QUANTUM ENTANGLEMENT OF N-LEVEL ATOMS UNDER THE INFLUENCE OF THERMAL ENVIRONMENTAL

by

S. Jamal ANWAR [a*], ***Muhammad IBRAHIM*** [a], ***S. Abdul KHALEK*** [b], ***Muhammad ILAYS*** [a], ***Haifa S. ALQANNAS*** [c], ***and M. Khalid KHAN*** [a]

[a] Department of Physics, Quaid-i-Azam University, Islamabad, Pakistan
[b] Department of Mathematics, College of Science, Taif University, Taif, Saudi Arabia
[c] Department of Physical Science, Faculty of Science, University of Jeddah, Jeddah, Saudi Arabia

Original scientific paper
https://doi.org/10.2298/TSCI2406955A

This study investigates the quantum features of entanglement in N-level atomic systems subjected to varying Stark effect (SE) and intrinsic decoherence (ID) parameters. The quantum entanglement (QE) diminishes with increasing SE parameter, while the Quantum Fisher information (QFI) exhibits complex dynamics with no consistent trend across N-levels. Notably, distinct phase factor values influence the QFI differently, with certain phase factors yielding higher QFI values. The ID proves influential, causing a decline in both QFI and von Neumann entropy (VNE) magnitudes. The QFI experiences oscillations, dampening with heightened decoherence, and decays more rapidly than VNE. Additionally, the VNE displays differential behaviors among N-level systems, with the 3-level system maintaining a sustained steady-state compared to the 4- and 5-level systems. The QFI and VNE exhibit periodic behavior across a range of Stark parameter values and phase factors. These findings contribute to a nuanced understanding of entanglement dynamics in multi-level atomic systems under various influencing factors.

Key words: *quantum entanglement, quantum Fisher information, Stark effect, von Neumann entropy*

Introduction

The QE, a sort of non-local correlation, is taken as fundamental in quantum theory [1, 2]. Recent years have seen several new developments in QE [3-5] which have made it a strong tool central to many quantum developments such as quantum thermodynamics [6, 7] and physics of solid states [8-12]. As a result, there has been a great deal of interest in further research into how QE is portrayed and evaluated [13]. Examined were notable bodily events such as QE's unplanned birth and QE's abrupt death [14, 15]. Consequently, during dynamics, the decoherence effect limits the way quantum information is managed and transferred. In that case, it is crucial to have conversations on dynamical decay, research, and defense of QE stabilization.

The concept of QFI has lately been crucial to theory of parameter measurement by specifying the limits of precision in quantum estimations [16]. Clock synchronisation and quantum frequency standards are two significant applications of it in quantum technology [17, 18],

* Corresponding author, e-mail: sjamalanwar@gmail.com

as well as the acceleration of gravity [19]. The QFI effectively characterizest he capacity of a parameter encoded in a quantum state to be statistically distinguished using the Cramer-Rao inequality to establish the parameter estimate precision. The statistically significant measurement stored in a quantum condition is effectively characterized by the Cramer-Rao inequality, which the QFI use to establish the parameter estimate precision [20-22], it shows an upper bound for which measurements may distinguish between quantum states and provides a particular bound for identifying members of the probability distribution family. It is necessary to examine the link among the concept of QFI and other significant occurrences like squeezing and QE. It was demonstrated that in this instance, the QFI can be utilised to understand the multi-partite states' QE [23, 24]. It was also demonstrated that the spin-squeezing phenomenon is not as strong as the QE requirement provided by the QFI. The QFI is taken into consideration in the context of finite systems to estimate the noise parameter of amplitude-damping with respect to open systems and non-unitary evolutions [25, 26] as well as depolarizing channels [27]. When there is a bosonic channel, the determination of the loss parameter can be enhanced by employing Gaussian squeezed probes [28]. Phase transitions have recently been explained by scientists using the parameter estimation issue of quantum systems [29, 30].

In atomic physics, a phenomena known as the SE is the division and shift of a spectral line into many parts when there is an electric field present. The degree of splitting is known as the SE [31]. It is similar to the Zeeman phenomenon, which occurs when there is a magnetic field and a spectral line break into several components. Its influence on certain phenomena in different quantum systems has been studied. Its influence on certain phenomena in different quantum systems has been studied. The interaction between a coherent field and an analysis of a two-level atom is done [32], where the influence on Wehrl entropy and field purity of the Kerr-like medium and SE parameter is investigated. The impact of SE on the radiation field coupling between two two-level atoms is addressed. [33]. How QFI works and QE are studied for three and four-level atomic systems collaborating with a coherent field and affected by the KLM and SE [34]. Furthermore, the QE of two three-level atomic systems under the influence of the KLM and the SE has been investigated [35]. The QE dynamics of an N-level atomic system coupling with a coherent field have been examined with both a KLM and SE present [36].

The Hamiltonian undergoes random and unknowable pertubations, which cause decoherence. If it is impossible to follow these pertubations precisely, experiments must average over them. The density operator's off-diagonal components' decay in the same basis causes the atom to effectively undergo irreversible evolution and suppress coherent quantum characteristics. In addition the off-diagonal matrix element's decay, the conjugate variables also get noise. Non-etheless, the topic of decoherence in quantum mechanics has drawn increased attention because of its possible uses in quantum computing and quantum measurement methods. In particular, Milburn [37] offered an accessible framework of ID, based on the assumption that the framework evolves stochastically in a succession of identical unitary conversions across sufficiently small-time steps, instead of continuously under unitary development. This model offers a basic modification of traditional quantum mechanics in which the quantum coherence naturally disappears as the quantum system gets larger. Milburn [37] considered only the natural evolution of elementary quantum systems. In this approach, quantum coherence occurs naturally disrupted as the quantum system progresses, providing a straightforward modification of standard quantum mechanics. Milburn [37] only took into account the simple quantum system's free development. Studies have been done on the ID in the JCM with a single quantized field mode.

