A NOTE ON APPROXIMATE SOLUTIONS TO ZELDOVICH'S EQUATION

by

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The double integral-balance approach and Barenblatt's assumed profile have been used to create approximate solutions to the Zeldovich equation, both linear and degenerate. The evaluation of the controlling dimensionless groups and proper dimensional scaling have been the main focus of the solution developments and analyses.

Keywords: reaction-diffusion, approximate solutions, integral-balance method, scaling

1 Introduction

This note concerns an approximate solution to the Zeldovich equation appearing originally in the combustion to the flame propagation [1, 2] but appearing also in population dynamics as a version of Fisher's type models [3]. Zeldovich's linear equation (with $D = const.$) in a general presentation

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + f_z(u), \quad f_z(u) = Gu^p(1 - u^q) \tag{1}$$

The growth function (a term coming from Fisher's models)$f_z(u)$, is related to heat generation during combustion and can be presented in several forms such as: $f_z(u) = bu - du^{p+1}$ [4] as an extended polynomial, or in a compact form (with $p = 2$ and $q = 1$) [5] coinciding with the original formulation $f_z(u) = u^2(1 - u)$ [1, 2], as well as $f_z(u) = qu^2 + ru^3$ [6].

This note aims to present approximate solutions to the Zeldovich equation by applying the double-integration technique [9] of the integral-balance method [7] and the Barenblatt parabolic profile [8, 9] known also a parabolic profile with an unspecified exponent [10], upon Dirichlet boundary condition in a semi-infinite domain. The study was motivated by the challenging properties of the integral-balance method to solve both linear and non-linear diffusion models and to find a solution that differs from the dominating approach of traveling waves [3, 4, 5, 6].

2 Solution

2.1 Model scaling and dimensionless groups

The model (1) [1, 2, 3, 4, 11] is semi-scaled, because $0 < u = U/U_{ref} \leq 1$ is dimensionless, while there are no defined length and time scales. A simple inspection of (1) reveals that the coefficient G has a dimension inverse of time ($\left[s^{-1}\right]$), and the length scale can be defined as $\sqrt{D/G}$. Hence, changing the variables as $t/(1/G) = Gt = Fo$ (the Fourier number, i.e. the dimensionless time) and $\bar{x} = x/\sqrt{(D/G)}$ we may write (1) completely in a dimensionless form, namely

$$\frac{\partial u}{\partial Fo}=\frac{\partial^2 u}{\partial x^2}+u^p\left(1-u^q\right) \tag{2}$$

The expression (2) has a form frequently used in the literature, with the only difference: the time is dimensionless, i.e. represented by the Fourier number Fo . We especially stress the attention on the scaling and characteristic length and time scales, because they intrinsically appear in the approximate solutions developed next. Last but not least, since this is important for the further analyses of the results developed, the products DG (the linear case) has a dimension $\left[m^2 s^{-2}\right]$ an this leads to a characteristic velocity $V_0=\sqrt{DG}$; the same is valid in the nonlinear case where the velocity scale is defined by $\sqrt{D_0 G}$.

2.2 The linear case

Consider the generalized linear formulation (1), with $D=const.$, and its growth function in an extended form

$$\frac{\partial u}{\partial t}=D\frac{\partial^2 u}{\partial x^2}+Gu^p-Gu^{p+q} \tag{3}$$

supposing $u\left(0,t\right)=1$, i.e. Dirichlet boundary condition (the diffusion process starts from the boundary $x=0$, but not at $x\rightarrow-\infty$ (as in the traveling wave solution). The double integration applied to (3) [9] results in the following integral equation

$$\int_0^\delta\int_x^\delta\frac{\partial u}{\partial t}dxdx=\int_0^\delta\int_x^\delta D\frac{\partial^2 u}{\partial x^2}dxdx+\int_0^\delta\int_x^\delta Gu^p dxdx-\int_0^\delta\int_x^\delta Gu^{p+q}dxdx \tag{4}$$

