (1+1) Burgers equation by incorporating extra spatial dimensions and higher-order fractional derivatives. This fractional PDE is frequently encountered in the analysis of fluid dynamics and non-linear wave phenomena. By accounting for third-order spatial fractional derivatives, it introduces additional complexity compared to its classical version. This equation effectively describes phenomena such as shock formation, turbulence, and non-linear wave propagation in two spatial dimensions, offering valuable insights into various physical systems governed by fluid dynamics and non-linear wave behavior.

Equations (9) can be written in another form by taking the transformation $M = D_{M,y}^{\alpha,\beta}N$. Substituting the potential eq. (10) in eq. (9), then eq. (9) transformed:

$$D_{M,t}^{\alpha,\beta} D_{M,y}^{\alpha,\beta} N = D_{M,x}^{\alpha\alpha,\beta} D_{M,y}^{\alpha,\beta} N + 2D_{M,y}^{\alpha,\beta} N D_{M,x}^{\alpha,\beta} D_{M,y}^{\alpha,\beta} N \tag{10}$$

Utilizing the fractional singular manifold method and analyzing the leading order, we can truncate the Painleve chain of eq. (9):

$$M = \varphi^{-1} M_0 + M_1, \;\; N = \varphi^{-1} N_0 + N_1 \tag{11}$$

with φ is the fractional singular manifold and $\{M_1, N_1\}$ is an arbitrary solution eq. (9), substituting eq. (11) into eq. (9) and equating the coefficients of like powers of φ gives

$$M_0 = D_{M,y}^{\alpha,\beta}\varphi, \;\; N_0 = D_{M,x}^{\alpha,\beta}\varphi$$

since φ satisfies:

$$D_{M,t}^{\alpha,\beta}\varphi = D_{M,x}^{\alpha\alpha,\beta}\varphi + 2M_1 D_{M,x}^{\alpha,\beta}\varphi \tag{12}$$

Equation (12) is called the equation of M-truncated fractional singular manifold. Equations (11) and (12) specify an auto-Backlund M-truncated fractional transformation for eq. (9). If we take:

$$M_1 = \varphi, \;\; D_{M,y}^{\alpha,\beta} N_1 = D_{M,x}^{\alpha,\beta}\varphi$$

then

$$M = \frac{1}{\varphi} D_{M,x}^{\alpha,\beta}\varphi + \varphi \tag{13}$$

where φ satisfies

$$D_{M,t}^{\alpha,\beta}\varphi = D_{M,x}^{\alpha\alpha,\beta}\varphi + 2\psi D_{M,x}^{\alpha,\beta}\varphi, \;\; D_{M,y}^{\alpha,\beta}\psi = D_{M,x}^{\alpha,\beta}\varphi \tag{14}$$

Equations (13) and (14) constitute another form of an auto-Backlund M-truncated fractional transformation for eq. (9). If $M_1 = 0$, $N_1 = 0$, then we can get the Cole-Hopf type M-truncated fractional transformation or hetero-Backlund fractional transformation:

$$M = \frac{1}{\varphi} D_{M,x}^{\alpha,\beta}\varphi$$

where φ satisfies:

$$D_{M,\mathrm{t}}^{\alpha,\beta}\varphi = D_{M,x}^{\alpha\alpha,\beta}\varphi \tag{15}$$

is given to the (2+1)-Burgers equation with M-truncated fractional space and time derivative. If we suppose a special solution:

$$M_1 = 0, \;\; N_1 = N_1\left(\frac{x^{\alpha}\Gamma(\beta+1)}{\alpha}, \frac{t^{\alpha}\Gamma(\beta+1)}{\alpha}\right) \tag{16}$$

where N_1 is the arbitrary function of indicated variables. Then we can obtain systematically that eq. (12) has the non-linear separation function solution:

$$\varphi = F\left(\frac{x^{\alpha}\Gamma(\beta+1)}{\alpha}, \frac{t^{\alpha}\Gamma(\beta+1)}{\alpha}\right)G\left(\frac{y^{\alpha}\Gamma(\beta+1)}{\alpha}\right) + H\left(\frac{y^{\alpha}\Gamma(\beta+1)}{\alpha}\right) \tag{17}$$

