On Generalized Local Fractal Calculus Associate with Gauge Integral and Applications

Hemanta Kalita[a,*], **Alireza K. Golmankhanehand**[b], **and Bipan Hazarika**[c]
[a]Mathematics Division, School of Advanced Sciences and Languages, VIT Bhopal University, Bhopal-Indore Highway, Sehore, Madhya Pradesh, India.
[b]Department of Physics, College of Science, Urmia Branch, Islamic Azad University, Urmia, Iran
[b] Department of Mathematics, Faculty of Sciences, Van Yuzuncu Yil University, Van 65080, Turkey
[c]Department of Mathematics, Gauhati University, Guwahati, Assam, India
Email: *hemanta30kalita@gmail.com;hemantakalita@vitbhopal.ac.in; alirezakhalili2002@yahoo.co.in; a.khalili@iaurmia.ac.ir; bh_rgu@yahoo.co.in; bh_gu@gauhati.ac.in*

Abstract

In this work, a new integral so called $*F^{\alpha}$-integral with respect to local fractal derivatives are introduced. Several properties of $*F^{\alpha}$-integrals are discussed. Fundamental theorem for $*F^{\alpha}$-integrable functions is also introduced. A relationship of F^{α} and $*F^{\alpha}$ integral is shown. Finally, as an application we solve fractal differential equation $D_F^{\alpha}\big(S_F^{\alpha}(x)\big) = f\big(t, S_F^{\alpha}(x)\big)$ with $S_F^{\alpha}(\tau) = \xi$ in sense of $*F^{\alpha}$-integral.

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1 Introduction

In the real-line domain, Riemann defined the integration of functions, which has several applications in science, physics, and engineering. It turned out that Riemann's definition of the integral was not without flaws. For instance, Riemann's formulation limits the integration of all derivatives. In order to address these shortcomings, Lebesgue redefined integration. Lebesgue's approach is intricate, requiring a significant amount of measure theory. Later J. Kurzweil, during the 1958's introduced a generalized version of Riemann Integral and Henstock during the 1960's made the first systematic study of this new integral. This new integration technique is so powerful that it includes every function the others can integrate and the added advantage is its simplicity compared to the other integrals [4]. Since both Henstock and Kurzweil gave independently minute yet ingenious modification to the classical Riemann Integral while obtaining equivalent real-valued integral, this integral is named the Henstock-Kurzweil integral. This integral can handle a wider class of functions, including those with unbounded variation and certain types of discontinuities that are not Lebesgue integrable. The Henstock-Kurzweil integral retains the intuitive appeal of the Riemann integral while extending its applicability significantly. Henstock and Kurzweil just modified the conventional $(\varepsilon - \delta)$ definition of the Riemann integral by substituting a strictly positive function known as the gauge for the constant δ. This is why Henstock-Kurzweil integrals are also known as gauge integrals.

A geometric object with intricate structure at arbitrary tiny scales is called a fractal [12]. Fractal geometry is a field that was founded by Benoit Mandelbrot and deals with shapes that have fractal dimensions larger than their topological dimensions [2, 10]. Complex fractals are self-similar and often show complicated and non-integer dimensions [14]. However, the analysis of fractals presents challenges, given that traditional geometric measures such as Hausdorff measure [3], length, surface area, and volume are typically applied to standard shapes [16]. Fractal sets are those whose Hausdorff dimension strictly surpasses the topological dimension [6]. In many engineering applications, including porous media modelling, nanofluids, fracture mechanics, and many more, fractals are used at the nanoscale [13]. In these applications, the fractal nature of the objects must be taken into consideration because different transport phenomena cannot be described by a smooth continuum approach. Since the structure of fractals is non-differentiable, standard calculus cannot be applied to them. A. Parvate et al. [15] discussed Riemann type integrals on fractal sets. Hausdorff measure is a suitable measure for fractal geometry, is not appropriate for fractal calculus. Alireza K. Golmankhaneh et al. [5] introduced a new measure using the gauge function on fractal sets that gives a finer dimension in comparison with the Hausdorff and box dimension. Recently, Alireza K. Golmankhaneh et al. [7] present a generalized fractal calculus for irregular functions on fractal sets in associate with gauge integrable functions.

We motivated to find more generalized definition of F^{α}-integrals of fractal calculus [7] without the assumption $Sch(f)$ is contained in α-perfect set.

The article is organized as follows: In Section 2, we recall several definitions and results of fractal calculus. In Section 3, we define generalized F^{α}-integral so called $*F^{\alpha}$ integrals on $[a,b]$ with out the restriction of $Sch(f)$ is contained in α-perfect set. Several properties of $*F^{\alpha}$ integrable functions have been covered. A suitable fundamental theorem of calculusare presented for $*F^{\alpha}$ integrable functions. Relationship of F^{α} and $*F^{\alpha}$ integrable functions are discussed. In Section 4, we solve fractal differential equation $D_F^{\alpha} S_F^{\alpha}(x) = f\big(t, S_F^{\alpha}(x)\big)$ with $S_F^{\alpha}(\tau) = \xi$ in sense of $*F^{\alpha}$-integral. We state counter example to show that our solutions are not necessarily F^{α} integrable functions.

2 Preliminaries

Throughout the article, we denote $\mathbb{N}$ be set of natural numbers, $\mathbb{R}$ set of real numbers, and $F \subset [a,b]$ be a fractal set. In our work $[a,b] \subset \mathbb{R}$. If f is not constant over any open interval (a,b) including x, then a point x is a point of change of a function f. The set of change of f is the set of all points of change of f; it is represented by the symbol $Sch(f)$. For better captures the scaling behavior of fractal sets and aligns with established fractal calculus principles we consider fractal summation measure. Let $0 < \alpha < 1$. The α-dimensional fractal summation measure of $[a,b] \subset \mathbb{R}$ is defined as

$$\mathcal{H}^{\alpha}([a,b]) = \inf\left\{ \sum_{i=1}^{\infty} \big(\Gamma(\alpha+1)(x_i - x_{i-1})^{\alpha} \text{ whenever } [a,b] \subset \bigcup_{i=1}^{\infty} [a_i, b_i] \right\}.$$

$\mathcal{H}^{\alpha}(.)$ is a Borel regular measure and

$$\mathcal{H}^{\alpha}([a,b]) = \inf\{\alpha : \ \mathcal{H}^{\alpha}(F,a,b) = 0\} = \sup\{\alpha : \ \mathcal{H}^{\alpha}(F,a,b) = \infty\}$$

is called α-dimension of $[a,b]$. We called any α-set $E \subset [a,b]$ to be $\mathcal{H}^{\alpha}$-measurable if $0 < \mathcal{H}^{\alpha}(E) < \infty$. Recall F-limits, and F-continuous function as below:

Definition 2.1 *[6, Definition 4.11] Let $F \subset \mathbb{R}$, $f : \mathbb{R} \to \mathbb{R}$ and $x \in F$. A number l is called the limit of f through the points of F, or simply F-limit of f, as $y \to x$, if for given any $\epsilon > 0$, there exists $\delta > 0$ such that $y \in F$, & $|y - x| < \delta$ implies $|f(y) - f(x)| < \epsilon$. If such a number exists, then it is denoted by $l = F\text{-}\lim_{y \to x} f(y)$.*

Definition 2.2 *[6, Definition 4.12] A given function $f\ :\ [a,b]\ \to\ \mathbb{R}$ is called F-continuous if $f(x)\ =\ F\text{-}\lim_{y\to x} f(y)$.*

When $x \in F \subset [a,b]$ and It is very clear that every continous function is F-continous but inverse may not true (see [15]). Let $f : [a,b] \times \mathbb{R} \to \mathbb{R}$. We call f is Carathódory function if

1. $f(t,x)$ is fractal summation measure;
2. $f(t,x)$ is F-continuous for almost every $t \in F \subset [a,b]$;
3. $f(t,x)$ is bounded on $[a,b]$.

Recall the definition of a flag function for F as below:

Definition 2.3 *[6, Definition 4.1] The flag function for a thin fractal sets F and a closed interval $I\ =\ [a,b],\ a < b \in \mathbb{R}$ is defined by*

$$\theta(F,I) = \begin{cases} 1, \text{ if } F \cap I \neq \emptyset \\ \\ 0, \text{ otherwise.} \end{cases}$$

Definition 2.4 *[6, Definition 4.3] For a given set F we consider a sub-division $P_{[a,b]}$, which is a finite set of points $\Big\{a = x_0, x_1, ..., x_n = b\Big\}$. The fractal summation is $\sigma^{\alpha}[F,I] = \sum_{i=1}^{n} \Gamma(\alpha+1)(x_i - x_{i-1})^{\alpha}\theta(F,[I_i])$, where $I_i = [x_{i-1}, x_i]$ & $0 < \alpha \le 1$, and $\sigma^{\alpha}[F,I] \le 1$.*

Definition 2.5 *[6, Definition 4.4] The mass function $\gamma^{\alpha}(F,a,b) = \lim_{\delta\to 0} \inf \lim_{P_{[a,b]}:\ |P|\le\delta} \sigma^{\alpha}[F,I]$ where $|P| = \max \lim_{1\le i\le n}(x_i - x_{i-1})$.*

Definition 2.6 *[6, 15] The integral staircase function $S_F^{\alpha}(x)$ corresponding for the set F is*

$$S_F^{\alpha}(x) = \begin{cases} \gamma^{\alpha}(F, a_0, x), \text{ if } x \ge a_0 \\ \\ -\gamma^{\alpha}(F, a_0, x), \text{ otherwise} \end{cases}$$

where a_0 is a fixed and real number. The γ-dimension of $F \cap [a,b]$ is defined as

$$\begin{aligned} dim_{\gamma}(F \cap [a,b])(F \cap [a,b]) &= \inf\Big\{\alpha :\ \gamma^{\alpha}(F,a,b) = 0\Big\} \\ &= \sup\Big\{\alpha :\ \gamma^{\alpha}(F,a,b) = \infty\Big\} \end{aligned}$$

where $\alpha = dim_{\gamma}F$ & $S_F^{\alpha}(x)$ is finite for all $x \in \mathbb{R}$.