Further, assuming an approximate profile as Barenblatt parabola $u_a=u_s(1-x/\delta\ )^n$ [8, 9, 10], thus defining a finite front of the solution $\delta\left(t\right)$ evolving in time and satisfying the Goodman conditions [7] $u_a\left(x=\delta\right)=\partial u_a\left(x=\delta\right)/\partial t\ =0$. The front divides the medium (a semi-infinite) along the positive axis, into two zones: $u_a>0$ for $0<x<\delta$, and $u_a=0$ for $\delta<x<\infty$. The Goodman conditions replace the commonly used in diffusion problems asymptotic assumption $u_{(x\rightarrow\infty)}=0$. Applying the Leibniz rule to the left-hand side as well as the Goodman boundary conditions during the integration, we get

$$\frac{d}{dt}\int_0^\delta\int_x^\delta u\left(x,t\right)dxdx=Du\left(0,t\right)+G\frac{u^p\left(0,t\right)}{\left(p+1\right)\left(p+2\right)}-G\frac{u^{p+q}\left(0,t\right)}{\left(\left(p+q\right)+1\right)\left(\left(p+q\right)+2\right)} \tag{5}$$

$$\begin{gathered}\frac{d\delta^2}{dt}=D\left(n+1\right)\left(n+2\right)+\delta^2 G\Phi_Z\left(n,p,q\right)\\ \Phi_Z\left(n,p,q\right)=\left[\frac{\left(n+1\right)\left(n+2\right)}{\left(np+1\right)\left(np+2\right)}-\frac{\left(n+1\right)\left(n+2\right)}{\left(n\left(p+q\right)+1\right)\left(n\left(p+q\right)+2\right)}\right]\end{gathered} \tag{6}$$

Equation (6) can be presented in a compact form

$$\frac{d\delta^2}{dt}=A+B\delta^2,\quad A=D\left(n+1\right)\left(n+2\right),\quad B=G\Phi_Z\left(n,p,q\right) \tag{7}$$

This is a separable equation with a solution $\delta^2=C_1 e^{Bt}-\frac{A}{B}$. With the physically motivated condition, $\delta\left(t=0\right)=0$ [7] the front position (penetration depth) is

$$\delta=\sqrt{\frac{A}{B}}\sqrt{e^{Bt}-1}\Rightarrow\delta=\sqrt{\frac{D}{G}}\sqrt{\frac{e^{Bt}-1}{\Phi_Z\left(n,p,q\right)}}\Rightarrow\delta=\sqrt{\frac{D}{G}}\sqrt{\frac{e^{\Phi_Z\left(n,p,q\right)Fo}-1}{\Phi_Z\left(n,p,q\right)}} \tag{8}$$

The product Bt in the exponential function defines the Fourier number Fo multiplied by the dimensionless factor $\Phi_Z(n,p,q)$.

For $t \to 0_+$ we can approximate $e^{Bt} \approx 1 + Bt$ and then from (8) it follows $\delta \approx \sqrt{At} \equiv \sqrt{Dt}$, and taking into account that $A \equiv D$, we have a Gaussian diffusion with a front moving with the square-root law $\delta \approx \sqrt{Dt}$.

For a long time when, i.e.,for $e^{Bt} >> 1$, the front dynamics follows an exponential law, namely

$$\delta(t) \approx \sqrt{\frac{D_0}{G}}\sqrt{\Phi_Z(n,p,q)\, e^{\Phi_Z(n,p,q)Fo}} \equiv \sqrt{\frac{D_0}{G}}\sqrt{e^{\Phi_Z(n,p,q)Fo}} \equiv \sqrt{\frac{D_0}{G}}e^{(\Phi_Z(n,p,q)Fo)/2} \tag{9}$$

For $p = 2$ and $q = 1$, assuming, for instance, $n = 2$, we have $\Phi_Z(n,p,q) \approx 0.185$.

Note: Here we especially restrict ourselves to using $n = 2$ because the focus is on the technology of solution development but not on its refinement, which is beyond the task of this work.

Hence, the approximate solution is

$$u_a(x,t) = \left(1 - \frac{x}{\sqrt{\frac{D}{G}}\sqrt{\frac{e^{\Phi_Z(n,p,q)Fo}-1}{\Phi_Z(n,p,q)}}}\right)^n = \left(1 - \frac{Z}{\sqrt{\frac{e^{\Phi_Z(n,p,q)Fo}-1}{\Phi_Z(n,p,q)}}}\right)^n, \quad Z = \frac{x}{\sqrt{\frac{D}{G}}} \tag{10}$$

where Z is a dimensionless distance because $\sqrt{D/G}$ defines the process length scale.