A quantum system experiences decoherence when connected to its surroundings, which specifically happens in an open quantum system and results in the loss of its distinctive quantum property. Information is lost due to the decrease of coherence, which is primarily characterized by the Lindblad master formula within a dynamic quantum system [37-39]. When interacting with the environment, information is permanently lost, and system characteristics are combined with those of the macroscopic scale. Without an external reservoir, Milburn suggested another straightforward method for researching this decaying dynamic process [40]. The ID systems and diminishing quantum correlations in physically significant systems have been the subject of extensive investigation in recent years. The JCM with Heisenberg exchange coupling demonstrated QE degradation beneath the influence of ID [41-44].

The present work aims at determining the QE dynamics of the N-level atomic system with moving atom under the effect of ID and SE. We examine the effects of the ID and the SE on the atomic systems dynamics of QFI and VNE that are moving at three, four, and five levels. It is evident that the ID and the SE predominate during the quantum system's temporal growth. When ID is present, the SE significantly affects QFI dynamics. Furthermore, the SE affects the VNE more strongly when the ID is present but there is no motion of atom. In the end, it is discovered that the N-level atomic system is extremely susceptible to these external influences.

Hamiltonian model

We study the scenario of the moving atoms at levels two, three, four, and five when ID is present and penetrate the cavity while it's the superposition condition bound by the SE. In the presence of the SE with ID, we study the cascade configuration of atom with two, three, four, and five levels in motion. Assuming the RWA, the system $\hat{H}_T$, can be expressed [45]:

$$\hat{H}_T = \hat{H}_{\text{atom-field}} + \hat{H}_I \tag{1}$$

where $\hat{H}_{\text{atom-field}}$ is the atom which is not interacting and field Hamiltonian and $\hat{H}_I$, – the coupling portion. Our writing for $\hat{H}_{\text{atom-field}}$ will be:

$$\hat{H}_{\text{atom-field}} = \sum_j \omega_j \hat{\sigma}_{j,j} + \nu \hat{a}^\dagger \hat{a} \tag{2}$$

where j^{th} is thelevel's represents population operators. For the non-resonant scenario, the $\hat{H}_I$ can be found [45]:

$$\hat{H}_I = \sum_{s=1}^{N} \Omega(t)\left[\hat{a}\mathrm{e}^{-i\Delta_s t}\hat{\sigma}_{s,s+1} + \left(\hat{a}\mathrm{e}^{-i\Delta_s t}\hat{\sigma}_{s,s+1}\right)^\dagger\right] \tag{3}$$

It depicts the two-level atom when $N = 1$, and the three-, four-, and five-level atoms when $N = 2$, 3, and 4.

When it comes to SE, the $\hat{H}_I$ is provided:

$$\hat{H}_I = \sum_{s=1}^{N} \Omega(t)\left[\hat{a}\mathrm{e}^{-i\Delta_s t}\hat{\sigma}_{s,s+1} + \left(\hat{a}\mathrm{e}^{-i\Delta_s t}\hat{\sigma}_{s,s+1}\right)^\dagger\right] + \beta \hat{a}^\dagger \hat{a} \tag{4}$$

The Stark shift parameter is denoted by χ. The term of atomic mobility is incorporated into the final, which is the DM obtained by the use of eq. (4). The basic system Hamiltonian models the mobility of atoms through the parameter $\Omega(t)$. Non-etheless, the time evolution unitry operator directs the dynamics of the entire atom field interacting system to determine the system's time-dependent density matrix for the investigation of information quantifiers such as QFI. The parameter for detuning is defined:

$$\Delta_s = \nu - \left(\omega_s - \omega_{s+1}\right) \tag{5}$$

The $\Omega(t)$ describes the moving atom in eq. (4), where atom and field are coupled through constant g [46] and along the z-axis, the atom is travelling:

$$\Omega(t) = g\sin\left(\frac{p\pi vt}{L}\right),\ p \neq 0 \tag{6}$$

$$\Omega(t) = g,\ \ p = 0 \tag{7}$$

where L is the cavity length along the z-direction, half of the mode's wavelengths in the cavity are represented by the symbol p, and v – the atomic motion velocity. Using $v = \mathrm{g}L/\pi$ as the velocity of an atom:

$$\begin{aligned}\Omega_1(t) &= \int_0^t \Omega(\tau)\,\mathrm{d}\tau = \frac{1}{p}\left(1-\frac{\cos(p\pi vt}{L}\right) \text{ for } \ p\neq 0\\ &= gt \ \text{ for } \ p=0\end{aligned} \tag{8}$$

The system's ideal input condition is provided:

$$\left|\Psi(0)\right\rangle_{\mathrm{Opt}} = \frac{1}{\sqrt{2}}\left(\left|0\right\rangle + \left|1\right\rangle\right)\otimes\left|\alpha\right\rangle \tag{9}$$

where the ground and excited states of an atom are represented by the symbols $|1\rangle$ and $|0\rangle$, and the coherent input state α is provided:

$$\Big|\left|\alpha\right\rangle = \sum_{n=10}^{\infty}\alpha^n\sqrt{\frac{\mathrm{e}^{-|\alpha|^2}}{n!}}\left|n\right\rangle \tag{10}$$

We examine a phase gate consisting of just one atom that presents the phase change:

$$\hat{U}_\varphi = \left|1\right\rangle\left\langle 1\right| + \mathrm{e}^{i\varphi}\left|0\right\rangle\left\langle 0\right| \tag{11}$$