where F, G, and H are arbitrary functions in indicated variables, if:

$$M_1 = \frac{D_{M,t}^{\alpha,\beta}F - D_{M,x}^{\alpha\alpha,\beta}F}{2D_{M,x}^{\alpha,\beta}F}$$

from eqs. (10), (11), (16), and (17) yields a general functional separation solution of the (2+1)-Burgers equation with M-truncated fractional space and time derivative:

$$N = \frac{FD_{M,y}^{\alpha,\beta}G + D_{M,y}^{\alpha,\beta}H}{FG + H} \tag{18}$$

where

$$F = F\left(\frac{x^{\alpha}\Gamma(\beta+1)}{\alpha}, \frac{t^{\alpha}\Gamma(\beta+1)}{\alpha}\right), \; G = G\left(\frac{y^{\alpha}\Gamma(\beta+1)}{\alpha}\right)$$

and

$$H = H\left(\frac{y^{\alpha}\Gamma(\beta+1)}{\alpha}\right)$$

are arbitrary functions of the specified variables. By employing the fractional singular manifold method alongside a separation of variables approach, we derive a solution that depends on three distinct functions of both time and space variables. This enables us to investigate a variety of solution forms for eq. (9) of the (2+1) Burgers equation, which includes fractional derivatives in both spatial and temporal dimensions. Careful selection of these arbitrary functions allows us to analyze a wide range of behaviors and patterns characteristic of the system. When $\alpha = 1$, $\beta = 1$, eqs. (10)-(18) correspond directly to eqs. (5)-(14) from the work of Peng and Yamba [29].

Multiple soliton solution

Now we discuss the multiple-kink wave solution. If we choose:

$$M = \mathrm{e}^{\Gamma(\beta+1)(k_i x^{\alpha} + r_i y^{\alpha} - c_i t^{\alpha})/\alpha} \tag{19}$$

By substituting into the linear term of eq. (9), we obtain the dispersion relation $c_i = -k_i^2$, then we get:

$$\theta_i = \frac{\Gamma(\beta+1)(k_i x^{\alpha} + r_i y^{\alpha} + k_i^2 t^{\alpha})}{\alpha} \tag{20}$$

By using the Cole-Hopf fractional transformation, the multiple-kink wave solution of eq. (9) is:

$$N = R\ln(f), \; M = Rf^{-1}D_{M,y}^{\alpha,\beta}f \tag{21}$$

For the one-kink wave solution:

$$f = 1 + \mathrm{e}^{\Gamma(\beta+1)(k_1 x^{\alpha} + r_1 y^{\alpha} + k_1^2 t^{\alpha})/\alpha} \tag{22}$$

Substituting eq. (21) into eq, (10) and then solving for R, we have $R = 1$:

$$N = \ln 1 + \mathrm{e}^{\Gamma(\beta+1)(k_1 x^\alpha + r_1 y^\alpha + k_1^2 t^\alpha)/\alpha}$$

Then the one-kink wave solution is:

$$M = \frac{r_1 \mathrm{e}^{\Gamma(\beta+1)(k_1 x^\alpha + r_1 y^\alpha + k_1^2 t^\alpha)/\alpha}}{1 + \mathrm{e}^{\Gamma(\beta+1)(k_1 x^\alpha + r_1 y^\alpha + k_1^2 t^\alpha)/\alpha}} \tag{23}$$

The two-kink wave solution takes the form:

$$f = 1 + \mathrm{e}^{\theta_1} + \mathrm{e}^{\theta_2} + a_{12} \mathrm{e}^{\theta_1 + \theta_2} \tag{24}$$

Using eq. (24) in eq. (21) and then substituting the result into eq. (10), we obtain $R = 1$, and there are no phase shifts $a_{12} = 0$, hence, we have:

Thus, we have:

$$a_{ij} = 0,\ 1 \le i < j \le 3 \tag{25}$$

$$N = \ln\left[1 + \mathrm{e}^{\Gamma(\beta+1)(k_1 x^\alpha + r_1 y^\alpha + k_1^2 t^\alpha)/\alpha} + \mathrm{e}^{\Gamma(\beta+1)(k_2 x^\alpha + r_2 y^\alpha + k_2^2 t^\alpha)/\alpha}\right]$$