Definition 2.7 *[15] Let $f : [a,b] \to \mathbb{R}$ be given function then the right and left D_{+F}^{α}-derivative of $f(x)$ at $x \in F$ is defined as follows: for given $\varepsilon > 0$ there exists $\delta_{\varepsilon}(t) > 0$ such that $y \in F,\ 0 < y - x < \delta_{\varepsilon}(x)$ implies $\Big|\frac{f(y)-f(x)}{S_F^{\alpha}(y)-S_F^{\alpha}(x)} - D_{+F}^{\alpha}f(x)\Big| < \varepsilon$. The left D_{-F}^{α}-derivative of $f(x)$ at $x \in F$ is defined as follows: for given $\varepsilon > 0$ there exists $\delta_{\varepsilon}(t) > 0$ such that $y \in F,\ 0 < x - y < \delta_{\varepsilon}(x)$ implies $\Big|\frac{f(y)-f(x)}{S_F^{\alpha}(y)-S_F^{\alpha}(x)} - D_{-F}^{\alpha}f(x)\Big| < \varepsilon$. If $D_{-F}^{\alpha}f(x) = D_{+F}^{\alpha}f(x)$ then $f(x)$ is local fractal differentiable and*

$$D_F^{\alpha}f(x) = \begin{cases} F - \lim_{y\to x} \frac{f(y)-f(x)}{S_F^{\alpha}(y)-S_F^{\alpha}(x)}, \text{ if } x \in F \\ \\ 0, \text{ otherwise,} \end{cases}$$

where $S_F^{\alpha}(x)$ is integral staircase function.

Theorem 2.8 *[15, Lemma 48] The F^{α}-derivative of a constant function $f:\mathbb{R}\to\mathbb{R},\ f(x)=k\in\mathbb{R}$ is zero (i.e. $D_F^{\alpha}(f)=0$).*

Theorem 2.9 *[15, Lemma 49] The derivative of the staircase integral itself is the characteristic function χ_F of F i.e. $\big(D_F^{\alpha}(S_F^{\alpha}(x))=\chi_F(x)\big)$.*

Recall F^{α}-integral of f on I from [6, 15] as follows:

Definition 2.10 *[15, Definition 37] Let f be a bounded function on F. We called f to be F^{α}-integrable on I if $\int_{\underline{a}}^{b} f d_F^{\alpha} = \int_a^{\bar{b}} f d_F^{\alpha}$ where $\int_{\underline{a}}^{b} f d_F^{\alpha} = \sup L^{\alpha}(f,F,P),\ L^{\alpha}(f,F,P)=\sum\limits_{i=1}^{n} m[f,F,[I_i]]$ and $\int_a^{\bar{b}} f d_F^{\alpha} = \inf U^{\alpha}(f,F,P)=\sum\limits_{i=1}^{n} M[f,F,[I_i]]$ of*

$$M[f,F,[I_i]]=\begin{cases}\sup\limits_{x\in F\cap[I_i]} f(x), & \textit{if } F\cap I\neq\emptyset\\ \\ 0, & \textit{otherwise}\end{cases}$$

and

$$m[f,F,[I_i]]=\begin{cases}\inf\limits_{x\in F\cap[I_i]} f(x), & \textit{if } F\cap I\neq\emptyset\\ \\ 0, & \textit{otherwise}.\end{cases}$$

F^{α}-integral is a Riemann type integrals. It is very clear from the definition that F^{α}-integral maintain linearity and sub-additive property. The F-derivative of a given function $f:[a,b]\to\mathbb{R}$ is defined as follows.

The F^{α}-integrable functions full fill following fundamental theorem of Calculus.

Theorem 2.11

1. *[15, Theorem 54]*(First fundamental theorem) Let $F\subset\mathbb{R}$ be an α-perfect set. If f is bounded and F-continuous on $F\cap[a,b]$ and $g(x)=\int_a^x f(x)d_F^{\alpha}(x)\ \forall\ x\in[a,b]$ then $D_F^{\alpha}(g(x))=f(x)\chi_F(x)$.

2. [15, Theorem 55] (Second fundamental theorem) Let $f:[a,b]\to\mathbb{R}$ be F-continous, F^{α}-differentiable such that $Sch(f)\subset F$ and $h:[a,b]\to\mathbb{R}$ be F-continuous such that $h(x)\chi_F(x)=D_F^{\alpha}(f(x))$. Then $\int_a^b h(x)d_F^{\alpha}x=f(b)-f(a)$.

3 $*F^{\alpha}$ integrable functions

In this Section, we define $*F^{\alpha}$-integral with more general sense. Several properties of $*F^{\alpha}$-integrable functions are discussed here. Further we establish relationship of F^{α} and $*F^{\alpha}$-integrable functions. In order to define $*F^{\alpha}$-integrable functions, let $\widehat{P}=\Big\{([a_i,b_i],t_i):\ i=1,2,...,n\Big\}=\Big\{([a_i,b_i],t_i)\Big\}_{i=1}^{n}$ be a partition on I. The partition $\widehat{P}$ is δ-fine if $[a_i,b_i]\subset[t_i-\delta(t_i),t_i+\delta(t_i)]$. Suppose $\widehat{P}$ be a δ-fine partition of I. We define the fractal summation

$$\sigma_*^{\alpha}[F,I]=\sum_{i=1}^{n}\Gamma(\alpha+1)(x_i-x_{i-1})^{\alpha}\theta(F,[I_i])\text{ provided the right side exists.}$$

Next, using gauge function, we define the generalized coarse grained mass function of $F\cap I$ by $*\gamma_{\delta(t)}^{\alpha}(F,a,b)=\inf\lim\limits_{|\widehat{P}|<\sup\{\delta(t_i):\ t_i\in[x_{i-1},x_i]}\sigma_*^{\alpha}[F,I]$, where $|\widehat{P}|=\max\lim\limits_{1\le i\le n}(x_i-x_{i-1})$. The generalized mass function

$*\gamma^{\alpha}(F,a,b) = \lim_{\sup\{\delta(t_i):\ t_i \in [x_{i-1},x_i]\} \to 0} *\gamma^{\alpha}_{\delta(t)}(F,a,b)$. By using the gauge function, the generalized integral staircase function $*S^{\alpha}_F(x)$ of order α for a given set F as

$$*S^{\alpha}_F(x) = \begin{cases} *\gamma^{\alpha}(F,a_0,x), \text{ if } x \geq a_0 \\ \\ -*\gamma^{\alpha}(F,a_0,x), \text{ otherwise} \end{cases}$$

The $*\gamma$-dimension of $F \cap I$, indicated as $dim_{*\gamma}(F \cap I)$ and is defined by

$$\begin{aligned} dim_{*\gamma}(F \cap I)(F \cap I) &= \inf\Big\{\alpha :\ *\gamma^{\alpha}(F,a,b) = 0\Big\} \\ &= \sup\Big\{\alpha :\ *\gamma^{\alpha}(F,a,b) = \infty\Big\}. \end{aligned}$$

It is known that $*\gamma$-dimension is less than γ-dimension [7, 15].

Let $*S^{\alpha}_F(x)$ be finite for $x \in F \cap I$, $\widehat{P}$ is a δ-finite partition on I. For a function $f : \mathbb{R} \to \mathbb{R}$ is contained in α-perfect set F, the generalised Riemann sum of f corresponding to $\widehat{P}$ is $S(f,\widehat{P}) = \sum_{i=1}^{n} f(t_i)\Big(*S^{\alpha}_F(x_i) - *S^{\alpha}_F(x_{i-1})\Big)$, $t_i \in [x_{i-1}, x_i]$.

Definition 3.1 *[7] A function $f : I \to \mathbb{R}$ having property that $Sch(f)$ is contained in α-perfect set F is called Gauge integrable* ($*F^{\alpha}$-integrable) on $F \cap I$, if there exists a number $C \in \mathbb{R}$ so that for every $\varepsilon > 0$ there exists a gauge δ_ε on I such that if $\widehat{P}$ is δ_ε-fine on I then $\big|S(f,\widehat{P}) - C\big| \leq \varepsilon$. Here $C = \int_a^b f(x) d^{\alpha}_F x$.