When the ideal input state is subjected to the phase gate operator, we obtain:

$$\hat{U}_\varphi\left|\Psi(0)\right\rangle_{\mathrm{Opt}} = \left|\Psi(0)\right\rangle = \frac{1}{\sqrt{2}}\left(\left|1\right\rangle + \mathrm{e}^{i\varphi}\left|0\right\rangle\right)\otimes\left|\alpha\right\rangle \tag{12}$$

$$\left|\Psi(0)\right\rangle = \frac{1}{\sqrt{2}}\left(\left|1\right\rangle + \mathrm{e}^{i\varphi}\left|0\right\rangle\right)\otimes\left|\alpha\right\rangle \tag{13}$$

where the parameter for the phase shift is φ. The state vector that varies with time is provided:

$$\left\|\Psi(t)\right\rangle = \hat{U}(t)\left|\Psi(0)\right\rangle \tag{14}$$

The time dependent DM is determined by applying the unity operator to the time dependent state vector, which is obtained using eq. (15). The Hamiltonian's eigenvalues and eigenvectors yield the unitary matrix in eq. (15). The precise form utilised to determine the unitary matrix is substituted for eq. (15). The DM expression:

$$\hat{\rho}(t) = \sum_{m,n}^{N}\left|\Psi_n(t)\right\rangle\left\langle\Psi_n(t)\right|\hat{\rho}(t)\left|\Psi_m(t)\right\rangle\left\langle\Psi_m(t)\right| \tag{15}$$

Moreover, the DM is provided as [47] in terms of ID:

$$\hat{\rho}(t) = \sum\exp\left(\frac{\gamma t}{2}\left(E_m - E_n\right)^2 - i\left(E_m - E_n\right)t\right)\times\left\langle\Psi_m\right|\hat{\rho}(0)\left|\Psi_n\right\rangle\left|\Psi_m\right\rangle\left\langle\Psi_n\right| \tag{16}$$

where $E_{m,n}$ and $\Psi_{m,n}$ are the eigenvalues and eigenvectors of H_I, and γ is the ID parameter.

The QFI expressed in terms of φ for a bipartite ρ_{AB} as [48, 49]:

$$I_{QF}(t) = I(\varphi,t) = \text{Tr}\left[\rho_{AB}(\varphi,t)\left\{L^2(\varphi,t)\right\}\right] \tag{17}$$

where the quantum score [50] SLD is represented by $L(\varphi, t)$, which may be found:

$$\frac{\partial\rho(\varphi,t)}{\partial\varphi} = \frac{1}{2}\left[L(\varphi,t)\rho_{AB}(\varphi,t) + \rho_{AB}(\varphi,t)L(\varphi,t)\right] \tag{18}$$

In a similar vein, the VNE is described:

$$S_A = -\text{Tr}\left(\rho_A \ln\rho_A\right) = -\sum_i r_i \ln r_i \tag{19}$$

where the atomic DM $\rho_A = \text{Tr}B(\rho_{AB})$ eigenvalues are denoted by r_i.

The ID parameter γ has time-inverse dimensions, while p has length-dimensions.

The following part now presents the impact of various environmental elements, γ, β, φ, and p, on the development of the QFI and VNE.

Discussions and numerical outcomes

We assume that the system interacts with the Coherent field and the decoherence effects of different strengths are also present. The system's level is Stark shifted with strength β. We solve the system dynamics numerically and have chosen a 0.1 time step size.

The VNE and QFI of N-level stationary Stark shifted atomic systems

Figures 1-3 show the dynamical behavior of the quantifiers for β = 0.3, 1, and 3, respectively. The decoherence parameter value is fixed at γ = 0.00001. For all the case of β, the QFI is increased with N for the phase factor $\varphi = \pi/4$ as compared to φ = 0. Whereas the VNE is not significantly changing with respect to this phase factor change.

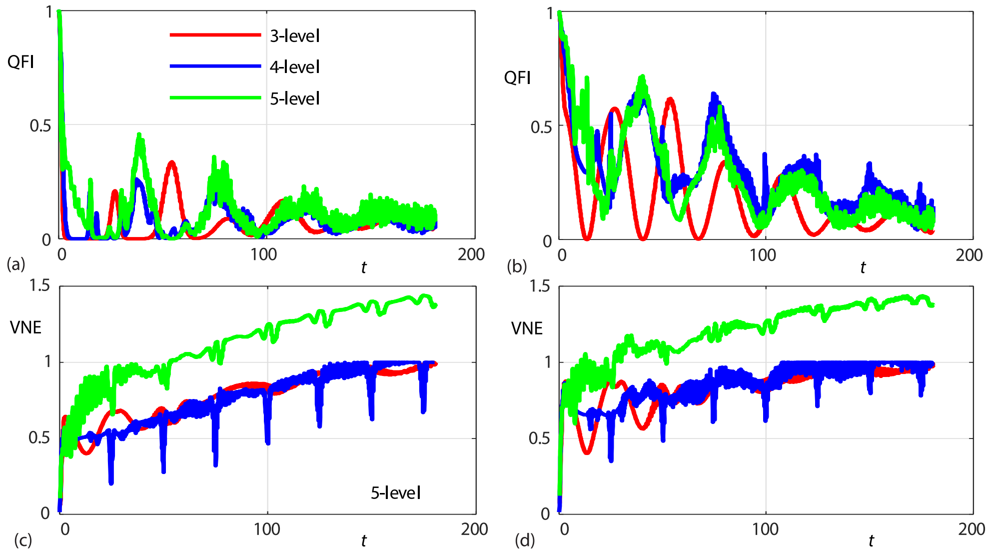

Figure 1. The QFI (a), (b) and VNE (c), (d) as a function of time, α = 6, β = 0.3, γ = 0.00001, φ = 0 (a), (c) and $\pi/4$ (b), (d) p = 0