From (8) we may define the speed of the front as

$$\frac{d\delta}{dt} = \sqrt{\frac{A}{B}}\frac{e^{Bt}}{2\sqrt{\frac{e^{Bt}-1}{\Phi_Z(n,p,q)}}} \Rightarrow \frac{\sqrt{AG}}{2}\frac{e^{\Phi_Z Gt}}{\sqrt{\frac{e^{\Phi_Z Gt}-1}{\Phi_Z(n,p,q)}}} = \frac{V_0}{2}\sqrt{\Phi_Z(n,p,q)}\frac{e^{\Phi_Z Gt}}{\sqrt{e^{\Phi_Z Gt}-1}} \tag{11}$$

because $AB \equiv \sqrt{DG} = V_0$; the scaled front speed $\frac{d\delta}{dt}/V_0$ is shown in fig. 1-panel a.

Without loss of generality, it is possible to accept that the characteristic velocity is defined as $\sqrt{DG}/2$, as it is used in the literature [11]; this will simplify the expressions without any effect on the physical meaning.

From eq.(11) we may see what are the asymptotes of the front speed: For $e^{\Phi_Z Fo} \ll 1$ and $e^{\Phi_Z Fo} \approx 1 + \Phi_Z Fo$ we have $\frac{d\delta}{dt} \propto \frac{V_0}{2}\frac{1}{\sqrt{Fo}}$ and for $Fo \to 0$ (i.e. at the onset of the diffusion process) the speed is infinite. This can be explained by the fact that upon the same approximation $\delta \equiv \sqrt{Dt}$ and therefore $d\delta/dt \equiv \sqrt{D}/2\sqrt{t}$. Further, for $e^{\Phi_Z Fo} \gg 1$ (i.e. long time), we get $\frac{d\delta}{dt} \propto \frac{V_0}{2}\sqrt{\Phi_z}e^{(\Phi_Z Fo)/2}$, i.e. an almost finite speed *because the time-dependent term has minimum at* $\Phi_Z Fo = ln2$. Upon the conditions imposed by the present calculations ($n = 2$, $p = 2$, and $q = 1$) we get $\frac{d\delta}{dt}/V_0 \propto 0.440 - 0.498$ if Fo is in the range $Fo = 10 \div 15$ (actually this happens for $Fo > 5$, see the plots in fig.1-panel a).

As commented above, for t_{0+} the product $\sqrt{\frac{D}{G}}\sqrt{e^{\Phi(n,p,q)Fo}} \to \sqrt{\frac{D}{G}}\sqrt{Fo} \to \sqrt{Dt}$ and then the ratio $\eta = \frac{x}{\sqrt{Dt}}$ defines the Boltzmann similarity variable. Now, we may suggest that the ratio $\eta_Z = \frac{x}{\sqrt{\frac{D}{G}}\sqrt{e^{\Phi(n,p,q)Fo}}} = \frac{Z}{\sqrt{e^{\Phi(n,p,q)Fo}}}$ defines a new, dimensionless variable η_Z. Then, the approximate profile can be presented as

$$u_a(\eta_Z, \Phi) = \left(1 - \eta_Z\sqrt{\Phi}\right)^n \tag{12}$$

and the front is defined by $1 - \eta_Z\sqrt{\Phi} = 0 \Rightarrow \eta_{Z(front)} = 1/\sqrt{\Phi}$; for short times, we have a transition $\eta_Z \to \eta$. The approximate solution $u_a(\eta_Z)$ and $V_a = 1 - u_a(\eta_Z)$ are shown in fig. 1-panel b.