The two-kink wave solution is:

$$M = \frac{r_1 \mathrm{e}^{\Gamma(\beta+1)(k_1 x^\alpha + r_1 y^\alpha + k_1^2 t^\alpha)/\alpha} + r_2 \mathrm{e}^{\Gamma(\beta+1)(k_2 x^\alpha + r_2 y^\alpha + k_2^2 t^\alpha)/\alpha}}{1 + \mathrm{e}^{\Gamma(\beta+1)(k_1 x^\alpha + r_1 y^\alpha + k_1^2 t^\alpha)/\alpha} + \mathrm{e}^{\Gamma(\beta+1)(k_2 x^\alpha + r_2 y^\alpha + k_2^2 t^\alpha)/\alpha}} \tag{26}$$

For the three-kink wave solution, we put:

$$f = 1 + \mathrm{e}^{\theta_1} + \mathrm{e}^{\theta_2} + \mathrm{e}^{\theta_3}$$

Proceeding as before, we have:

$$N = \ln\left[1 + \mathrm{e}^{\Gamma(\beta+1)(k_1 x^\alpha + r_1 y^\alpha + k_1^2 t^\alpha)/\alpha} + \mathrm{e}^{\Gamma(\beta+1)(k_2 x^\alpha + r_2 y^\alpha + k_2^2 t^\alpha)/\alpha} + \mathrm{e}^{\Gamma(\beta+1)(k_3 x^\alpha + r_3 y^\alpha + k_3^2 t^\beta)/\alpha}\right]$$

This gives the three-kink wave solution:

$$M = \frac{r_1 \mathrm{e}^{\Gamma(\beta+1)(k_1 x^\alpha + r_1 y^\alpha + k_1^2 t^\alpha)/\alpha} + r_2 \mathrm{e}^{\Gamma(\beta+1)(k_2 x^\alpha + r_2 y^\alpha + k_2^2 t^\alpha)/\alpha} + r_3 \mathrm{e}^{\Gamma(\beta+1)(k_3 x^\alpha + r_3 y^\alpha + k_3^3 t^\alpha)/\alpha}}{1 + \mathrm{e}^{\Gamma(\beta+1)(k_1 x^\alpha + r_1 y^\alpha + k_1^2 t^\alpha)/\alpha} + \mathrm{e}^{\Gamma(\beta+1)(k_2 x^\alpha + r_2 y^\alpha + k_2^2 t^\alpha)/\alpha} + \mathrm{e}^{\Gamma(\beta+1)(k_3 x^\alpha + r_3 y^\alpha + k_3^2 t^\alpha)/\alpha}} \tag{27}$$

For the N-kink wave solution, we put $f = 1 + \mathrm{e}^{\theta_1} + \mathrm{e}^{\theta_2} + ... + \mathrm{e}^{\theta_n}$, proceeding as before, we have:

$$N = \ln\left[1 + \mathrm{e}^{\Gamma(\beta+1)(k_1 x^\alpha + r_1 y^\alpha + k_1^2 t^\alpha)/\alpha} + \mathrm{e}^{\Gamma(\beta+1)(k_2 x^\alpha + r_2 y^\alpha + k_2^2 t^\alpha)/\alpha} + ... + \mathrm{e}^{\Gamma(\beta+1)(k_n x^\alpha + r_n y^\alpha + k_n^2 t^\alpha)/\alpha}\right]$$

This gives the N-kink wave solution:

$$M = \frac{r_1 \mathrm{e}^{\Gamma(\beta+1)(k_1 x^\alpha + r_1 y^\alpha + k_1^2 t^\alpha)/\alpha} + r_2 \mathrm{e}^{\Gamma(\beta+1)(k_2 x^\alpha + r_2 y^\alpha + k_2^2 t^\alpha)/\alpha} + ... + r_n \mathrm{e}^{\Gamma(\beta+1)(k_n x^\alpha + r_n y^\alpha + k_n^2 t^\alpha)/\alpha}}{1 + \mathrm{e}^{\Gamma(\beta+1)(k_1 x^\alpha + r_1 y^\alpha + k_1^2 t^\alpha)/\alpha} + \mathrm{e}^{\Gamma(\beta+1)(k_2 x^\alpha + r_2 y^\alpha + k_2^2 t^\alpha)/\alpha} + ... + \mathrm{e}^{\Gamma(\beta+1)(k_n x^\alpha + r_n y^\alpha + k_n^2 t^\alpha)/\alpha}} \tag{28}$$