We start the section with Cousin's Lemma in our setting.

Lemma 3.2 *If δ_ε is a gauge on $F \cap I$ then there exists a δ-fine partition of $F \cap I$.*

Proof: The proof is similar to [11, Theorem 2.31] that used on the real line.

Let us remove the restriction $Sch(f) \subset F$. Then for a function $f : I \to \mathbb{R}$, with α-perfect set $F \subset I$, the generalized Riemann sum of f corresponding to $\widehat{P}$ is $S(f,\widehat{P}) = \sum_{i=1}^{n} f(t_i)\Big(*S^{\alpha}_F(x_i) - *S^{\alpha}_F(x_{i-1})\Big)$, $t_i \in [x_{i-1}, x_i]$ is remain same. If the generalized Riemann sum exists, we refine the definition of $*F^{\alpha}$-integrable function as below.

Definition 3.3 *A function $f : I \to \mathbb{R}$ called Gauge integrable* ($*F^{\alpha}$-integrable) on $F \cap I$, if there exists a number $C \in \mathbb{R}$ so that for every $\varepsilon > 0$ there exists a gauge δ_ε on I such that if $\widehat{P}$ is δ_ε-fine on I then $\big|S(f,\widehat{P}) - C\big| \leq \varepsilon$. Here $C = \int_a^b f(x) d^{\alpha}_F x$.

The uniqueness of $*F^{\alpha}$-integrability of $f : I \to \mathbb{R}$ is follows from the definition. Let $f : I \to \mathbb{R}$ be $*F^{\alpha}$-integrable functions on I. Let $[a,x] \subset F \cap I$, then $\Xi(x) = \int_a^x f(x) d^{\alpha}_F x$ is called a primitive of f on $[a,x]$. It is not hard to see $\Xi(x) = *S^{\alpha}_F(x)$. It is clear from the definition: F^{α}-integrable function are $*F^{\alpha}$-integrable. The following example shows that $*F^{\alpha}$-integrable function are not necessarily F^{α}-integrable.

Example 3.4 *Consider a discontinuous function as*

$$f(x) = \begin{cases} 1, \ \textit{if } x \in \{I \cap \mathbb{Q} \cap F \ (\textit{cantor set})\} \\ \\ 0, \ \textit{otherwise} \end{cases}$$

where $\mathbb{Q}$ is set of rational number and $f(x)$ is discontinuous at every point of I.

Definition 3.5 *[9, Definition 7] A function $f : I \to \mathbb{R}$ is consider to be fractal absolutely continuous on I if for a given $\varepsilon > 0$, there exists a $\delta > 0$ such that whenever a finite sequence of pairwise disjoint sub intervals $[a_k, b_k]$ of I is given, where $w^{-1}(a_k) < t_k = w^{-1}(b_k)$ the following condition holds: $\sum_k \big| *S_F^{\alpha}(b_k) - *S_F^{\alpha}(a_k)\big| < \delta$ implies $\sum_k \big|f(b_k) - f(a_k)\big| < \varepsilon$. We denote fractal absolutely continuity is AC_F.*

We define generalized fractal absolutely continuous, fractal bounded variation and generalized fractal absolutely continuous functions in restricted sense as follows.

Definition 3.6

1. *A function $f : I \to \mathbb{R}$ is called generalized fracal absolutely continuous on $[a_k, b_k]$ if $f_{|_{[a_k,b_k]}}$ is F-continuous on $[a_k, b_k]$ and $[a_k, b_k]$ can be written as a countable union of sets on I in which f is fractal absolute continuous.*

2. *Let us define the oscillation of $*S_F^{\alpha}$ on I by $\mathtt{h}\Big(*S_F^{\alpha}, [a_k, b_k]\Big) = \sup\Big\{| *S_F^{\alpha}(b_k) - *S_F^{\alpha}(a_k)| : \ a \leq a_k \leq b_k \leq b\Big\}$. The fracal weak variation and fracatal strong variation of S_F^{α} are defined by*

$$V_F(*S_F^{\alpha}, [a_k, b_k]) = \sup\Big\{\sum_{k=1}^{n} | *S_F^{\alpha}(b_k) - *S_F^{\alpha}(a_k)|\Big\}$$

and

$$V_{F,*}(*S_F^{\alpha}, [a_k, b_k]) = \sup\Big\{\sum_{k=1}^{n} \mathtt{h}\Big(S_F^{\alpha}, [a_k, b_k]\Big)\Big\}$$

where the supremum is taken over all possible finite collections of non-overlapping sub intervals $[a_k, b_k]$. A given function $f : I \to \mathbb{R}$ fractal bounded variation on $[a_k, b_k] \subset I$ if $V_F(*S_F^{\alpha}, [a_k, b_k])$ is finite. The given function $f : I \to \mathbb{R}$ is called fractal bounded variation in the restricted sense on $[a_k, b_k]$ if $V_{F,*}(*S_F^{\alpha}, [a_k, b_k])$ is finite.

3. *We say f is fractal absolutely continuous on $[a_k, b_k]$ in the restricted sense $AC_{F,*}$, if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that for every sequence of pairwise disjoint intervals $\Big\{[a_k, b_k] : \ 1 \leq k \leq n\Big\}$ ends points are in $[a_k, b_k]$ and $\sum_k (*S_F^{\alpha}(b_k) - *S_F^{\alpha}(a_k)) < \delta$ then $\sum_k \mathtt{h}\big(*S_F^{\alpha}, [a_k, b_k]\big) < \varepsilon$.*

4. *We say f is generalized fracal absolutely continuous in restricted sense $(ACG_{F,*})$ in $[a_k, b_k]$ if $f_{|_{[a_k,b_k]}}$ is F-continuous and $[a_k, b_k]$ is a countable union of sets $\Big\{[a_k, b_k]\Big\}_{k=1}^{n}$ such that f is $AC_{F,*}$ on each $[a_k, b_k]$.*

Theorem 3.7 *Let $f : I \to \mathbb{R}$ and $E \subseteq I$.*

1. *If f is $AC_F(ACG_F)$ on E, then f is $BV_F(BVG_F)$ on E.*

2. *If f is $AC_{F,*}(ACG_{F,*})$ on E then f is $BV_{F,*}(BVG_{F,*})$ on E.*

3. *If f is $BV_{F,*}$ on E, then f is $BV_{F,*}$ on $\overline{E}$, where $\overline{E}$ is closure of E.*

4. *Suppose $f_{|_{\overline{E}}}$ is F-continuous on $\overline{E}$. If f is $BV_F, AC_F, AC_{F,*}$ on E, then f is $BV_F, AC_F, AC_{F,*}$ on $\overline{E}$.*

Proof: For (1) : Corresponding to $\varepsilon > 0$, choose $\delta > 0$ so that $\sum\limits_{k=1}^{n} |*S_F^\alpha(b_k) - *S_F^\alpha(a_k)| < \varepsilon$ whenever $\left\{[a_k, b_k] : \ 1 \leq k \leq n\right\}$ is any collection of non-overlapping intervals in I satisfying $\sum\limits_{k=1}^{n} |f(b_k) - f(a_k)| < \delta$. Since $V_F(S_F^\alpha, I) = \sum\limits_{k=1}^{n} V_F(*S_F^\alpha, [a_k, b_k])$. So, by [15, Theorem 16]

$$\begin{aligned} V_F(S_F^\alpha, I) &= \sup\left\{ \sum_{k=1}^{n} | * S_F^\alpha(b_k) - *S_F^\alpha(a_k)| \right. \\ &= \sup\left\{ \sum_{k=1}^{n} | * \gamma^\alpha(F, a_k, b_k)| \right. \end{aligned}$$

whenever $a_k, b_k \in (a, b) \subset I$ with $a_k < b_k$. Since $F \cap (a_k, b_k) = \emptyset$, by [15, Lemma 9], $\sup\left\{ \sum\limits_{k=1}^{n} | * \gamma^\alpha(F, a_k, b_k)| = 0 < \infty \right.$. So, f is of BV_F on E. Similar way we can proof for ACG_F.

The proof of (2) is analogous to (1).