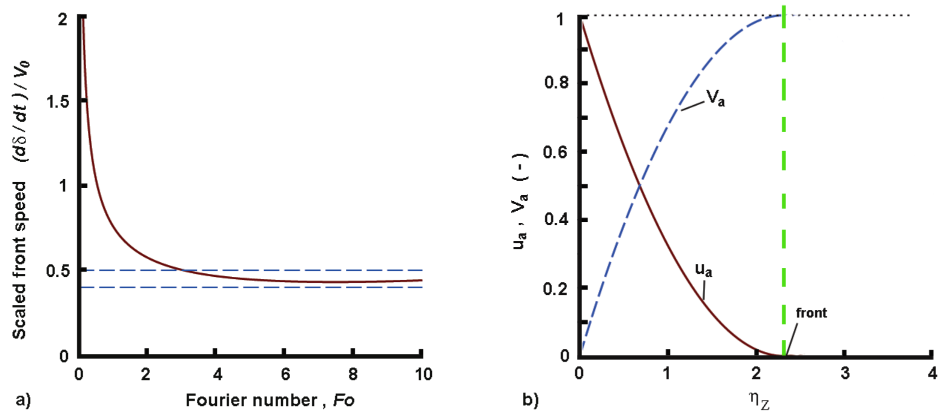

Figure 1 – Approximate solution (with $n = 2$) in the linear case for $p = 2$ and $q = 1$: a) Scaled front speed $\frac{d\delta}{dt}/V_0$ as a function of the Fourier number; b) The approximate solution $u_a(\eta_Z)$ and the related function $V_a = 1 - u_a(\eta_Z)$: Note: in this specific case $\eta_Z(front) \approx 2.320$

2.3 An important point that should be clarified

For the sake of clarity, we have to discuss common moments between the present approach and the well-known traveling wave method [3, 4, 5, 11]. Applying the traveling wave approach to (1), where the solution is looked for as a simple wave $u = ax + bt$ (existing only of $b \neq 0$), there is only one exact solution (when $p = 2$ and $q = 1$, and $b = \frac{1}{\sqrt{2}}$) [12] (upon traveling wave boundary conditions [13])

$$u_e\left(x,t\right) = \frac{1}{1+e^{\tau}}, \quad \tau = \left(-\frac{1}{\sqrt{2}}t \pm x\right)\frac{1}{\sqrt{2}}, \quad u_{e(x\to-\infty)} = 0, \quad u_{e(x\to+\infty)} = 1 \tag{13}$$

In the assumed Barenblatt profile, we have $u_a\left(x=0\right) = 1.$, $u_a\left(x=\delta\right) \approx u_a\left(x\to\infty\right)$ and these boundary conditions are just the opposite to (13); thus, we have to clarify where the main difference comes from. For this, if we can construct the function $V_e\left(x,t\right) = 1 - u_e\left(x,t\right)$ then we have $V_{e(x\to-\infty)} = 1$ and $V_{e(x\to+\infty)} = 0$. The behaviors of u_e and $V_e\left(x,t\right) = 1 - u_e\left(x,t\right)$ are shown in fig. 2-panel a. In a similar way, with the function $V_a\left(x,t\right) = 1 - u_a\left(x,t\right)$, based on the assumed profile, we have $V_{a(x\to-\infty)} = 0$ and $V_{a(x\to+\infty)} = 1$ (see the plots in fig. 2-panel b).

An important point is that with the approximate solution (the Barenblatt parabolic profile), the condition $u_{e(x\to-\infty)} = 0 \Rightarrow V_{e(x\to-\infty)} = 1$ is replaced by $u_{a(x=0)} = 1$, as well as $u_{e(x\to+\infty)} = 1 \Rightarrow V_{e(x\to+\infty)} = 0$ is replaced by $u_{a(x=\delta)} = 0 \Rightarrow V_{a(x=\delta)} = 1$; as a result of the sharp front concept, there are no smooth transitions of the approximate profiles towards the x axis. Therefore, the approximate solution with the parabolic profiles $u_a\left(x,t\right)$ represents *the density profile evolution* in time along the x axis (this confirms the analysis made by Newman [11]). The function $V_a = 1 - u_a$, satisfies the boundary conditions in (13), and, to a greater extent, models the pulse approaching unity at $x = \delta$ (a condition replacing the second one in (13) at $x \to +\infty$).