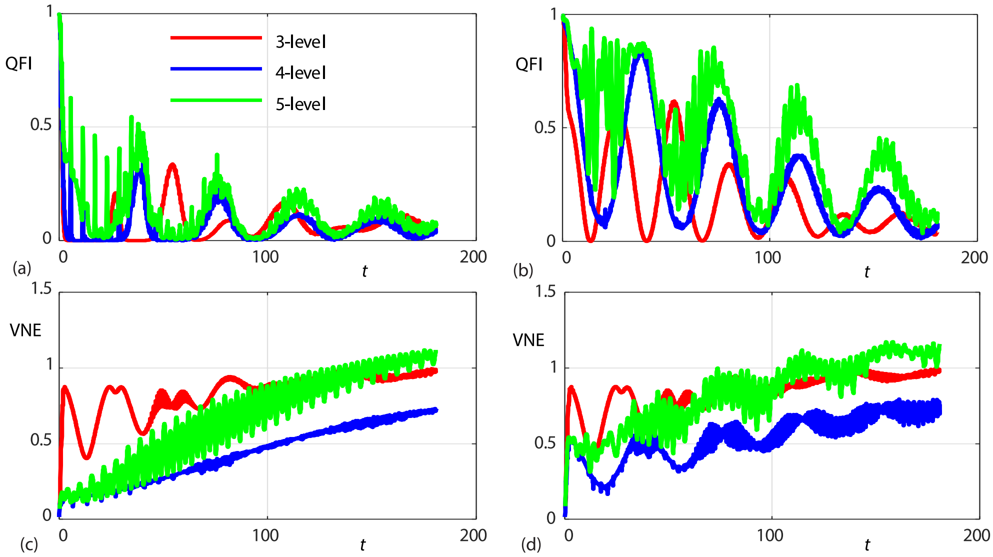

Figure 2. The QFI (a), (b) and VNE (c), (d) as a function of time, $\alpha = 6$, $\beta = 1$, $\gamma = 0.00001$, $\varphi = 0$ (a), (c) and $\pi/4$ (b), (d) $p = 0$

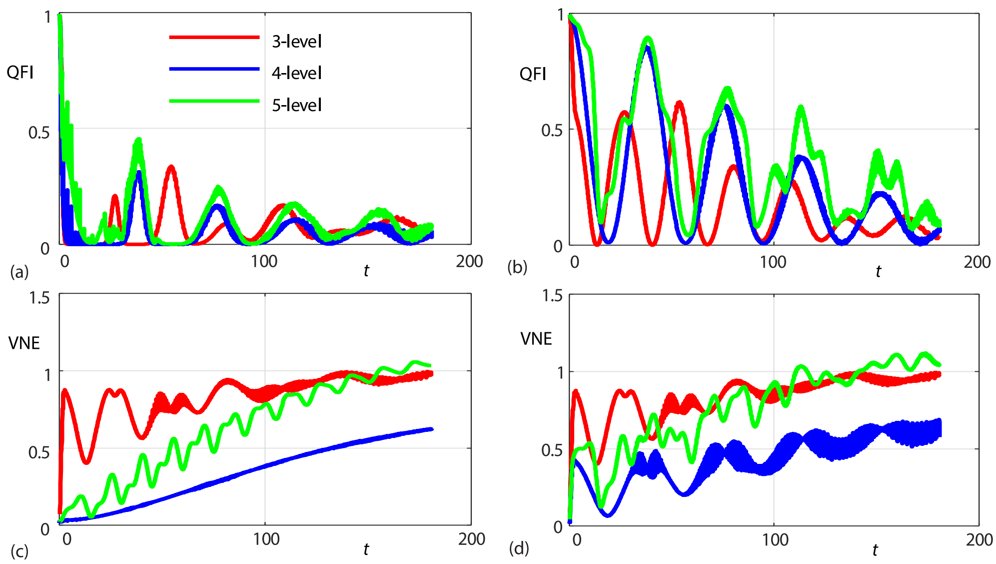

Figure 3. The QFI (a), (b) and VNE (c), (d) as a function of time, $\alpha = 6$, $\beta = 3$, $\gamma = 0.00001$, $\varphi = 0$ (a), (c) and $\pi/4$ (b), (d) $p = 0$

For $\beta = 0.3$, there is no specific trend of changing observed by N for the QFI for separate cases of both φ. The VNE magnitude for different N values and $\beta = 0.3$ shows that the VNE dynamics of 5-level is greater than 3-level, and 4-level atomic system. Moreover, for this Case 3-level and 4-level systems have nearly equal value of the VNE. For $\beta = 1$ case, the QFI shows no specific increasing or decreasing behavior for all N-levels. Additionally, for phase factor $\varphi = \pi/4$, the N-level atomic systems have more QFI value as compared to $\varphi = 0$. The VNE magnitude is decreased for both phase factor values as compared to $\beta = 0.3$ case. Interestingly, the VNE dynamics value for 5-level system is less than the VNE dynamics value for 4-level initially in time evolution. As time progresses, the VNE magnitude for 5-level increases. For β = 1, we observe no behavior change in the QFI and VNE dynamics for *N*-level system as compared to β = 0.3. The case of β = 1 has more rapid oscillation of the dynamics as compared to. For all values of Stark parameter, we observe that the QFI value decreases in the dynamics for all *N*-level systems, whereas the magnitude of the VNE increases in the dynamics.

Figures 4-6 show the dynamical behavior of the QFI and VNE of the system at intrinsic decoherence γ = 0.0001 for β = 0.3, 1, and 3, respectively. For β = 0.3, the QFI magnitude reduces as time progresses for all *N*-level systems for both values of φ. For $\varphi = \pi/4$, the magnitude of QFI dynamics is greater than the QFI magnitude at φ = 0. On the other hand, for the case of β = 0.3, the VNE value is increased and attains a steady-state value for all *N*-level cases. Whereas, for the case of γ = 0.00001, the VNE value in the dynamics does not get a steady-state value in the given time scale. Additionally, the 5-level atomic system has more VNE as compared to the 3- and 4-level atomic systems. For β = 1 and $\varphi = \pi/4$ case, the QFI shows increasing magnitude as levels increase. In the initial time evolution, the VNE dynamics for 5-level atomic system has less magnitude as compared to 4- and 3-level systems, and as time progress the VNE value improves and increases for 5-level system. For the case of β = 3, the magnitude of the QFI and VNE of all *N*-values except 5-level does not change significantly as compared to the β = 1 case. For the 5-level atomic system, the dynamics of the QFI and VNE exhibit the broadening of the line for β = 1 case. Comparing the results with γ = 0.00001, we have seen that the QFI magnitude is slightly decreased for both φ cases. The magnitude of the VNE is also decreased as the intrinsic decoherence value is increased and the system dynamics gets steady-state value for *N*-level systems.