Multiple singular soliton solution

We assume that the singular soliton solution of eq. (9) takes a similar form to that described earlier, with the auxiliary function defined:

$$f=1-e^{\Gamma(\beta+1)(k_1x^\alpha+r_1y^\alpha+k_1^2t^\alpha)/\alpha} \tag{29}$$

Substituting eq. (29) into eq. (10) and then solving for R, we have $R = 1$. So:

$$N=\ln\left[1-e^{\Gamma(\beta+1)(k_1x^\alpha+r_1y^\alpha+k_1^2t^\alpha)/\alpha}\right]$$

Then the one-kink wave solution is:

$$M=-\frac{r_1e^{\Gamma(\beta+1)(k_1x^\alpha+r_1y^\alpha+k_1^2t^\alpha)/\alpha}}{1-e^{\Gamma(\beta+1)(k_1x^\alpha+r_1y^\alpha+k_1^2t^\alpha)/\alpha}} \tag{30}$$

The two singular kink wave solution takes the form:

$$f=1-e^{\theta_1}-e^{\theta_2}$$

Thus, we have:

$$N=\ln\left[1-e^{\Gamma(\beta+1)(k_1x^\alpha+r_1y^\alpha+k_1^2t^\alpha)/\alpha}-e^{\Gamma(\beta+1)(k_2x^\alpha+r_2y^\alpha+k_2^2t^\alpha)/\alpha}\right]$$

The two singular kink wave solution is:

$$M=-\frac{r_1e^{\Gamma(\beta+1)(k_1x^\alpha+r_1y^\alpha+k_1^2t^\alpha)/\alpha}+r_2e^{\Gamma(\beta+1)(k_2x^\alpha+r_2y^\alpha+k_2^2t^\alpha)/\alpha}}{1-e^{\Gamma(\beta+1)(k_1x^\alpha+r_1y^\alpha+k_1^2t^\alpha)/\alpha}-e^{\Gamma(\beta+1)(k_2x^\alpha+r_2y^\alpha+k_2^2t^\alpha)/\alpha}} \tag{31}$$

For the three-kink wave solution, we put:

$$f=1-e^{\theta_1}-e^{\theta_2}-e^{\theta_3}$$

Proceeding as before, we have:

$$N=\ln\left[1-e^{\Gamma(\beta+1)(k_1x^\alpha+r_1y^\alpha+k_1^2t^\alpha)/\alpha}-e^{\Gamma(\beta+1)(k_2x^\alpha+r_2y^\alpha+k_2^2t^\alpha)/\alpha}-e^{\Gamma(\beta+1)(k_3x^\alpha+r_3y^\alpha+k_3^2t^\alpha)/\alpha}\right]$$

This gives the three-kink wave solution:

$$M=-\frac{r_1e^{\Gamma(\beta+1)(k_1x^\alpha+r_1y^\alpha+k_1^2t^\alpha)/\alpha}+r_2e^{\Gamma(\beta+1)(k_2x^\alpha+r_2y^\alpha+k_2^2t^\alpha)/\alpha}+r_3e^{\Gamma(\beta+1)(k_3x^\alpha+r_3y^\alpha+k_3^2t^\alpha)/\alpha}}{1-e^{\Gamma(\beta+1)(k_1x^\alpha+r_1y^\alpha+k_1^2t^\alpha)/\alpha}-e^{\Gamma(\beta+1)(k_2x^\alpha+r_2y^\alpha+k_2^2t^\alpha)/\alpha}-e^{\Gamma(\beta+1)(k_3x^\alpha+r_3y^\alpha+k_3^2t^\alpha)/\alpha}} \tag{32}$$