For (3): Since f is $BV_{F,*}$ on E, it is bounded on E. Say it is bounded by M. Let $\left\{[a_k, b_k] : \ 1 \leq k \leq n\right\}$ be a finite collection of non-overlapping intervals that have end points in $\overline{E}$. Let $\bigcup\limits_{k=1}^{n} [a_k, b_k] = I$. Let us construct $\left\{[a_r, b_r] : \ 1 \leq r \leq p\right\}$ be the collection of intervals $[a_k, b_k]$, in increasing order, that intersect E, and let $\left\{K_j : \ 1 \leq j \leq q\right\}$ be the remaining intervals, also in increasing ordr. For each r, choose a point v_r in $E \cap [a_r, b_r]$. By our construction, no two $K_j's$ are adjacent. Hence, for each j there exists a unique integer k_j such that $K_j \subseteq [v_{k_j}, v_{k_{j+1}}]$. Let $I_r = [\rho, \beta]$ for some r in $\left\{2, 3, ..., p-1\right\}$. It is observed that

$$\begin{aligned} h(*S_F^\alpha, [a_r, b_r]) &\leq h(*S_F^\alpha, [\rho, v_k]) + h(*S_F^\alpha, [v_k, \beta]) \\ &\leq h(*S_F^\alpha, [v_{k-1}, v_k]) + h(*S_F^\alpha, [v_k, v_{k+1}]). \end{aligned}$$

We have

$$\begin{aligned} \sum_{k=1}^{n} h(*S_F^\alpha, [a_k, b_k]) &= \sum_{j=1}^{q} h(*S_F^\alpha, K_j) + \sum_{r=1}^{q} h(*S_F^\alpha, [a_r, b_r]) \\ &\leq \sum_{j=1}^{q} h(*S_F^\alpha, [v_{k_j}, v_{k_{j+1}}]) + h(*S_F^\alpha, [a_1, b_1]) \\ &+ h(*S_F^\alpha, [a_p, b_p]) + h(*S_F^\alpha, [v_1, v_2]) + 2\sum_{k=2}^{p-2} h(*S_F^\alpha, [v_k, v_{k+1}]) + h(*S_F^\alpha, [v_{p-1}, v_p]) \\ &\leq V_{F,*}(*S_F^\alpha, [a_k, b_k]) + 2h(*S_F^\alpha, [a_k, b_k]) + 2V_{F,*}(*S_F^\alpha, [a_k, b_k]) \\ &\leq 3V_{F,*}(*S_F^\alpha, [a_k, b_k]) + 4M. \end{aligned}$$

So, $V_{F,*}(*S_F^\alpha, \overline{E})$ is finite and the function f is $BV_{F,*}$ on $\overline{E}$.

For (4): Consider AC_F case: Let $\varepsilon > 0$ and choose $\delta > 0$ so that $\sum\limits_{k=1}^{n} | * S_F^\alpha(b_k) - *S_F^\alpha(a_k)| < \frac{\varepsilon}{2}$ whenever

$\Big\{[a_k, b_k] : 1 \leq k \leq n\Big\}$ is a finite collection of non overlapping intervals that have end points in E with $\sum\limits_{k=1}^{n} |f(b_k) - f(a_k)| < \delta$. Let $\Big\{[a'_k, b'_k] : 1 \leq k \leq n\Big\}$ be a finite collection of non overlapping intervals that have end points in $\overline{E}$ and satisfy $\sum\limits_{k=1}^{n} |f(b'_k - f(a'_k)| < \delta$. Clearly E is dense in $\overline{E}$. Since $f_{|_{\overline{E}}}$ is F-continuous on $\overline{E}$, there exists a finite collection $\Big\{[a_k, b_k] : 1 \leq k \leq n\Big\}$ of non overlapping intervals that have end points in E such that $\sum\limits_{k=1}^{n} |f(b_k)-f(a_k)| < \delta$, $\sum\limits_{k=1}^{n} |{*S_F^\alpha}(a'_k)-{*S_F^\alpha}(a_k)| < \frac{\varepsilon}{4}$, & $\sum\limits_{k=1}^{n} |{*S_F^\alpha}(b'_k)-{*S_F^\alpha}(b_k)| < \frac{\varepsilon}{4}$. Then we have

$$
\begin{aligned}
\sum_{k=1}^{n} |*S_F^\alpha(b'_k) - *S_F^\alpha(a'_k)| &\leq \sum_{k=1}^{n} |*S_F^\alpha(b'_k) - *S_F^\alpha(b_k)| \\
&+ \sum_{k=1}^{n} |*S_F^\alpha(b_k) - *S_F^\alpha(a_k)| + \sum_{k=1}^{n} |*S_F^\alpha(a_k) - *S_F^\alpha(a'_k| \\
&< \frac{\varepsilon}{4} + \frac{\varepsilon}{2} + \frac{\varepsilon}{4} = \varepsilon.
\end{aligned}
$$

Let $E \subseteq I$ such that $a = \inf E,\ b = \sup E$. If we compress the sequence of intervals $\Big\{[a_k, b_k]\Big\}$ to $(a, b)\backslash E$, then $\Big\{[a_k, b_k]\Big\}$ is called sequence of intervals contiguous to E in I. We denote $\sum\limits_{k=1}^{\infty} h(*S_F^\alpha, \widehat{[a_k, b_k]})$ be series of oscillation of f on the intervals contiguous to E in I.

Lemma 3.8 *Let $E \subseteq I$ with $\inf E = a,\ b = \sup E$. Let us construct $\Big\{[a_k, b_k]\Big\}$ be the sequence of disjoint intervals contibuous to E in I, then for a given function $f : I \to \mathbb{R}$, $f\Big(* S_F^\alpha, I\Big) \leq V_F(*S_F^\alpha, E) + 2\sum\limits_{k=1}^{\infty} f(*S_F^\alpha, \widehat{[a_k, b_k]})$.*

Proof: Let $m = \inf\Big\{f(t) : t \in E\Big\}$ and $M = \sup\Big\{f(t) : t \in E\Big\}$. Clearly $m \leq f(t) \leq M\ \forall\ t \in E$. Consequently, $m - h(*S_F^\alpha, \widehat{[a_k, b_k]}) \leq f(t) \leq M + h(*S_F^\alpha, \widehat{[a_k, b_k]})\ \forall\ t \in [a_k, b_k]$, and $m - h(*S_F^\alpha, \widehat{[a_k, b_k]}) \leq f(t) \leq M + h(*S_F^\alpha, \widehat{[a_k, b_k]})\ \forall\, t \in [a, b]$. So,

$$
\begin{aligned}
f(*S_F^\alpha, I) &\leq M - m + 2\sum_{k=1}^{\infty} h(*S_F^\alpha, \widehat{[a_k, b_k]}) \\
&\leq V_F(f, E) + 2\sum_{k=1}^{\infty} h(*S_F^\alpha, \widehat{[a_k, b_k]}).
\end{aligned}
$$

Theorem 3.9 *Let E be a bounded, closed set of I and let $f : I \to \mathbb{R}$ be BV_F. Then f is $BV_{F,*}$ on E if and only if the series of the oscillation $\sum\limits_{k=1}^{n} h\Big(* S_F^\alpha, [a_k, b_k]\Big)$ on E is convergent.*

Proof: Let $\Big\{[a_k, b_k]\Big\}_{k=1}^{n}$ be the sequence of intervals contiguous to E in F. If f is $BV_{F,*}$ on E then $\sum\limits_{k=1}^{n} h\Big(* S_F^\alpha, [a_k, b_k]\Big) < V_{F,*}(f, E)$ for all n. Clearly $\sum\limits_{k=1}^{n} h\Big(* S_F^\alpha, [a_k, b_k]\Big)$ is finite.

Conversely, suppose $\sum\limits_{k=1}^{n} h\Big(*S_F^{\alpha}, [a_k, b_k]\Big)$ is convergent on E. Let f be BV_F on E and let $\Big\{K_j : 1 \leq j \leq p\Big\}$ be a finite collection of non-overlapping intervals that have end points in E. Let $\bigcup\limits_{j=1}^{p} K_j = I$. Next,

$$\begin{aligned}\sum_{j=1}^{p} h\Big(*S_F^{\alpha}, K_j\Big) &\leq \sum_{j=1}^{p} V_F\Big(f, E \cap K_j\Big) + 2 \sum_{\big\{[a_k,b_k]\big\}_{k=1}^{n} \subseteq K_j} h\Big(*S_F^{\alpha}, \Big\{\widehat{[a_k, b_k]}\Big\}_{k=1}^{n}\Big) \\ &= V_F(*S_F^{\alpha}, E)_2 \sum_{k=1}^{\infty} h\Big(*S_F^{\alpha}, \Big\{\widehat{[a_k, b_k]}\Big\}_{k=1}^{n}\Big) \\ &< \infty.\end{aligned}$$

So, f is $BV_{F,*}$ on E.

3.1 Properties of $*F^{\alpha}$-integral

In this Section, several properties of $*F^{\alpha}$-integrable functions are discussed. The following properties of $*F^{\alpha}$-integral can be prove from the definition of $*F^{\alpha}$-integral.

Theorem 3.10 *1. Let $a < b$ and f be an $*F^{\alpha}$-integrable function on I. Let $c \in (a, b)$. Then f is $*F^{\alpha}$-integrable on $[a, c]$ and $[c, b]$. Further, $\int_a^b f(x)d_F^{\alpha}x = \int_a^c f(x)d_F^{\alpha}x + \int_c^b f(x)d_F^{\alpha}x$.*

2. If f is $*F^{\alpha}$-integrable on I, λ is a real number, then $\int_a^b \lambda f(x)d_F^{\alpha}x = \lambda \int_a^b f(x)d_F^{\alpha}x$.

3. If f and g are $*F^{\alpha}$-integrable on I, then $\int_a^b \big(f(x) + g(x)\big)d_F^{\alpha}x = \int_a^b f(x)d_F^{\alpha}x + \int_a^b g(x)d_F^{\alpha}x$.

4. If f and g are $*F^{\alpha}$-integrable over I, and $f(x) \geq g(x)$ for all $x \in F \cap I$, then $\int_a^b f(x)d_F^{\alpha}x \geq \int_a^b g(x)d_F^{\alpha}x$.

