THE APPLICATION OF BERNOULLI TRIALS TO THE THEORY OF APPROXIMATION

by

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Original scientific paper
https://doi.org/10.2298/TSCI2406991B

In recent years, q-Bernstein polynomials and α-Bernstein polynomials have emerged as prominent topics in approximation theory. Numerous studies have examined the convergence criteria of these polynomials, highlighting their importance and usefulness. The main idea underlying the article is that the kernels of these polynomials depend on the probabilities of the variable parameter binomial distribution. Taking advantage of this property, we developed a simplified form of these polynomials regarding binomial dependence. This facilitated the calculation of moments for a binomial variable with variable parameters. This method both simplifies the computational processes and allows us to better understand the convergence properties of these polynomials. By examining these reduced forms, important information has been obtained regarding the structure of the underlying distribution. The findings underscore the versatility and power of q-Bernstein and α-Bernstein polynomials in approximation theory and provide a deeper understanding of their mathematical foundations and potential applications.

Key words: *approximation theory, Bernstein polynomial, Bernoulli process, convergence, expected values*

Introduction

The polynomials stand out with their various advantages in many fields, from mathematical analysis to engineering. In essence polynomials are fundamental tools for modelling and solving countless problems in different disciplines. They are easy to analyze thanks to their continuous and differentiable structure, which offers great practicality in operations such as differentiation and integration. In addition, the structural simplicity and flexibility of polynomials make them a powerful tool in many fields such as mathematical modelling, statistical analysis and signal processing. In these aspects, polynomials offer a versatile benefit by providing ease of calculation and flexibility in both theoretical studies and practical applications.

This study focuses on Bernstein polynomials, a special class of polynomials known for their significant applications. Bernstein polynomials are widely used due to their simple structure and valuable properties. They are particularly important in approximation theory, which deals with approximating complex functions using simpler and more manageable functions. This theory is crucial because working directly with functions with unknown or complicated properties can be challenging. By approximating these functions with well-known, simpler functions, researchers can derive useful results more easily.

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Approximation theory addresses whether it is possible to convert complex functions into polynomial functions and how closely such approximations can represent the original functions. This process of approximation is vital in various fields, providing a way to work with functions that are otherwise difficult to handle. By using polynomial approximations, researchers and practitioners can simplify their analyses and obtain more tractable solutions to complex problems. It can be said that approximation by polynomials is perhaps the most important branch of approximation theory. Today, Bernstein polynomials are mostly applied in the field of approximation theory [1, 2]. Recent studies on the approximation properties of blending-type modified Bernstein-Durrmeyer operators have demonstrated that these operators possess strong approximation characteristics [3]. In a recent study, a new type of coupled Bernstein operators for Bezier basis functions was introduced, demonstrating their approximation properties, including the establishment of a local approximation theorem and a convergence theorem for Lipschitz continuous functions [4]. In a recent study, a novel method for approximating the Koopman operator using Bernstein polynomials was proposed. This approach provides a finite-dimensional approximation and characterizes approximation errors with upper bounds expressed in the uniform norm, covering various contexts including univariate and multivariate systems [5]. In a recent study, a new class of Bernstein polynomials based on Bezier basic functions with a shape parameter $\lambda \in [-1, 1]$ was examined. The study provides a Korovkin-type approximation theorem and demonstrates improvements in error estimation in some cases by comparing these operators with classical Bernstein operators [6].

Feller's [7] basic book contains a lot of the most recent work on accomplishment studies in the Bernoulli trials. Let be the sum of the n independents' Bernoulli trial victories. The majority of the distribution's characteristics and associated theorems are well-known and covered in numerous statistical publications and studies when the trials are identical. The probability mass function plays a crucial role in the q-Bernstein polynomial, as demonstrated by Charalambides [8]. The variance and anticipated value were computed using this probability function.

There are two potential results from the single experiment in the Bernoulli procedure. When the intended circumstance materializes, it is deemed a success; when the undesirable circumstance materializes, it is deemed a failure. Let k represent how many of the n separate Bernoulli tests were successful:

$$p_{n,k}\left(x\right)=\binom{n}{k}x^{k}\left(1-x\right)^{n-k}, \quad k=0,1,\ldots,n \tag{1}$$

where x is the probability of success in a single Bernoulli trial and $p_{n.k}(x)$ is the probability of k successes in n trials [9].