For the N-singular kink wave solution, we put:

$$f=1-e^{\theta_1}-e^{\theta_2}-\ldots-e^{\theta_n}$$

Proceeding as before, we have:

$$N=\ln\left[1-e^{\Gamma(\beta+1)(k_1x^\alpha+r_1y^\alpha+k_1^2t^\alpha)/\alpha}-e^{\Gamma(\beta+1)(k_2x^\alpha+r_2y^\alpha+k_2^2t^\alpha)/\alpha}-\ldots-e^{\Gamma(\beta+1)(k_nx^\alpha+r_ny^\alpha+k_n^2t^\alpha)/\alpha}\right]$$

This gives the N-singular kink wave solution:

$$M=-\frac{r_1e^{\Gamma(\beta+1)(k_1x^\alpha+r_1y^\alpha+k_1^2t^\alpha)/\alpha}+r_2e^{\Gamma(\beta+1)(k_2x^\alpha+r_2y^\alpha+k_2^2t^\alpha)/\alpha}+\ldots+r_ne^{\Gamma(\beta+1)(k_nx^\alpha+r_ny^\alpha+k_n^2t^\alpha)/\alpha}}{1-e^{\Gamma(\beta+1)(k_1x^\alpha+r_1y^\alpha+k_1^2t^\alpha)/\alpha}-e^{\Gamma(\beta+1)(k_2x^\alpha+r_2y^\alpha+k_2^2t^\alpha)/\alpha}-\ldots-e^{\Gamma(\beta+1)(k_nx^\alpha+r_ny^\alpha+k_n^2t^\alpha)/\alpha}} \tag{33}$$

Conclusions

In this paper, we investigate the (2+1) fractional Burgers equation, uncovering a variety of solutions, including multiple kink and singular kink wave solutions. Notably, we observe that several non-linear fractional equations, even those incorporating higher fractional derivatives, produce identical solutions. This finding highlights the intricate and interconnected nature of non-linear phenomena within the realm of fractional Burgers equations. The flexibility in selecting three arbitrary functions allows us to analyze diverse properties of these solutions, offering new physical insights into the problem. Additionally, the variability introduced by arbitrary fractional orders results in significantly richer structural complexities, empowering researchers to explore a wide range of solution behaviors and delve deeper into the underlying phenomena.

When the fractional order is set to one, our results align precisely with previous findings by Peng and Yamba [29] and Wazwaz [28], reinforcing the consistency and validity of our work within the existing literature. This convergence underscores the robustness of our analytical framework and affirms the credibility of our findings. We believe this methodology has great potential for application in a variety of other non-linear fractional differential models. The versatility and effectiveness demonstrated in solving the (2+1) fractional Burgers equation suggest that similar approaches could be effectively applied to a wide array of non-linear fractional differential equations, paving the way for exciting explorations across various scientific disciplines.