Lemma 3.11 *If $\chi_F(x)$ is the characteristic function of $F \subset \mathbb{R}$, then $\int_a^b \chi_F(x)d_F^{\alpha}x = *S_F^{\alpha}(b) - *S_F^{\alpha}(a)$.*

Theorem 3.12 *A function $f : I \to \mathbb{R}$ is $*F^{\alpha}$-integrable on $F \cap I$ if and only if for a given $\varepsilon > 0$ there exists a gauge δ_ε on I such that $\big|S(f, \widehat{P}) - S(f, \widehat{Q})\big| < \varepsilon$ for each pair of δ_ε-fine tagged partitions $\widehat{P}$ and $\widehat{Q}$ of I.*

Proof: Let $\varepsilon > 0$ be given. Since f is $*F^{\alpha}$-integrable on $F \cap I$, there exists a number $C \in \mathbb{R}$ so that for every $\varepsilon > 0$ there exists a gauge δ_ε on I so that

$$\big|S(f, \widehat{P}_1) - \int_a^b f d_F^{\alpha}\big| < \frac{\varepsilon}{2} \tag{1}$$

whenever $\widehat{P}_1$ is δ_ε-fine partition of I. Let $\widehat{P}$ and $\widehat{Q}$ are δ_ε-fine partitions of I, and the triangle inequality(1) yield

$$\begin{aligned}\big|S(f, \widehat{P}) - S(f, \widehat{Q})\big| &\leq \big|S(f, \widehat{P}) - \int_a^b f d_F^{\alpha}\big| + \big|S(f, \widehat{Q}) - \int_a^b f d_F^{\alpha}\big| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

Conversely, let for each $n \in \mathbb{N}$, let δ_n be a gauge on I so that $\big|S(f, \widehat{Q}_n) - S(f, \widehat{R}_n)\big| < \frac{1}{n}$ for each pair of δ_n-fine partition $\widehat{Q}_n$ and $\widehat{R}_n$ of I. Let us define $\Delta = \min\big\{\delta_1, \delta_2, ..., \delta_n\big\}$. Using Lemma 3.2; $\widehat{P}_n$

is tagged partition of I sub-ordinate to δ_ε. Let $\varepsilon > 0$ be given and $\frac{1}{N} < \varepsilon$ where $N \in \mathbb{N}$. Consider $n_1,\ n_2 \in \mathbb{N}$, and $m = \min\{n_1, n_2\} \geq N$. Clearly $\widehat{P}_{n_1}$ and $\widehat{P}_{n_2}$ are Δ_m-fine tagged partition of I so that $|S(f, \widehat{P}_{n_1}) - S(f, \widehat{P}_{n_2})| < \frac{1}{N} < \varepsilon$. So, $\left\{S(f, \widehat{P}_n)\right\}_{n=1}^{\infty}$ is Cauchy sequence of real numbers. Say $\left\{S(f, \widehat{P}_n)\right\}_{n=1}^{\infty} \to A$ as $n \to \infty$. Let $\widehat{P}$ be a Δ_N-fine partition of I. Since $\{\Delta_n\}_{n=1}^{\infty}$ is non increasing. It is easy to see Δ_n-fine partition $\widehat{P}_n$ is Δ_N-fine for every integers $n \geq N$. Thus

$$\begin{aligned}\left|S(f, \widehat{P}) - A\right| &= \lim_{n\to\infty} \left|S(f, \widehat{P}) - S(f, \widehat{P}_n)\right| \\ &\leq \frac{1}{N} = \varepsilon.\end{aligned}$$

So, f is $*F^\alpha$-integrable and $A = \int_a^b f d_F^\alpha$.

Next, we prove a Saks Henstock type lemma for $*F^\alpha$ integrable functions.

Lemma 3.13 *Let $f : I \to \mathbb{R}$ be $*F^\alpha$-integrable on I and $\Xi(x) = \int_a^x f d_F^\alpha x,\ x \in I$. Let $\varepsilon > 0$ be given. Also let δ_ε be a choosen positive function on I so that $\left|S(f, \widehat{P}) - \Xi(b)\right| < \varepsilon$ whenever $\widehat{P} = \left\{(t_i, [x_{i-1}, x_i] :\ 1 \leq i \leq N\right\}$ is sub-ordinate to δ_ε on I. Then $\left|S(f, \widehat{P}) - \int_{\widehat{P}} f d_F^\alpha\right| \leq \varepsilon$ and $\sum_{i=1}^{N} \left|S(f, \widehat{P}) - [\Xi(x_i) - \Xi(x_{i-1})]\right| \leq 2\varepsilon$.*

Proof: Let $\left\{K_j :\ 1 \leq j \leq m\right\}$ be the collection of closed intervals I that are contiguous to $\widehat{P}$. Let $\nu > 0$ and for each j, let $\widehat{P}_j$ be tagged partition of K_j sub-ordinate to δ_ε and satisfies $\left|S(f, \widehat{P}_j) - \int_{\widehat{P}_j} f d_F^\alpha\right| < \frac{\nu}{m}$. Let $\widehat{Q} = \bigcup_{j=1}^{m} \widehat{P}_j$. Then $\widehat{Q}$ is a tagged partition of I sub-ordinate to δ_ε and

$$\begin{aligned}\left|S(f, \widehat{P}) - \int_{\widehat{P}} f d_F^\alpha\right| &= \left|S(f, \widehat{P}) + \sum_{j=1}^{m} S(f, \widehat{P}_j) - \int_{\widehat{P}} f d_F^\alpha - \sum_{j=1}^{m} \int_{\widehat{P}_j} f d_F^\alpha + \sum_{j=1}^{m} \left[\int_{\widehat{P}_j} f d_F^\alpha - S(f, \widehat{P}_j)\right]\right| \\ &\leq \left|S(f, \widehat{Q}) - \int_{\widehat{Q}} f d_F^\alpha\right| + \sum_{j=1}^{m} \left|S(f, \widehat{P}_j) - \int_{\widehat{P}_j} f d_F^\alpha\right| \\ &< \varepsilon + \nu.\end{aligned}$$

Consequently, $\left|S(f, \widehat{P}) - \int_{\widehat{P}} f d_F^\alpha\right| \leq \varepsilon$. Similarly, we can prove second inequality.

Some more properties of $*F^\alpha$- integrable functions are as follows:

Theorem 3.14 1. *Let $f : I \to \mathbb{R}$ be $*F^\alpha$-integrable on I and $E \subset I$. If $\Xi(x) = \int_a^x f d_F^\alpha x,\ x \in I$. Then Ξ is F-continuous on E.*

2. *Let $f : I \to \mathbb{R}$ be $*F^\alpha$-integrable on I and $E \subset I$. . If $\Xi(x) = \int_a^x f d_F^\alpha x,\ x \in I$. Then Ξ is $BVG_{F,*}$ on E.*

Proof: For (1) : Let $t \in E \subset I$. Let $\varepsilon > 0$ be given and δ_ε be a positive function on I so that $\left|S(f, \widehat{P}) - \int_a^b f d_F^\alpha\right| < \varepsilon$. Let $\nu = \min\{\delta(t), (1 + |f(t)|)^{-1}\}$. Consider $s \in I \cap [t, t + \nu)$. Let $\widehat{P}_1$ be tagged partition on $[a,t]$, and $\widehat{Q} = \widehat{P}_1 \cup (t,[t,s])$. Then $\widehat{Q}$ is sub-ordinate to $[a,s]$. Let $s \to t$. By Lemma 3.13

$$\begin{aligned}\Big|\Xi(s) - \Xi(t)\Big| &= \Big|\Xi(s) - S(f,\widehat{Q}) + f(t)(s-t) + S(f,\widehat{P}_1) - \Xi(t)\Big| \\ &< \Big|S(f,\widehat{Q}) - \Xi(s)\Big| + |f(t)|\nu + \Big|S(f,\widehat{P}_1) - \Xi(t)\Big| \\ &< \varepsilon + |f(t)|\nu + \varepsilon = 3\varepsilon.\end{aligned}$$

Hence Ξ is F-continuous on E as $s \to t$.