Many studies have been conducted on the approximation of real-valued continuous functions for an extended period. One of the key theorems commonly used in functional analysis is the Weierstrass Approximation Theorem. This theorem states that any function that is continuous over a closed interval can be uniformly approximated by polynomials. Bernstein [10] demonstrated this approximation theorem in its most basic form in 1912.

Researchers now favor Bernstein operators since they are simpler and have substantially distinct approach features. Lupas [11] introduced the q-Bernstein theory as a scientific advance. Acu [12] has researched many Bernstein operator generalizations. Cardenas-Morales [13] introduced the new series of linear Bernstein-type operators and studied the q-generalization of these operators. It is also helpful to note that there are a lot more generalizations of the Benstein operators that can advance science.

Positive operators obtained with the help of the binomial distribution

Bernstein's Weierstrass theorem was proved in 1912 using Bernstein polynomials, which were subsequently employed in the proving of numerous other theorems. One common example is the Korovkin theorem.

Let $f: [0, 1] \to R$. The Bernstein polynomial of f:

$$B_n(f;x)=\sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k}x^k(1-x)^{n-k}=\sum_{k=0}^{n} f\left(\frac{k}{n}\right)p_{n,k}(x) \tag{2}$$

where $B_n(f;x)$ is called Bernstein operators of order n for f [14].

Statistical convergence, lacunary statistical convergence, and statistical summability $(C, 1)$ were used in [15] to prove a few Korovkin-type approximation theorems. The Korovkin approximation theorem [16] will now be discussed.

Theorem 1. Let (T_n) be a sequence of positive linear operators from $C[a, b]$, into $C[a, b]$. Then $\lim_n \|T_n(f, x) - f(x)\|_\infty = 0$, for all $f \in C[a, b]$ if and only if $\lim_n \|T_n(f_i, x) - f_i(x)\|_\infty = 0$, for $i = 0, 1, 2$ where $f_0(x) = 1$, $f_1(x) = x$, and $f_2(x) = x^2$ [17].

Convergence criteria

Definition 1. Let the function f be continuous in the interval $[a, b]$. The $\omega(\delta)$ function:

$$\omega(\delta)=\sup_{|x_1-x_2|\le\delta}\left|f(x_1)-f(x_2)\right| \tag{3}$$

with $x_1, x_2 \in [a, b]$ for the real number $\delta > 0$, is the modulus of continuity of the function f [18].

This function will take values based on f, the interval $[a, b]$ and the chosen $\delta > 0$. Let's continue the features of $\omega(\delta)$ modulus of continuity:

– For $0 < \delta_1 < \delta_2$, $\omega(\delta_1) \le \omega(\delta_2)$.
– The $\lim_{\delta \to 0} \omega(\delta) = 0$. When function f is continuous in the interval $[a, b]$.
– The $\omega(\lambda\delta) \le (1 + \lambda)\, \omega(\delta)$ for the real number $\lambda > 0$.

Let us express the theorem that enables to evaluate the difference $|B_n(f; x) - f(x)|$ with the help of the modulus of continuity.

Approximation theorem

Let $\{X_n : n = 1, 2, ...\}$ be a sequence of independent random variables where X_n has a distribution with parameters (n, x). Where n represents the number of trials, and x represents the probability of success. Let f be the real-valued function defined on the real interval $[a, b]$ such that $f^{(m)} \in C[a, b]$ and $L_n(f, x) = Ef(X_n) < \infty$. Then for any $x \in [a, b]$ and any $\delta > 0$:

$$\left|L_n(f,x)-\sum_{k=0}^{m}\frac{f^{(k)}(x)}{k!}b_k\right| \le \begin{cases} \dfrac{1}{m!}\left(\sqrt{b_{2m}}+\delta b_{m+1}\right)\omega_m\left(\dfrac{1}{\delta}\right), & \text{if } m \text{ is odd} \\ \dfrac{1}{m!}\left(b_m+\delta\sqrt{b_2 b_{2m}}\right)\omega_m\left(\dfrac{1}{\delta}\right), & \text{if } m \text{ is even} \end{cases} \tag{4}$$

where $b_k = b_k(n, x) = E(X_n - x)^k$ for $k = 0, 1, ..., m$ is k^{th} moment of random variable X_n around x, $\omega_m(1/\delta)$ is the modulus of continuity of $f^{(m)}$ [19].