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References

[1] Kilbas, A. A., *et al*., *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, The Netherlands, 2006
[2] Hilfer, R., *Applications of Fractional Calculus in Physics*, World Science, Singapore, Singapore, 2000
[3] Manolis, G. D., Rangelov, T. V., Non-Homogeneous Elastic Waves in Soils: Notes on the Vector Decomposition Technique, *Soil Dynamics and Earthquake Engineering*, *26* (2006), 10, pp. 952-959
[4] Rizvi, S. T., *et al*., Study of Multiple Lump and Rogue Waves to the Generalized Unstable Space Time Fractional Non-Linear Schrodinger Equation, *Chaos, Solitons and Fractals*, *151* (2021), 111251
[5] Rizvi, S. T., *et al.*, Rogue, Multi-Wave, Homoclinic Breather, M-Shaped Rational and Periodic-Kink Solutions for a Non-Linear Model Describing Vibrations, *Results in Physics*, *29* (2021), 104654
[6] Akhmetov, A., Long Current Loops as Regular Solutions of the Equation for Coupling Currents in a Flat Two-Layer Superconducting Cable, *Cryogenics*, *43* (2003), 3-5, pp. 317-322
[7] Carslaw, H. S., Jaeger, J. C., *Conduction of Heat in Solids*, Oxford University Press, Oxford, UK, 1959
[8] Ozisik, M. N., *Heat Conduction*, 2nd ed., John Wiley and Sons, New York, USA, 1993
[9] Bejan, A., *Convection Heat Transfer*, 4th ed., John Wiley and Sons, New York, USA, 2013
[10] Turcotte, D. L., Schubert, G., *Geodynamics*, 2nd ed., Cambridge University Press, Cambridge, UK, 2002
[11] Peck, L., *Heat as a Tool for Studying the Earth: Thermal Modelling in Geosciences*, Cambridge University Press, Cambridge, UK, 2016
[12] Hartmann, D. L., *Global Physical Climatology*, 2nd ed., Academic Press, Cambridge, Mass., USA, 2016
[13] Abdel-Salam, E. A.-B., Periodic Structures Based on the Symmetrical Lucas Function of the (2+1)-Dimensional Dispersive Long-Wave System, *Zeitschrift für Naturforschung A*, *63* (2008), 10-11, pp. 671-678
[14] He, J. H., Approximate Analytical Solution for Seepage Flow with Fractional Derivatives in Porous Media, *Comput. Methods Appl. Mech. Eng.*, *167* (1998), 1-2, pp. 57-68
[15] Momani, S., Odibat, Z., Analytical Approach to Linear Fractional Partial Differential Equations Arising in Fluid Mechanics, *Phys. Lett. A*, *355* (2006), 4-5, pp. 271-279
[16] Yousif, E. A., *et.al*., On the Solution of the Space-Time Fractional Cubic Non-Linear Schrodinger Equation, *Results in Physics*, *8* (2018), Mar., pp. 702-708
[17] Abdel-Salam, E. A.-B., Mourad, M. F., Fractional Quasi AKNS-Technique for Non-Linear Space-Time Fractional Evolution Equations, *Math. Meth. Appl. Sci*, *42* (2018), 18, pp. 5953-5968
[18] Abdel-Salam, E. A.-B., *et al.*, Geometrical Study and Solutions for Family of Burgers-Like Equation with Fractional Order Space Time, *Alexandria Engineering Journal*, *61* (2022), 1, pp. 511-521

[19] Nouh, M. I., *et al.*, Modelling Fractional Polytropic Gas Spheres Using Artificial Neural Network, *Neural Comput. and Applic.*, *33* (2021), Aug., pp. 4533-4546
[20] Nouh, M. I., Abdel-Salam, E. A.-B., Analytical Solution The Fractional Polytropic Gas Spheres, *Eur. Phys. J. Plus*., *133* (2018), 149
[21] Abdel-Salam, E. A.-B., Nouh, M. I., Conformable Fractional Polytropic Gas Spheres, *New Astronomy*, *76* (2020), 101322
[22] Abdel-Salam, E. A.-B., *et al*., Analytical Solution of the Space-Time Fractional Non-Linear Schrodinger Equation, *Reports on Mathematical Physics*, *77* (2016), 1, pp. 19-34
[23] Kilany, A. A., *et al*. Non-Integer Order Analysis of Electro-Magneto-Thermoelastic with Diffusion and Voids Considering Lord-Shulman and Dual-Phase-Lag Models with Rotation and Gravity, *Waves in Random and Complex Media*, *11* (2022), pp. 1-31
[24] Khalil, R., *et al*., A New Definition of Fractional Derivative, *J. Comput. Appl. Math*., *264* (2014), 65
[25] da C. Sousa, J. V., *et al*., A New Truncated M-Fractional Derivative Type Unifying Some Fractional Derivative Types with Classical Properties, *Int. J. Anal. Appl*., *16* (2018), 1, pp. 83-96
[26] Hassan, G. F., *et al*., Approximation of Functions by Complex Conformable Derivative Bases in Frechet Spaces, *Mathematical Methods in the Applied Sciences*, *46* (2023), 2, pp. 2636-2650
[27] Zayed, M, *et al*., On the Convergence of Series of Fractional Hasse Derivative Bases in Frechet Spaces, *Mathematical Methods in the Applied Sciences*, *47* (2024), 11, pp. 8366-8384
[28] Wazwaz, A. M., A Study on the (2+1)-Dimensional and the (2+1)-Dimensional Higher-Order Burgers Equations, *Applied Mathematics Letters*, *25* (2012), 10, pp. 1495-1499
[29] Peng, Y. Z., Yomba, E., New Applications of the Singular Manifold Method to the (2+1)-Dimensional Burgers Equations, *Applied Mathematics and computations*, *183* (2006), 1, pp. 61-67

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