For (2) : Let us construct a positive function δ_ε on I so that $0 < \delta(t) < 1$ for all $t \in I$ and $\Big|S(f,\widehat{P}) - \int_a^b f d_F^\alpha\Big| < 1$ whenever $\widehat{P}$ is sub-ordinate to δ on I. Let $F_{j_m} = \Big\{t \in I: \; j-1 \le |f(t)| < j \;\&\; \frac{1}{m+1} \le \delta(t) \le \frac{1}{m}\Big\}$. Clearly $I = \bigcup\limits_{j=1}^{\infty}\bigcup\limits_{m=1}^{\infty} F_{j_m}$. Let $\Big\{[a_k,b_k]: \; 1 \le k \le N\Big\}$ be a collection of non-overlapping intervals having end points in F_{j_m}. Since Ξ is F-continuous. We can select points $a'_k,\ b'_k \in [a_k,b_k] \subset E$ with $a'_k < b'_k$ such that $\big|\Xi(b'_k) - \Xi(a'_k)\big| = h\big(S_F^\alpha,[a_k,b_k]\big)$ whenever $(a'_k,[a'_k,a_k]),\ (b'_k,[b'_k,b_k])$ are sub-ordinate to δ. Using Lemma 3.13, we have

$$\begin{aligned}\sum_{k=1}^{N} h\big(S_F^\alpha,[a_k,b_k]\big) &= \sum_{k=1}^{N} |\Xi(b'_k) - \Xi(a'_k)| \\ &\le \sum_{k=1}^{N}\Big\{|\Xi(a'_k) - \Xi(a_k) - f(a_k)(a'_k - a_k)| \\ &+ |f(a_k)(a'_k - a_k)| + |\Xi(a_k) - \Xi(b_k) + f(a_k)(b_k - a_k)| \\ &+ |f(a_k)(b_k - a_k)| + |\Xi(b_k) - \Xi(b'_k) - f(b_k)(b_k - b'_k)| + |f(b_k)(b_k - b'_k)|\Big\} \\ &= \sum_{k=1}^{N} |D(b_k) - \Xi(a_k) - f(a_k)(b_k - a_k)| \\ &+ \sum_{k=1}^{N}\Big\{|\Xi(a'_k) - \Xi(a_k) - f(a_k)(a'_k - a_k)| + |\Xi(b_k) - \Xi(b'_k) - f(b_k)(b_k) - b'_k)|\Big\} \\ &+ \sum_{k=1}^{N}\Big\{|f(a_k)|(a'_k - a_k) + |f(b_k)|(b_k - b'_k)\Big\} + \sum_{k=1}^{N} |f(a_k)|(b_k - a_k) < \infty.\end{aligned}$$

Therefore Ξ is $BV_{F,*}$ on each E_{j_m}. So, Ξ is $BVG_{F,*}$ on I through $E \subset I$.

Next, we are construct an example to show that the first fundamental theorem of calculus not hold for $*F^\alpha$-integrable functions.

Example 3.15 *Let $C \subset [0,1]$ be the tenary Contor set. Clearly C is an α-set for $\alpha = \log_3^2$. Let $f: [0,1] \to \mathbb{R}$ by*

$$f(x) = \begin{cases} 0, \ \textit{if}\, x \le \frac{1}{3} \\ 3x - 1, \ \textit{if}\, \frac{1}{3} < x < \frac{2}{3} \\ 1, \ \textit{if}\, x \ge \frac{2}{3} \end{cases}$$

has D_F^α-derivative $\left(D_F^\alpha f\right)$ exists on null of C, and $\left(D_F^\alpha f\right)$ is $*F^\alpha$-integrable functions but $\int_C \left(D_F^\alpha f\right) d_F^\alpha x = 0 \neq F_C^\alpha(1) - F_C^\alpha(0) = 1$.

The best suitable fundamental theorem for $*F^\alpha$-integrable function can be state as follows:

Theorem 3.16 *Let $E \subseteq I \subset \mathbb{R}$ be a closed α-set with $a = \min I,\ b = \max I$ and let $\left\{(a_j, b_j)\right\}_{j\in\mathbb{N}}$ be the contiguous intervals of I. If $\Xi : I \to \mathbb{R}$ is α-differentiable at each $x \in I$ and if $\sum_{j=1}^{\infty} |\Xi(b_j) - \Xi(a_j)| < \infty$ then $\left(D_F^\alpha \Xi\right)$ is $*F^\alpha$-integrable on I and $\int_I \left(D_F^\alpha \Xi\right) d_F^\alpha x = \Xi(b) - \Xi(a) - \sum_{j=1}^{\infty} \left(\Xi(b_j) - \Xi(a_j)\right)$.*

Proof: Let $\varepsilon > 0$ be given and $N \in \mathbb{N}$ so that $\sum_{j=N}^{\infty} |\Xi(b_j) - \Xi(a_j)| < \frac{\varepsilon}{2}$. Let $m = \inf\left\{|b_j - a_j| :\ j = 1, 2, ..., n-1\right\}$. For simple notation let $\left\{(a_j, b_j)\right\}_{j\in\mathbb{N}} = E$ and $x \in E$. By α-differentiability of Ξ at x, there exists $0 < \delta(x) < m$ such that

$$\left|\Xi(u) - \Xi(x) - (D_F^\alpha \Xi)\mathcal{H}^\alpha([u, x])\right| \leq \frac{\varepsilon \mathcal{H}^\alpha([u, x])}{2\mathcal{H}^\alpha(E)} \tag{2}$$

Next for each $u \in E \cap (x - \delta(x), x + \delta(x))$, consider $\left\{[u_i, v_i], x_i\right\}_{i=1}^{n}$ be a δ_ε-fine partition of E. By inequality(2), we have

$$\begin{aligned}
\left|\Xi(v_i) - \Xi(u_i) - (D_F^\alpha \Xi)(x_i)\mathcal{H}^\alpha([u_i, v_i])\right| &\leq \left|\Xi(v_i) - \Xi(x_i) - (D_F^\alpha \Xi)(x_i)\mathcal{H}^\alpha([x_i, v_i])\right| \\
&\quad + \left|\Xi(x_i) - \Xi(u_i) - (D_F^\alpha \Xi)(x_i)\mathcal{H}^\alpha([u_i, x_i])\right| \\
&\leq \varepsilon\left(\frac{\mathcal{H}^\alpha([u_i, x_i])}{2\mathcal{H}^\alpha(E)} + \frac{\mathcal{H}^\alpha([x_i, v_i])}{2\mathcal{H}^\alpha(E)}\right) \\
&\leq \varepsilon \frac{\mathcal{H}^\alpha([u_i, v_i])}{2\mathcal{H}^\alpha(E)}.
\end{aligned}$$

Therefore

$$\left|\sum_{i=1}^{n}(D_F^\alpha \Xi)(x_i)\mathcal{H}^\alpha([u_i, v_i]) - \sum_{i=1}^{n}(\mathcal{F}(v_i) - \mathcal{F}(u_i)\right| \leq \frac{\varepsilon}{2\mathcal{H}^\alpha(E)} \sum_{i=1}^{n} \mathcal{H}^\alpha([u_i, v_i]) = \frac{\varepsilon}{2}. \tag{3}$$

Since $u_i, v_i \in E$ and $\{(a_j, b_j)\}_{j\in\mathbb{N}}$ is the sequence of all contiguous intervals of E then

$$\Xi(b) - \Xi(a) = \sum_{i=1}^{n}(\Xi(v_i) - \Xi(u_i)) + \sum_{(a_j, b_j) \not\subseteq \bigcup_{i=1}^{b}[u_i, v_i]} \left(\Xi(b_j) - \Xi(a_j)\right). \tag{4}$$

Since $[a_j, b_j] \subset [u_i, v_i]$, this implies $|b_j - a_j| \leq |v_i - u_i| < 2\delta(x_i) < m$. Consequently $j \geq N$, hence

$$\sum_{[a_j, b_j] \subset \bigcup_{i=1}^{n}[u_i, v_i]} \left|\Xi(b_j) - \Xi(a_j)\right| \leq \sum_{i=N}^{\infty} \left|\Xi(b_j) - \Xi(a_j)\right| \leq \frac{\varepsilon}{2}. \tag{5}$$

Finally by Eqn(3), (4) and (5), we have

$$\begin{aligned}
&\Big|\sum_{i=1}^{n}(D_F^{\alpha}\Xi)\mathcal{H}^{\alpha}([u_i,v_i]) - \Big(\Xi(b)-\Xi(a)-\sum_{j=1}^{\infty}(\Xi(b_i)-\Xi(a_i)\Big)\Big| \\
&\leq \Big|\sum_{i=1}^{n}(D_F^{\alpha}\Xi)(x_i)\mathcal{H}^{\alpha}([u_i,v_i]) - \sum_{i=1}^{n}(\Xi(v_i)-\Xi(u_i))\Big| \\
&+\Big|\sum_{i=1}^{n}\big(\Xi(v_i)-\Xi(u_i)\big) - \big(\Xi(b)-\Xi(a)-\sum_{j=1}^{\infty}(\Xi(b_j)-\Xi(a_j))\Big| \\
&\leq \frac{\varepsilon}{2} + \Big|\sum_{[a_j,b_j]\subset\bigcup_{i=1}^{n}[u_i,v_i]}\Big(\Xi(b_j)-\Xi(a_j)\Big)\Big| \leq \varepsilon.
\end{aligned}$$

If one agreed the definition of $*F^{\alpha}$ integral of [7], the Fundamental Theorem $*F^{\alpha}$ integrable function can be formulate as follows.

Theorem 3.17 *If $f : I \to \mathbb{R}$ is F-continuous and α-differentiable at each point of $E \subseteq I$ and if $Sch(f) \subseteq E$, then $\int_E (D_F^{\alpha} f) d_F^{\alpha}(t) = f(b) - f(a)$.*

Proof Since $Sch(f) \subseteq E,\ f$ is a constant on each contiguous interval (a_k, b_k) of E. It is clear that $f(a_k) = f(b_k)$ for $k \in \mathbb{N}$. Also F-continuity of f gives $\sum_{j=1}^{\infty} |f(b_j) - f(a_j)| < \infty$. Using Theorem 3.16, we can find complete proof.