In next section of the paper, a theorem stating that the kernel of the Bernstein polynomial $P(S_n = k)$ is equal to $p_{n,k}(x)$ is presented, which is different from the one in source [8]. The proof of this theorem is carried out using the method of induction. The expected value and variance of this probability function have also been calculated. Using the approximation theorem, an upper bound for the convergence of this operator has been determined. This upper bound indicates how quickly the operator converges under certain conditions, which is important for practical applications.

In the section *Reduced form α-Bernstein polynomials and probabilistic interpretation*, information is first provided about the a-Bernstein operator, which was introduced by Chen *et al*. [20]. Using this operator, the expected value and variance of the resulting probability function have been obtained. Again, using the approximation theorem, an upper bound for the convergence of this operator has been determined.

Reduced form of *q*-bernstein polynomials and probabilistic interpretation

Philips [21] introduced a generalization of the Bernstein polynomial q, which varies according to the integer values of q. Following this development, numerous authors have explored this topic from various perspectives. Given that q-Bernstein polynomials are positive linear operators on $C[0, 1]$, the case of $0 < q < 1$ is typically the focus of investigation. For each positive integer n, $B_n(f, q; x)$ q-Bernstein polynomials:

$$B_n\left(f,q;x\right)=\sum_{k=0}^{n} f\left(\frac{[k]}{[n]}\right)\begin{bmatrix} n \\ k \end{bmatrix} x^k \prod_{s=0}^{n-k-1}\left(1-q^s x\right) \tag{5}$$

when $q = 1$, $B_n(f, q; x)$ is the classical Bernstein operator. The q-Bernstein polynomial shares the shape-preserving properties of the classical Bernstein polynomial.

Let $q > 0$. For each non-negative integer l, the q-integer $[l]$, q-factorial $[l]!$, and q-binomial $\left[\begin{smallmatrix} n \\ r \end{smallmatrix}\right]$ $(n \geq 1 \geq 0)$ are defined:

$$[l]:=[l]_q:=\begin{cases}\left(1-q^l\right)/\left(1-q\right) & q\neq 1\\ l & q=1\end{cases}$$

$$[l]!:=\begin{cases}[l][l-1]\ldots[1] & q\neq 1\\ l! & q=1\end{cases}\quad \begin{bmatrix} n \\ l \end{bmatrix}:=[n]!/\left([n-l]![l]!\right) \tag{6}$$

respectively [22]. Additionally, $[0]! := 1$.

Theorem 2. Let a sequence $\{q_n\}$ satisfy $0 < q_n < 1$ and $q_n \rightarrow 1$ as $n \rightarrow \infty$. If $f \in C[0, 1]$ then: $B_n(f, q; x) \rightrightarrows f(x)$ for $x \in [0, 1]$ as $n \rightarrow \infty$ [23].

Direct calculations show that for $0 < q < 1$:

$$B_n\left(t^2,q;x\right)\rightrightarrows x^2+\left(1-q\right)x\left(1-x\right)\neq x^2,\ \ x\in[0,1]\ \text{ as }\ n\rightarrow\infty$$

Therefore, in general, the sequence $\{B_n(f, q; x)\}$ is not an approximating one for the function f [9].

Probabilistic properties of q-Bernstein polynomials

When discussing probability questions, the likelihood of a desired circumstance or event can always be discussed. The experiment is known as the Bernoulli test if two outcomes, such as successful or failed, are obtained for a trial and the trial may be repeated under the same circumstances. Discrete distributions are based on the Bernoulli trial.

Let's express it as the sum of the events of Bernoulli such that $S_n = X_1 + X_2 + ... + X_n$. Since the sum of the Bernoulli trials will give the binomial distribution, $S_n \sim p_{n,k}(x)$ can be written. So the expression can be written for S_n:

$$P\left(S_n=k\right)=\binom{n}{k}x^k\left(1-x\right)^{n-k} \tag{7}$$

Since the kernel of the Bernstein polynomial provides the properties $p_{n,k}(x) \geq 0$ and

$$\sum_{k=0}^{n} p_{n,k}(x) = 1 \quad \text{for} \quad 0 < x < 1$$

it can be regarded as the probability function of a random variable. From the equation $P(S_n = k) = p_{n,k}(x)$, $k = 0, 1, \ldots, n$, the Bernstein polynomial can be written in the form of the expected value of the S_n random variable with the help of the expected value operator E:

$$B_n(f;x) = Ef\left(\frac{S_n}{n}\right) \tag{8}$$

Kernel distributions of the q-Bernstein polynomial

If the Bernstein polynomial is shown as in eq. (2), the term q must be added on success possibilities of $X_1, X_2, \ldots, X_n$ Bernoulli in order to express the q-Bernstein polynomials in the same way. Accordingly:

$$\begin{aligned} P\left(X_j^* = 1\right) &= q^s x \\ P\left(X_j^* = 0\right) &= 1 - q^s x \end{aligned} \tag{9}$$

to define the number of j experiments can be written. Also s denotes the number of unsuccessful attempts in trials up to $j\text{-}1$.

*Example 1***.** The probability values obtained for $n = 3$ using the probability expression in eq. (9) can be derived as follows.

For $n = 3$, k values are 0,1,2 and 3, respectively:

$$P(S_3^* = 0) = P(S_3^* = 0 \mid X_1^* = 0, X_2^* = 0)P(X_1^* = 0, X_2^* = 0) +$$
$$+ P(S_3^* = 0 \mid X_1^* = 0, X_2^* = 1)P\left(X_1^* = 0, X_2^* = 1\right) + P(S_3^* = 0 \mid X_1^* = 1, X_2^* = 0)P\left(X_1^* = 1, X_2^* = 0\right) +$$
$$+ P(S_3^* = 0 \mid X_1^* = 1, X_2^* = 1)P\left(X_1^* = 1, X_2^* = 1\right) = (1 - q^2 x)(1 - qx)(1 - x)$$

$$P(S_3^* = 1) = P(S_3^* = 1 \mid X_1^* = 0, X_2^* = 0)P(X_1^* = 0, X_2^* = 0) +$$
$$+ P(S_3^* = 1 \mid X_1^* = 0, X_2^* = 1)P(X_1^* = 0, X_2^* = 1) +$$
$$+ P(S_3^* = 1 \mid X_1^* = 1, X_2^* = 0)P(X_1^* = 1, X_2^* = 0) + P(S_3^* = 1 \mid X_1^* = 1, X_2^* = 1)P(X_1^* = 1, X_2^* = 1) =$$
$$= q^2 x(1 - qx)(1 - x) + (1 - qx)qx(1 - x) + (1 - qx)x(1 - x)$$

$$P(S_3^* = 2) = P(S_3^* = 2 \mid X_1^* = 0, X_2^* = 0), P\left(X_1^* = 0, X_2^* = 0\right) +$$
$$+ P(S_3^* = 2 \mid X_1^* = 0, X_2^* = 1)P(X_1^* = 0, X_2^* = 1) +$$
$$+ P(S_3^* = 2 \mid X_1^* = 1, X_2^* = 0)P(X_1^* = 1, X_2^* = 0) +$$
$$+ P(S_3^* = 2 \mid X_1^* = 1, X_2^* = 1)P(X_1^* = 1, X_2^* = 1) =$$
$$= q^2 x^2 (1 - x) + qx(1 - x)x + (1 - x)x^2$$

$$P(S_3^*=3)=P(S_3^*=3\big|X_1^*=0,X_2^*=0),P(\big|X_1^*=0,X_2^*=0)+$$
$$+P(S_3^*=3\big|X_1^*=0,X_2^*=1)P(X_1^*=0,X_2^*=1)+$$
$$+P(S_3^*=3\big|X_1^*=1,X_2^*=0)P(X_1^*=1,X_2^*=0)+$$
$$+P(S_3^*=3\big|X_1^*=1,X_2^*=1)P(X_1^*=1,X_2^*=1)=x^3$$

It appears that:

$$P\left(S_3^*=0\right)+P\left(S_3^*=1\right)+P\left(S_3^*=2\right)+P\left(S_3^*=3\right)=1$$

Theorem 3. Let $0 < q < 1$ and $S_n^* = X_1^* + X_2^* +...+ X_n^*$ be the sum of Bernoulli events. Then:

$$P\left(S_n^*=k\right)=p_{n,k}\left(x\right)=\begin{bmatrix}n\\k\end{bmatrix}x^k\prod_{s=0}^{n-k-1}\left(1-q^sx\right)$$