The effect of increasing intrinsic decoherence parameter in the presence of variable Stark parameter strengths on *N*-level atomic systems has interesting results. We have seen that

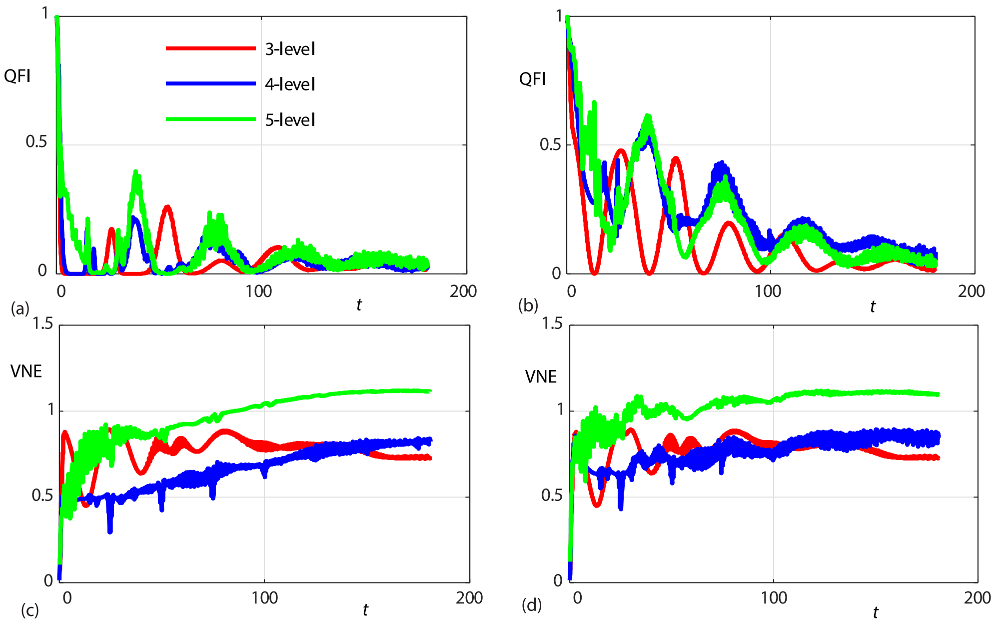

Figure 4. The QFI (a), (b) and VNE (c), (d) as a function of time, α = 6, β = 0.3, γ = 0.0001, φ = 0 (a), (c) and $\pi/4$ (b), (d) p = 0

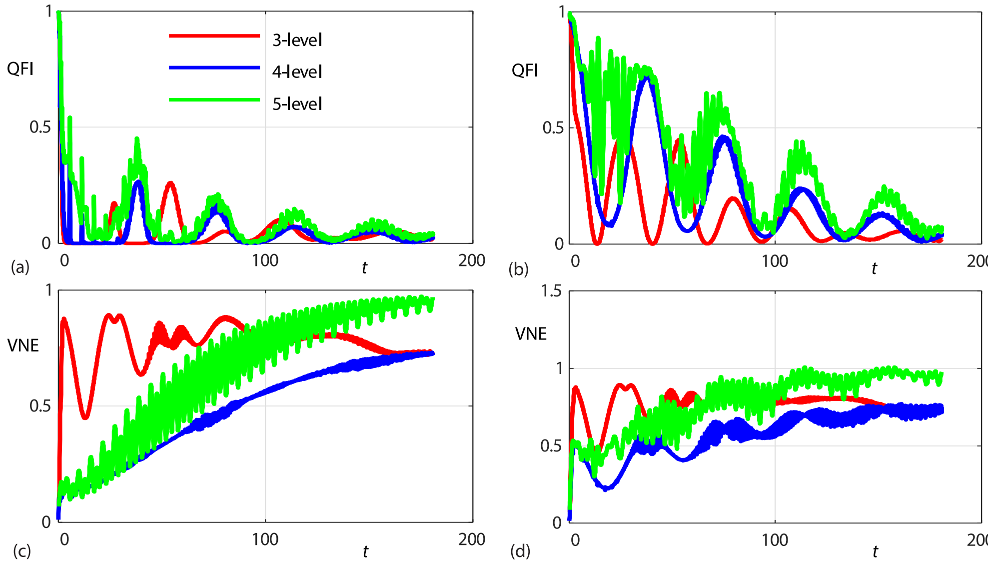

Figure 5. The QFI (a), (b) and VNE (c), (d) as a function of time, $\alpha = 6$, $\beta = 1$, $\gamma = 0.0001$, $\varphi = 0$ (a), (c) and $\pi/4$ (b), (d) $p = 0$

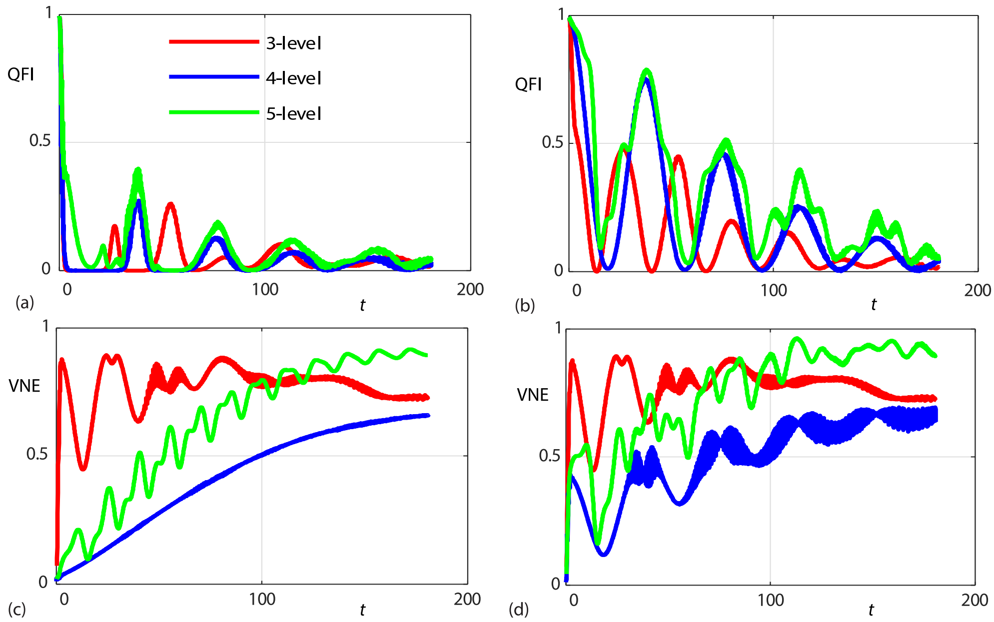

Figure 6. The QFI (a), (b) and VNE (c), (d) as a function of time, $\alpha = 6$, $\beta = 3$, $\gamma = 0.0001$, $\varphi = 0$ (a), (c) and $\pi/4$ (b), (d) $p = 0$