Lemma 3.18 *Let $\Xi : I \to \mathbb{R}$ is $ACG_{F,*}$ on I and $F \subset I$. If $\mathcal{H}^{\alpha}(F) = 0$ then for each $\varepsilon > 0$ there exists a positive function δ on F such that $|\Xi(\widehat{P})| < \varepsilon$ whenever $\widehat{P}$ is sub-ordinate to δ.*

Proof: Let $F = \bigcup_{k=1}^{\infty}[a_k, b_k]$ where $[a_k, b_k]'s$ are disjoint and Ξ is $AC_{F,*}$ on each $[a_k, b_k]$. Consider $|\Xi(\widehat{P})| < \frac{\varepsilon}{2^k}$ where $\widehat{P}$ is tagged partition on $[a_k, b_k]$ and $\mathcal{H}^{\alpha}(\widehat{P}) < \nu$ where $\nu > 0$. Let $\widehat{P}_k \subset \widehat{P}$ that has tags in $[a_k, b_k]$ and $\mathcal{H}^{\alpha}(\widehat{P}_k) < \mathcal{H}^{\alpha}(O_k) < \nu$ where $[a_k, b_k] \subseteq O_k,\ O_k$ are open set with $\mathcal{H}^{\alpha}(O_k) < \nu$. Hence $|\Xi(\widehat{P})| \leq \sum_{k=1}^{\infty} |\Xi(\widehat{P}_k)| < \varepsilon$.

Theorem 3.19 *A function f is $*F^{\alpha}$-integrable on I if and only if there exists F-continuous function Ξ which is $ACG_{F,*}$ on I such that $D_F^{\alpha}\Xi(x) = f(x)$ a.e.*

Proof: Let $f : I \to \mathbb{R}$ be $*F^{\alpha}$-integrable on I and let $\Xi(x) = \int_a^x f d_F^{\alpha} x$ for each $x \in I$. Clearly Ξ is F-continuous. Next we shall prove $D_F^{\alpha}\Xi(x) = f(x)$ a.e. on I. Let $A = \{x \in [a, b) :\ D_F^{\alpha}\Xi(x) \neq f(x)\}$. For each $x \in A$, there exists $\nu_x > 0$ and for each $h > 0$ there exists a point $v_h^x \in I \cap (x, x + h)$ such that $|\Xi(v_h^x) - \Xi(x) - f(x)(v_h^x - x)| \geq \nu_x(v_h^x - x)$. For each positive integer n, let $A_n = \{x \in A :\ n_x \geq \frac{1}{n}\}$. Next, we prove $\mathcal{H}^{\alpha}(A_n) = 0$ for each n. For fix n, let $\varepsilon > 0$. Since Ξ is $*F^{\alpha}$-integrable of f, there exists a positive function δ on I such that $|S(f, \widehat{P}) - \Xi(\widehat{P})| < \varepsilon$ whenever $\widehat{P}$ is δ_ε fine tagged partition of I. Let $\mathcal{J} = \Big\{[x, v_h^x] :\ x \in A_n;\ 0 < h < \delta(n)\Big\}$. Clearly $\mathcal{J}$ form a closed Vitali cover of A_n. By [2, Theorem 1.10], there exists a countable disjoint sequence of Borel sets $\{\mathcal{U}_k\}$ from $\mathcal{J}$ such that $\mathcal{H}^{\alpha}(\mathcal{J}) \leq \sum_k |\mathcal{U}_k|^{\alpha} + \frac{\varepsilon}{2}$. By Lemma 3.13, $\sum_k |\mathcal{U}_k|^{\alpha} < \frac{\varepsilon}{2}$. Hence $\mathcal{H}^{\alpha}(\mathcal{J}) < \varepsilon$. Since ε is an arbitrary, we can see $\mathcal{H}^{\alpha}(\mathcal{J}) = 0$. Hence $D_F^{\alpha}\Xi(x) = f(x)$ a.e..

Next let for each positive integer n, $\mathcal{J}_k = \left\{ x \in I : \ n-1 \leq |f(x)| < k \right\}$. Let us make k is fixed. Consider $\varepsilon > 0$. Since f is $*F^\alpha$ integrable on I, by definition $|S(f, \widehat{P}) - \int_a^b f d_F^\alpha| \leq \varepsilon$ whenever $\widehat{P}$ is tagged partition on I. If $\widehat{P}$ is δ_ε sub-ordinate to $\mathcal{J}_k$ and $\mathcal{H}^\alpha(\widehat{P}) < \frac{\varepsilon}{2k}$. By Lemma 3.13,

$$\begin{aligned} |\Xi(\widehat{P})| &\leq |\Xi(\widehat{P}) - S(f, \widehat{P})| + |S(f, \widehat{P})| \\ &< \frac{\varepsilon}{2} + k\mathcal{H}^\alpha(\widehat{P}) < \varepsilon. \end{aligned}$$

So, Ξ is $AC_{F,*}$ on $\mathcal{J}_k$. Since $I = \bigcup_{k=1}^{\infty} \mathcal{J}_k$, clearly D is $ACG_{F,*}$ on I.

Conversely, suppose there exists an $ACG_{F,*}$ function Ξ on I through F such that $D_F^\alpha \Xi = f$ a.e. on I. Let $E = \{x \in I : \ D_F^\alpha \Xi(x) \neq f(x)\}$ and let $\varepsilon > 0$. For each $x \in I \setminus E$, choose $\delta(x) > 0$ so that $|\Xi(y) - \Xi(x) - f(x)(y-x)| < \varepsilon|y-x|$ whenever $|y - x| < \delta(x)$ and $y \in I$. By Lemma 3.18, for $\delta(x) > 0$ on E so that $|S(f, \widehat{P})| < \frac{\varepsilon}{2}$ and $|\Xi(\widehat{P})| < \frac{\varepsilon}{2}$ whenever $\widehat{P}$ is sub-ordinate to δ on E. Let $\widehat{P}_E \subset \widehat{P}$ tagges in E. If $\widehat{P}_d = \widehat{P} \setminus \widehat{P}_E$, then we have

$$\begin{aligned} |S(f, \widehat{P}) - \Xi(\widehat{P})| &\leq |S(f, \widehat{P}_d) - \Xi(\widehat{P}_d)| + |S(f, \widehat{P}_E)| + |\Xi(\widehat{P}_E)| \\ &< \varepsilon. \end{aligned}$$

So, f is $*F^\alpha$ integrable on I and $\int_a^b f d_F^\alpha = \Xi(b) - \Xi(a)$.

Next theorem gives a relationship of F^α and $*F^*$ integrable functions.

Theorem 3.20 *If f is positive, $*F^\alpha$ integrable on $I \cap F$ and F-derivative of its primitive is bounded then f is F^α integrable therein.*

Proof: Let f is nonnegative on $I \cap F$, then $\Xi(x) = \int_a^x f d_F^\alpha x$ is nondecreasing on $I \cap F$. Since $\Xi(x)$ is nondecreasing and $D_F^\alpha \Xi(x)$ is bounded, then $D_F^\alpha \Xi(x)$ is F^α-integrable on $I \cap F$. From Theorem 3.19, $D_F^\alpha \Xi(x) = f$ a.e. on $I \cap F$. Hence f is F^α-integrable on $I \cap F$.

Next we discuss controlled type convergence theorem for $*F^\alpha$-integrals. In order to prove controlled type convergence theorem, we introduce equi-F-continuous $*F^\alpha$ integral as follows.

Definition 3.21 *Let $f_n : I \to \mathbb{R}, \ n = 1, 2, ...$ be given $*F^\alpha$-integrable functions with integral $\int_a^b f_n(s) d_F^\alpha s$ for every $n = 1, 2, ...$ The sequence (f_n) of $*F^\alpha$-integrable function is called equi-F-continuous if for every $\varepsilon > 0$ there is a gauge δ on I such that $\left|S(f_n, \widehat{P}) - \int_a^b f_n(s) d_F^\alpha s\right| < \varepsilon$ whenever $\widehat{P}$ is tagged partition on I.*

The following theorem is a convergence theorem for $*F^\alpha$ integral.

Theorem 3.22 *If a sequence of $*F^\alpha$ integrable functions $\{f_k\}$ satisfied the followings:*

1. *$f_k(x) \to f(x)$ a.e. in I as $k \to \infty$;*
2. *The primitives $\Xi_k(x) = \int_a^x f_k(s) d_F^\alpha s$ of f_k are $ACG_{F,*}$ uniformly in k;*
3. *The primitives Ξ_k are equi-F-continuous on I;*

then f is $*F^\alpha$-integrable on I and $\int_a^b f_k \to \int_a^b f$ as $n \to \infty$.