Proof. Now let's show that:

$$P\left(S_n^*=k\right)=\begin{bmatrix}n\\k\end{bmatrix}x^k\prod_{s=0}^{n-k-1}\left(1-q^sx\right)$$

using the method of mathematical induction. For $n = 1$, k values are 0 and 1, respectively:

$$P\left(S_1^*=0\right)=\left(1-x\right)$$
$$P\left(S_1^*=1\right)=x$$

Let's assume that:

$$P\left(S_n^*=k\right)=\begin{bmatrix}n\\k\end{bmatrix}x^k\prod_{s=0}^{n-k-1}\left(1-q^sx\right)\ \text{ for }\ n=k$$

Now let's show that for $n = k +1$:

$$P\left(S_{n+1}^*=k\right)=\begin{bmatrix}n+1\\k\end{bmatrix}x^k\prod_{s=0}^{n-k}\left(1-q^sx\right)$$

$$P\left(S_{n+1}^*=k\right)=P\left(S_{n+1}^*=k\big|S_n^*=k-1\right)P\left(S_n^*=k-1\right)+P\left(S_{n+1}^*=k\big|S_n^*=k\right)P\left(S_n^*=k\right)=$$
$$=q^{n-k+1}x\left(\begin{bmatrix}n\\k-1\end{bmatrix}x^{k-1}\prod_{s=0}^{n-k}\left(1-q^sx\right)\right)n+\left(1-q^{n-k}x\right)\left(\begin{bmatrix}n\\k\end{bmatrix}x^k\prod_{s=0}^{n-k-1}\left(1-q^sx\right)\right)=$$
$$=\frac{[n]![n+1]}{[k]![n-k+1]!}x^k\prod_{s=0}^{n-k}\left(1-q^sx\right)=\begin{bmatrix}n+1\\k\end{bmatrix}x^k\prod_{s=0}^{n-k}\left(1-q^sx\right)$$

Thus it is proved that:

$$P\left(S_n^*=k\right)=\begin{bmatrix}n\\k\end{bmatrix}x^k\prod_{s=0}^{n-k-1}\left(1-q^sx\right)$$

Approximation using the distributions

To obtain an approximate value, it is necessary to calculate the $E(X_n^* - x)^2$. In this case eq. (10) is obtained:

$$E\left(X_n^* - x\right)^2 = b_2\left(n,x\right) = E\left(\frac{S_n}{n} - x\right)^2 = \frac{1}{n^2}E\left(S_n^2 - 2nxS_n + n^2x^2\right) = b_2\left(n,x\right) = \frac{1}{n}xq \tag{10}$$

In approximation theorem eq. (3), if $m = 0$ is taken specially:

$$\left|\left|L_n\left(f,x\right) - f\left(x\right)\right| \le \left(1 + \delta\sqrt{b_2}\right)\omega\left(\frac{1}{\delta}\right) \tag{11}$$

is given. According to this, when $L_n = B_{n,k}$ and $\delta = (n)^{1/2}$ are taken, eq. (12) is obtained:

$$\begin{gathered}\left|B_n\left(f,q;x\right) - f\left(x\right)\right| \le \left(1 + \delta\sqrt{\frac{1}{n}xq}\right)\omega\left(\frac{1}{\delta}\right)\\ B_n\left(f,q;x\right) = f\left(x\right) + o\left(\frac{1}{n}\right),\ \ n \to \infty\end{gathered} \tag{12}$$

Reduced form of *α*-Bernstein polynomials and probabilistic interpretation

The *α*-Bernstein polynomials, like the *q*-Bernstein polynomials, are a derivative of the Bernstein polynomials. This new generalized *α*-Bernstein operators is defined.

Definition 2. Given a function $f(x)$ on [0,1], for each positive integer n and any fixed real α, we define α- Bernstein operator for $f(x)$:

$$T_{n,\alpha}\left(f,x\right) = \sum_{i=0}^{n} f_i p_{n,i}^{(\alpha)}\left(x\right) \tag{13}$$

where $f_i = f(i/n)$. For $i = 0, 1, 2,\ldots, n$, the *α*-Bernstein polynomial $p_{n,i}^{(\alpha)}(x)$ of degree n is defined by $p_{1,i}^{(\alpha)}(x) = 1 - x$, $p_{1,1}^{(\alpha)}(x) = x$:

$$p_{n,i}^{(\alpha)}\left(x\right) = \left[\binom{n-2}{i}\left(1-\alpha\right)x + \binom{n-2}{i-2}\left(1-\alpha\right)\left(1-x\right) + \binom{n}{i}\alpha x\left(1-x\right)\right]x^{i-1}\left(1-x\right)^{n-i-1} \tag{14}$$

where $n \ge 2$, $x \in [0, 1]$ and the binomial coefficients $\binom{k}{l}$ are given:

$$\binom{k}{l} = \begin{cases}\dfrac{k!}{l!\left(k-l\right)!}, & \text{if } 0 \le l \le k\\ 0, & \text{else}\end{cases}$$