both QFI and VNE magnitude decreases for both phase factors if γ is increased. The QFI oscillations and amplitude for all N-level systems decay with the increase in γ. The QFI dies out more quickly with the increase in γ as compared to the VNE. For the smaller value of γ, the VNE magnitude increases as time progresses. While for an increased value of γ, the VNE shows a decaying behavior in the dynamics. We have seen that the 3-level system exhibits a sustained steady-state value of VNE as compared to 4- and 5-level systems. The oscillatory behavior of the VNE for all N-level systems damp out with the increase in γ. The decay behavior of QFI by the increase in γ is observed for all β values.

The VNE and QFI of N-level moving Stark Shifted atomic systems

In this section we explore the dynamics of the N-level atomic system which is moving inside the cavity containing a coherent field. We consider the cases of increasing the magnitude of intrinsic decoherence and Stark parameter and study the dynamics of QFI and VNE of the system.

The dynamics of N-level moving atomic system in the presence of intrinsic decoherence strength $\gamma = 0.0001$ for $\beta = 0.3$, 1, and 3 is shown in figs. 7-9, respectively. For all the values of β, we observe the periodic behavior of the QFI and VNE for both phase factor of initial state. For the case of $\beta = 0.3$, the 3-level atomic system has more amplitude of periodic oscillation as compared to the 4- and 5-level atomic systems.

This difference becomes more prominent for the case of $\varphi = \pi/4$. For 5-level atomic system has negligible amplitude of oscillations as compared to the 3-level system. For the case of $\beta = 0.3$, the VNE exhibit least oscillation amplitude of 5-level system for phase factor $\varphi = 0$. On the other hand, for the case of $\varphi = \pi/4$, the VNE has least magnitude of oscillation of 4-level system. For $\beta = 1$ case, the QFI's oscillation amplitude does not significantly change for all N-level cases as compared to $\beta = 0.3$. But this amplitude reduces for the case of $\varphi = \pi/4$. For the case of VNE, the 4- and 5-level atomic systems have nearly the same amplitude of oscillations. While for the 3-level system, the oscillations amplitude is more as compared to the 4- and 5-level systems. For $\beta = 1$ and $\varphi = 0$ case, the 4- and 5-level system oscillations amplitude is more suppressed as compared to $\beta = 0.3$ and $\varphi = 0$. The QFI oscillation magnitude remains nearly

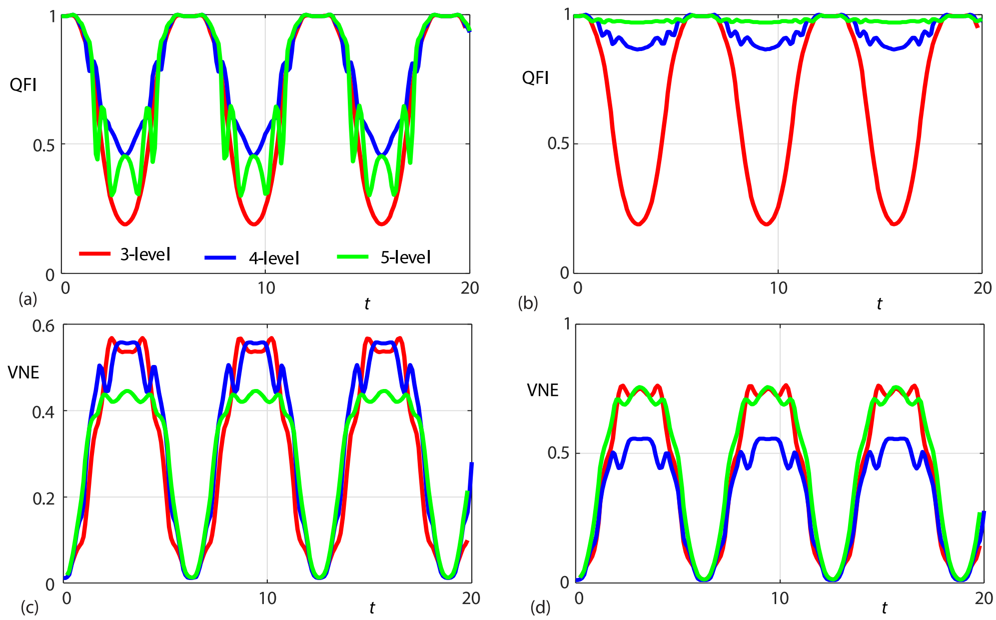

Figure 7. The QFI (a), (b) VNE (c), (d) as a function of time, $\alpha = 6$, $\beta = 0.3$, $\gamma = 0.00001$, $\varphi = 0$ (a), (c) and $\pi/4$ (b), (d) $p = 0$