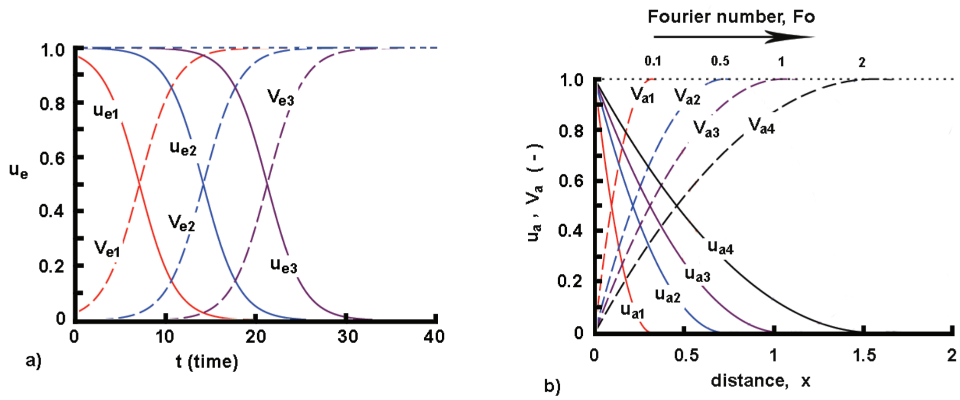

Figure 2 – A qualitative explanation of the boundary conditions to (13) and their effect on both the exact solution (13)-panel a) and the approximate solution-panel b). The linear case [12]: a) in time at three positions at the x axis: $x = 5$, $x = 10$, and $x = 15$.; b) Along the x axis for three different times represented by the Fourier number.

2.4 The non-linear (degenerate) case

We now discuss a more general form of (1) with a power-law diffusivity $D(u) = D_0 u^m$ for the sake of this note's completeness. It is well-known that the degenerate diffusion equations have solutions with finite speeds and sharp fronts such as the assumptions used when the integral-balance approach is applied [9] (and the references therein for completeness of the available solutions of such problems).

The important point in this solution is that the diffusion term has to be rearranged (before the integration)

$$\frac{\partial}{\partial x}\left(D_0 u^m \frac{\partial u}{\partial x}\right) = \frac{D_0}{m+1}\frac{\partial^2 u^{m+1}}{\partial x^2} \tag{14}$$

Then, applying the double-integration, as in the preceding example, and taking into account that $u^{m+1}(0,t) = 1$, the equation about the front propagation is

$$\frac{d\delta^2}{dt} = \frac{D_0}{m+1}(n+1)(n+2) + \delta^2 G\Phi_Z(n,p,q) \tag{15}$$

$$\Phi_Z(n,p,q) = \left[\frac{(n+1)(n+2)}{(np+1)(np+2)} - \frac{(n+1)(n+2)}{(n(p+q)+1)(n(p+q)+2)}\right] \tag{16}$$

That is, we got the same equation as (7) (only the term A is modified).

$$\frac{d\delta^2}{dt} = A + B\delta^2, \quad A = \frac{D_0(n+1)(n+2)}{m+1}, \quad B = G\Phi_Z(n,p,q) \tag{17}$$

$$\delta_m = \sqrt{\frac{A}{B}}\sqrt{e^{Bt}-1} \Rightarrow \delta_m = \sqrt{\frac{D_0}{G}}\sqrt{\frac{e^{Bt}-1}{(m+1)\Phi_Z(n,p,q)}} \Rightarrow \delta_m = \sqrt{\frac{D}{G}}\sqrt{\frac{e^{\Phi_Z(n,p,q)Fo}-1}{(m+1)\Phi_Z(n,p,q)}} \tag{18}$$

The increase in the degeneracy of the diffusion term, by an increase in the value of $m > 1$, results in shorter penetration depths. The analysis made about the behavior of $\delta_m(t)$ for short and long times, about equation

(10) is valid here too. The approximate solution in this case is

$$u_a\left(x,t\right)=\left(1-\frac{x}{\sqrt{\frac{D_0}{G}}\sqrt{\frac{e^{\Phi_Z(n,p,q)Fo}-1}{(m+1)\Phi_Z(n,p,q)}}}\right)^n=\left(1-\frac{Z_m}{\sqrt{\frac{e^{\Phi_Z(n,p,q)Fo}-1}{\Phi_Z(n,p,q)\Phi_Z(n,p,q)}}}\right)^n,\quad Z_m=\frac{x}{\sqrt{\frac{D_0}{G}}\sqrt{m+1}} \tag{19}$$

It is important to mention that in cases with $m>0$ (slow diffusion), the density profiles are convex, with steep fronts, and are modeled by an exponent $n=1/m$ [8] (see also [9] for more solutions in detail).