For example:

$$p_{2,0}^{(\alpha)}\left(x\right) = \left(1-\alpha x\right)\left(1-x\right),\ \ p_{2,1}^{(\alpha)}\left(x\right) = 2\alpha x\left(1-x\right)$$

$$p_{3,0}^{(\alpha)}\left(x\right) = \left(1-\alpha x\right)\left(1-x\right)^2,\ \ p_{3,1}^{(\alpha)}\left(x\right) = \left(1+2\alpha-3\alpha x\right)\left(1-x\right)\ldots$$

The class of *α*-Bernstein operators contains the classical Bernstein ones, for $\alpha = 1$ [20].

For each function $f(x)$, there is a sequence of *α*-Bernstein operators. The *α*-Bernstein operator maps a function f defined on [0, 1] to the function $T_{n,\alpha}(f)$ where $T_{n,\alpha}(f)$ evaluated at x is denoted by$T_{n,\alpha}(f, x)$.

The authors gave some elementary properties and proved the uniform convergences of the sequence of the α-Bernstein operators to $f \in C[0, 1]$.

Now we will construct the α-Bernstein operators in a different form. With the help of the approximation theorem mentioned earlier in the paper, we will obtain the upper bound required for the convergence of this operator.

Approximation using the distributions

Let K_n be the binomial trial with the same last two trials. In eq. (14), the last two trials are kept the same so that the middle expression is 0. Thus, the first term provides the expression snnnn and the last term provides the expression sn. Thus, the distribution is obtained as in eq. (15). In this case, equality in (13):

$$p_{n,i}^{(\alpha)}(x) = P(K_n = i) = (1-\alpha)P(S_n^* = i) + \alpha P(S_n = i) \tag{15}$$

can be written. Now let's calculate the expected value and

$$E\left(\frac{K_n}{n} - x\right)^2$$

of probability function apply the aforementioned approximation theorem:

$$E(K_n) = (1-\alpha)E(S_n^*) + \alpha E(S_n) \tag{16}$$

For this, let's first calculate the expected value

$$\left(\frac{K_n}{n} - x\right)^2$$

$$E\left(\frac{K_n}{n} - x\right)^2 = b_2 = E\left[(1-\alpha)S_n^* + \alpha S_n\right]^2 \tag{17}$$

In approximation theorem, if $m = 0$ is taken specially:

$$\left|L_n(f,x) - f(x)\right| \le \left(1 + \delta\sqrt{b_2}\right)\omega\left(\frac{1}{\delta}\right) \tag{18}$$

is given. Let $L_n = T_{n,\alpha}$. The aim here is to make the right-hand side of the approximation expression approach 0. Accordingly, if we choose δ such that $\omega(1/\delta)$ approaches 0:

$$\begin{gathered} \left|T_n(f,\alpha;x) - f(x)\right| \le (1 + \delta\sqrt{\frac{1}{2n}(b_2)}\omega\left(\frac{1}{\delta}\right) \\ T_n(f,\alpha;x) = f(x) + o\left(\frac{1}{n}\right), \ n \to \infty \end{gathered} \tag{19}$$

is obtained.

Conclusion

In this article, q-Bernstein polynomials based on q integers are introduced. These polynomials are constructed in a novel form, allowing for a fresh perspective on their application. By utilizing an approximation theorem, we derive an upper bound for the convergence of these operators. This result highlights the versatility of operators with kernel probability functions in approximation theory. Specifically, it demonstrates that any operator with a probabilistic kernel component can be adapted and utilized in various forms to achieve approximation results. This broadens the scope of approximation theory, suggesting that a wide range of operators can be modified to suit different analytical needs and can still provide effective approximations. The findings underscore the potential for innovative applications and further exploration within the field, paving the way for new methods and techniques in the study of polynomial approximations.

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Paper submitted: July 2, 2024
Paper revised: October 14, 2024
Paper accepted: November 1, 2024