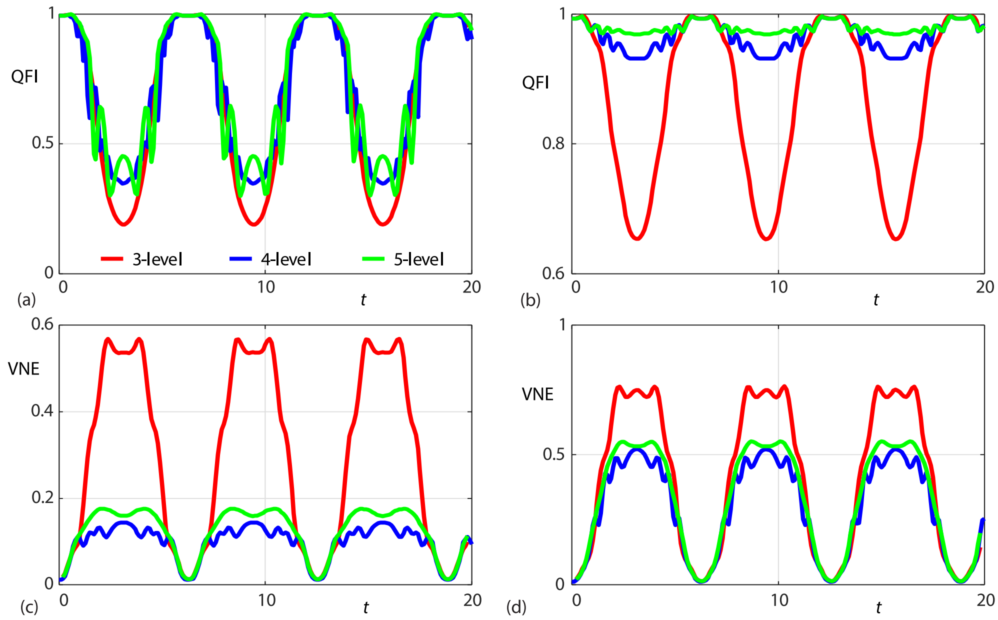

Figure 8. The QFI (a), (b) and VNE (c), (d) as a function, $\alpha = 6$, $\beta = 1$, $\gamma = 0.00001$, $\varphi = 0$ (a), (c) and $\pi/4$ (b), (d) $p = 0$

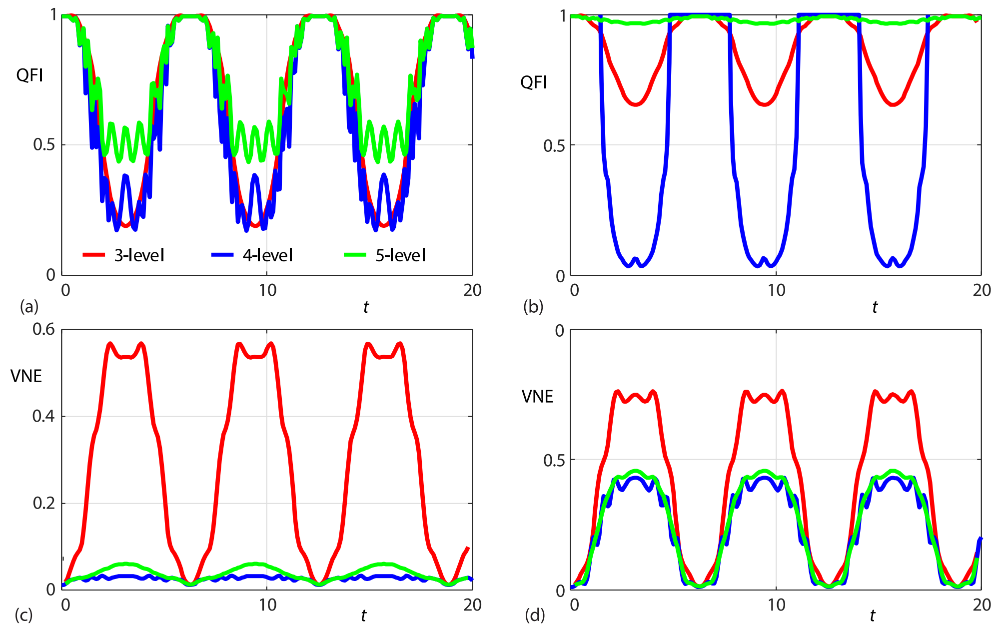

Figure 9. The QFI (a), (b) and VNE (c), (d) as a function of time, $\alpha = 6$, $\beta = 3$, $\gamma = 0.0001$, $\varphi = 0$ (a) and $\pi/4$ (b) $p = 0$

same for $\beta = 1$ and $\beta = 3$ for $\varphi = 0$. For $\varphi = \pi/4$ and $\beta = 3$, the 4-level system has more oscillation strength. For $\beta = 3$, the VNE, oscillations are reduced for the 4- and 5-level systems for both φ cases, but reduction is more noticeable for the case of $\varphi = 0$.

Conclusions

In summary, our investigation into the entanglement dynamics of *N*-level atomic systems under the influence of Stark shift and intrinsic decoherence parameters has revealed intriguing behaviors. The decrease in entanglement magnitude with increasing Stark shift parameter signifies the sensitivity of entanglement to external perturbations. The complex dynamics of QFI and VNE with varying *N*-levels, phase factors, and intrinsic decoherence values highlight the intricate nature of quantum systems. The differential influence of phase factors on QFI underscores the importance of initial conditions in shaping entanglement evolution. Intrinsic decoherence emerges as a critical factor, causing both QFI and VNE magnitudes to decline, with QFI exhibiting faster decay than VNE. The sustained steady-state of VNE in the 3-level system compared to higher-level systems indicates the impact of system dimensionality on entanglement stability.

The observed periodic behavior of QFI and VNE across Stark parameter values and phase factors adds another layer of complexity to the entanglement dynamics. Furthermore, the eventual demise of both QFI and VNE for higher intrinsic decoherence values highlights the fragility of entanglement in the face of environmental interactions. These findings contribute valuable insights into the intricate interplay of factors influencing entanglement in *N*-level atomic systems, with implications for quantum information processing and quantum communication. Future research may delve into more intricate system-environment interactions and explore potential applications of these findings in the development of robust quantum technologies.