From eq.(18) the front speed is

$$\frac{d\delta_m}{dt}=\frac{\sqrt{D_0G}}{2}\frac{1}{\sqrt{m+1}}\frac{\sqrt{\Phi_Z\left(n,p,q\right)}e^{\Phi_Z(n,p,q)Fo}}{2\sqrt{e^{\Phi_Z(n,p,q)Fo}-1}}=\frac{V_0}{2}\frac{1}{\sqrt{m+1}}\frac{\sqrt{\Phi_Z\left(n,p,q\right)}e^{\Phi_Z(n,p,q)Fo}}{2\sqrt{e^{\Phi_Z(n,p,q)Fo}-1}} \tag{20}$$

Hence, in general, the front speed is reduced $\sqrt{m+1}$ times (see the plots in fig.3-panel a).

For t_{0+} the behavior is the same as when $m=0$. However, when $e^{Bt}\gg 1$ it is possible to approximate $\frac{d\delta_m}{dt}\approx\frac{V_0}{2}\frac{1}{\sqrt{m+1}}\sqrt{\Phi_Z}$, i.e. there is a limit in the front speed for long time (high Fourier number) as it shown in fig.3-panel a. The approximate solution u_a, parallel to the function $V_a=1-u_a$, for various Fourier

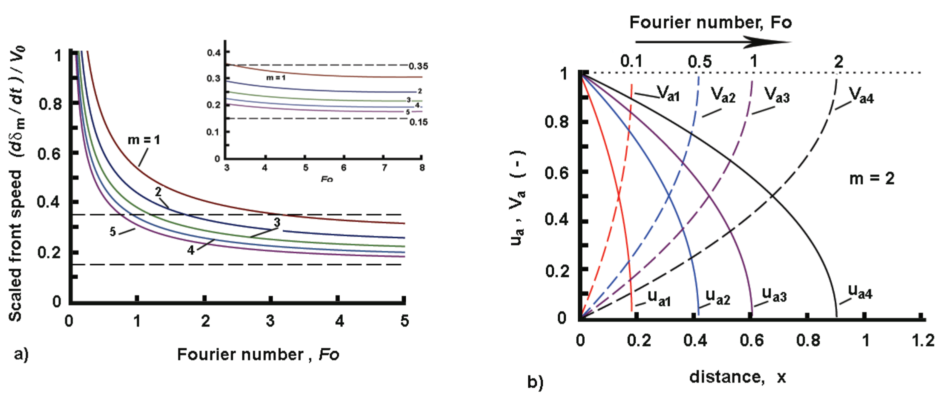

Figure 3 – Approximate solution (with $m=2$) to the degenerate case for $p=2$ and $q=1$: a) Scaled front speed $\frac{d\delta_m}{dt}/V_0$ as a function of the Fourier number; b) The approximate solution u_a and the related function $V_a=1-u_a$ along the x axis at various Fourier numbers: Note: in this specific case $n=1/m$

numbers, along the x axis, are shown fig.3-panel b. The general behavior is the same as with the linear case, but the only difference is that the density profiles are convex (a special feature of the Barenblatt profile when $n<1$ [9].

Taking into account that $\frac{d\delta}{dt}/V_0=\frac{1}{2}L\left(n,m,p,q\right)\frac{e^{\Phi_zFo}}{\sqrt{e^{\Phi_zFo}-1}}$, where $\frac{1}{2}L\left(n,m,p,q\right)$ is the cumulative prefactor of all time-independent terms, and $\lim_{Fo\to\infty}\frac{e^{\Phi_zFo}}{\sqrt{e^{\Phi_zFo}-1}}=1$, it follows that the limited front speed is predetermined by the prefactor $\frac{1}{2}L\left(n,m,p,q\right)$. The plots in fig. 3-panel a reveal that this finite speed could be attained at least for $Fo>5$.

3 Conclusions

An attempt to develop integral-balance solutions to the linear and degenerate Zeldovich's equations resulted in successful and physically relevant approximate density profiles. The solutions developed especially stressed the attention on correct dimensional scaling and evaluation of the controlling dimensionless group. It was clearly demonstrated that the Zeldovich equation has its own time, length, and velocity scales—a problem that is, in general, missing in the dominating studies involved in traveling-wave analyses.

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