Proof: Let $f_k(x) \to f(x)$ everywhere as $k \to \infty$. Since each f_k is $*F^*$-integrable on I, given $\varepsilon > 0$ there is $\delta_k(\varepsilon) > 0$ such that for any tagged partition $\widehat{P}$ sub-ordinate to $\delta_n(\xi)$, we have $|S(f_k, \widehat{P}) - \int_a^b f_k d_F^\alpha| < \varepsilon$. Now we define $\delta(\xi) = \delta_m(\varepsilon)$ so that $|f_m - f| < \varepsilon$. Clearly $m = m(\xi, \varepsilon)$ is a function of ξ and ε. Let us denote $\Xi_n([a_k, b_k])$ denote the integral of f_n on $[a_k, b_k]$ and $\Xi([a_k, b_k])$ be the F-limit of $\Xi_n([a_k, b_k])$ as $n \to \infty$. Let $\widehat{P}_k$ be a tagged partition on $[a_k, b_k]$ sub-ordinate to δ_k, consequently sub-ordinate to δ. Then

$$\begin{aligned}\big|S(f, \widehat{P}) - \int_a^b f d_F^\alpha\big| &\le \big|S(f, \widehat{P}_k) - S(f_k, \widehat{P}_k\big| \\ &\quad + \big|S(f_k, \widehat{P}_k) - \sum \Xi_k([a_k, b_k])\big| + \big|\sum \Xi_k([a_k, b_k]) - \int_a^b f d_F^\alpha\big| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.\end{aligned}$$

4 Existence of solution of the equation $D_F^\alpha\big(*S_F^\alpha(x)\big) = f(t, *S_F^\alpha(x))$

In this Section, let τ and ξ be fixed and let $f(t, *S_F^\alpha(x))$ be a Carathéodory function defined on a rectangle $R:\ |t - \tau| \le a,\ |*S_F^\alpha(x) - \xi| \le b$ i.e. f is F-continuous in $*S_F^\alpha(x)$ for almost all t and measurable in t for each fixed $*S_F^\alpha(x)$.

Theorem 4.1 *Let f be a function as our assumption. Consider $g(t)$ and $h(t)$ are $*F^\alpha$-integrable functions on $|t - r| \le a$ such that $g(t) \le f(t, *S_F^\alpha(x)) \le h(t)\ \forall\ *S_F^\alpha(x)$ and almost all t with $(t, *S_F^\alpha(x)) \in R$ then there exists a solution of $D_F^\alpha\big(*S_F^\alpha(x)\big) = f(t, *S_F^\alpha(x))$ on some interval $|t - r| \le \beta,\ \beta > 0$ with $\phi(\tau) = \xi$.*

Proof: Given $g(t) \le f(t, S_F^\alpha(x)) \le h(t)$ for all $*S_F^\alpha(x)$ and almost all t with $(t, *S_F^\alpha(x)) \in R$. Clearly $0 \le f(t, *S_F^\alpha(x)) - g(t) \le h(t) - g(t)$. By Theorem 3.20, $h - g$ is F^α-integrable. Let $\Xi(t, *S_F^\alpha(x)) = f\big(t, *S_F^\alpha(x) + \int_\tau^t g(s) d_F^\alpha s\big) - g(t)$. Since $f(t, *S_F^\alpha(x))$ is a Carathéodory function, so $\Xi(t, S_F^\alpha(x))$ is also Carathéodory function. Moreover, $0 \le \Xi(t, *S_F^\alpha(x)) \le h(t) - g(t)$ for all $(t, *S_F^\alpha(x)) \in R'$ where R' is sub-rectangle of R with $\big|*S_F^\alpha(x) + \int_\tau^t g(s) d_F^\alpha s - \xi\big| \le b$ for all $(t, *S_F^\alpha(x)) \in R'$. By Carathéodory existence theorem for fractal there is a function Φ on some $|t - r| \le \beta \subset F \cap I$ such that $D_F^\alpha \Phi(t) = \Xi(t, \Phi(t))$ almost everywhere in $|t - r| \le \beta$ and $\Phi(\tau) = \xi$. Let $\phi(t) = \Phi(t) + \int_\tau^t g(s) d_F^\alpha s$. Then for almost all t,

$$\begin{aligned} D_F^\alpha \phi(t) &= D_F^\alpha \Phi(t) + g(t) \\ &= \Xi(t, \Phi(t)) + g(t) \\ &= f\left(t, \Phi(t) + \int_\tau^t g(s) d_F^\alpha s\right) - g(t) + g(t) \\ &= f(t, \phi(t)). \end{aligned}$$

So, $\phi(\tau) = \Phi(\tau) + \int_r^\tau g(s) d_F^\alpha s = \xi$.

We present an example to validation of Theorem 4.1.

Example 4.2 *Let $D_F^\alpha\big(*S_F^\alpha(x)\big) = f(t, *S_F^\alpha(x)) = \S(t, *S_F^\alpha(x)) + h(t)$ where $|\S(t, *S_F^\alpha(x))| \le \S_1(t)$ for all $|t| \le 1,\ |x| \le 1$ and $\S_1(t)$ is F^α-integrable on $|t| \le 1$ and let*

$$h(t) = \begin{cases} D_F^\alpha\left(\frac{t^2}{sint^2}\right),\ if\, t \ne 0 \\ 0,\ if\, t = 0. \end{cases}$$

Here $\tau = \xi = 0$. Clearly h is $*F^{\alpha}$-integrable but not F^{α}- integrable. By Theorem 4.1, there exists a solution of $D_F^{\alpha}\big(* S_F^{\alpha}(x)\big) = f(t, *S_F^{\alpha}(x))$ with $x(0) = 0$. Moreover if $\S(t, *S_F^{\alpha}(x)) = t^2 * S_F^{\alpha}(x)$, then $\phi(t) = e^{\frac{t^3}{3}} \int_0^t e^{-\frac{s^3}{3}} h(s) d_F^{\alpha} s$ is a solution of $*F^{\alpha}$-integrable function which is not a F^{α}-integrable.

Finally, we prove the extension of Theorem 4.1 with the help of Theoem 3.22.

Theorem 4.3 *Let $f(t, *S_F^{\alpha}(x))$ be a Carathéodory function defined on a rectangle $R:\ |t-\tau| \leq a;\ | * S_F^{\alpha}(x) - \xi| \leq b$. Let $f(t, S_F^{\alpha}(x))$ be defined on $|t-\tau| \leq a$ for any step functions $u(t),\ v(t)$ defined by $|t-\tau| \leq a$ with values in $| * S_F^{\alpha}(x) - \xi| \leq b$ so that $f(t, u(t)) \leq f(t, *S_F^{\alpha}(x)) \leq f(t, v(t))$. Let $Z_u(t) = \int_\tau^t f(s, u(s)) d_F^{\alpha} s$ where $\big\{Z_u :\ u \text{ is step function}\big\}$ is $ACG_{F,*}$ uniformly in u and equi-F-continuous on $|t-\tau| \leq a$, then there exists a solution of $D_F^{\alpha}\big(* S_F^{\alpha}(x)\big) = f(t, *S_F^{\alpha}(x))$ on some interval $|t-\tau| \leq \beta$.*

Proof: Let $\{\mathfrak{F}_k(t)\}$ be a sequence of step function defined on $|t-\tau| \leq a$ with values in $| * S_F^{\alpha}(x) - \xi| \leq b$ such that $\mathfrak{F}_k(t) \to u(t)$ as $k \to \infty$. Then $f(t, \mathfrak{F}_k(t)) \to f(t, u(t))$ a.e. as $k \to \infty$. Let $Z_u(t) = \int_\tau^t f(s, \mathfrak{F}_k(s)) d_F^{\alpha} s$. Then $\{Z_u(t)\}$ is $ACG_{F,*}$ is uniformly in u and equi-F-continuous. By Theorem 3.22, $f(t, u(t))$ is $*F^{\alpha}$ integrable. Similar way we can see $f(t, v(t))$ is $*F^{\alpha}$-integrable. Clearly $f(t, *S_F^{\alpha}(x))$ is $*F^{\alpha}$ integrable. By Theorem 4.1, $D_F^{\alpha}\big(* S_F^{\alpha}(x)\big) = f(t, *S_F^{\alpha}(x))$ has a solution in some interval $|t-\tau| \leq \beta$.

5 Conclusion

We introduced F^{α}-type integrals with gauge functions called $*F^{\alpha}$-integral without assuming $Sch(f)$ contained in a fractal set $F \subset I$. Several properties of $*F^{\alpha}$ integrable functions are discussed. We have shown fundamental theorem of calculus for $*F^{\alpha}$ integrable functions when $Sch(f)$ not necessarily in F is slightly different from the fundamental theorem of calculus for $*F^{\alpha}$ integrable functions when $Sch(f) \subset F$. In the last Section, we have shown existence of solutions of fractal differential equation $D_F^{\alpha}\big(* S_F^{\alpha}(x)\big) = f(t, *S_F^{\alpha}(x))$ in the sense of $*F^{\alpha}$ integrable functions. An example has been provided to show that there are $*F^{\alpha}$ integrable solutions of the fractal differential equation $D_F^{\alpha}\big(S_F^{\alpha}(x)\big) = f(t, *S_F^{\alpha}(x))$ that are not necessarily F^{α} integrable.

One can use the generalized measure of [5] to construct Lebesgue-type F^* integrals, as well as $*F^{\alpha}$-type integrals to find the existence of solutions of several retarded fractal differential equations and their numerical techniques. In coming days, we shall compute numerical integration of F^*-integral and class of $*F^{\alpha}$ integral to find their practical applications in various physical models.

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