Acknowledgment

The authors extend their appreciation Taif University, Saudi Arabia, for supporting this work through Project No. (TU-DSPP-2024-08)

References

[1] Nielsen, M. A., Chuang, I.. L., *Quantum Computation and Information*, Cambridge University Press, Cambridge, UK, 2000
[2] Alber, G., *et al*., *Quantum Information*, Springer, Berlin, Germany, 2001 Chapter 5
[3] Joo, J., *et al.*, *Quantum Metrology with Entangled Coherent States*, *Phys. Rev. Lett*., *107* (2011), 083601
[4] Berrada, K., *et al*., Quantum Metrology With Entangled Spin-Coherent States of Two Modes, *Phys. Rev. A*, *86* (2012), 33823
[5] Berrada, K., Quantum Metrology with SU (1, 1) Coherent States in the Presence of Non-Linear Phase Shifts. Phys. Rev. A, *88* (2013), 013817
[6] Goold, J., *et al*., The Role of Quantum Information in Thermodynamics-A Topical Review, *J. Phys. A Math. Theor*., *49* (2016), 143001
[7] Kibe, T., *et al*., Quantum Thermodynamics of Holographic Quenches and Bounds on the Growth of Entanglement from the Quantum Null Energy Condition, *Phys. Rev. Lett*., *128* (2022), 191602
[8] Liu, C., *et al*., Towards Entanglement Distillation between Atomic Ensembles Using High-Fidelity Spin Operations, *Commun. Phys*., *5* (2022), 67
[9] Castelano, L. K., *et al*., Open Quantum System Description of Singlet-Triplet Qubits in Quantum Dots, *Phys. Rev. B*, *94* (2016), 235433
[10] Pfaff, W., *et al*., Demonstration of Entanglement-by-Measurement of Solid-State Qubits, *Nat. Phys*., *9* (2013), 29
[11] Aldaghfag, S. A., *et al*., Entanglement and Photon Statistics of two Dipole-Dipole Coupled Superconducting Qubits with Kerr-like Non-Linearitie, *Results Phys*., *16* (2020), 102978
[12] Abdel-Khalek, S., *et al*., Quantum Correlations and Non-Classical Properties for Two Superconducting Qubits Interacting with a Quantized Field in the Context of Deformed Heisenberg Algebra, *Chaos Solitons Fractals*, *143* (2021), 110466
[13] Horodecki, R., Quantum Entanglement, *Rev. Mod. Phys*., *81* (2009), 865

[14] Eberly, J. H., Yu, T., *The End of an Entanglement*, AAA of Science, 2007, Vol. 316, No. 5824, pp. 555-557
[15] Yu, T., and Eberly, J. H., Finite-Time Disentanglement Via Spontaneous Emission, *Phys. Rev. Lett*., *97* (2006), 140403
[16] Fisher, R. A., Theory of Statistical Estimation, *Math. Proc. Camb. Philos. Soc*., *22* (1925), 700
[17] Huelga, S. F., Improvement of Frequency Standards with Quantum Entanglement, *Phys. Rev. Lett.*, 79 (1997), 3865
[18] Jozsa, R., *et al*., Quantum Clock Synchronization Based on Shared Prior Entanglement, *Phys. Rev. Lett*., *85* (2000), 2010
[19] Peters, A., *et al*., Measurement of Gravitational Acceleration by Dropping Atoms, *Nature*, *400* (1999), 849
[20] Helstrom, C. W., *Quantum Detection and Estimation Theory*, Academic Press, Inc., New York, N. Y, USA, 1976
[21] Braunstein, S. L., Caves, C. M., Statistical Distance and the Geometry of Quantum States, *Phys. Rev. Lett*., *72* (1994), 3439
[22] Braunstein, S. L., *et al*., Generalized Uncertainty Relations: Theory, Examples, and Lorentz Invariance, *Ann. Phys*., *247* (1996), 135
[23] Boixo, S., Monras, A., Operational Interpretation for Global Multipartite Entanglement, *Phys. Rev. Lett*., *100* (2008), 100503
[24] Pezze, L., Smerzi, A., Entanglement, Non-Linear Dynamics, And The Heisenberg Limit, *Phys. Rev. Lett*., *102* (2009), 100401
[25] Berrada, K., Non-Markovian Effect on the Precision of Parameter Estimation, *Phys. Rev. A*, *88* (2013), 035806
[26] Ji, Z., *et al*., Parameter Estimation Of Quantum Channels, *IEEE Trans. Inf. Theory*, *54* (2008), 5172
[27] Fujiwara, A. Quantum Channel Identification Problem, *Phys. Rev. A*, *63* (2001), 042304
[28] Monras, A., Paris, M. G. A., Optimal Quantum Estimation of Loss In Bosonic Channels, *Phys. Rev. Lett*., *98* (2007), 160401
[29] Invernizzi, C., Optimal Quantum Estimation in Spin Systems at Criticality, *Phys. Rev. A*., *78* (2008), 042106
[30] Ma, J., Wang, X., Fisher Information and Spin Squeezing in the Lipkin-Meshkov-Glick Model, *Phys. Rev. A*, *80* (2009), 012318
[31] Alliluev, S. P., Malkin, I. A., Calculations of the Stark Effect in Hydrogen Atoms by Using the Dynamical Symmetry O(2, 2) O(2), *Zh. Eksp. Teor. Fiz*., *66* (1974), 1283
[32] Al Naim, A. F., *et al*., Effects of Kerr Medium and Stark Shift Parameter on Wehrl Entropy and the Field Purity for Two-Photon Jaynes-Cumminges Model under Dispersive Approximation, *J. Russ. Laser. Res*., *40* (2019), 20
[33] Hilal, E. M. A., Khalil, E. M., Quantum Statistical Aspects of Interactions between the Radiation Field and Two Entangled Two-Level Atoms in the Presence of Stark Shift Terms, *J. Russ. Laser. Res*., *39* (2018), 207
[34] Ibrahim, M., *et al*., Dynamics of N Two-Level Moving Atoms under the Influence of the Non-Linear Kerr Medium, *Optical and Quantum Electronics*, *55* (2023), 1009
[35] Anwar, S. J., *et al*., Effect of Stark-and Kerr-Like Medium on the Entanglement Dynamics of Two Three-Level Atomic System, *Quantum Inf. Process*., *18* (2019), 192
[36] Anwar, S. J., *et al*., Stark and Kerr Effects on the Dynamics of Moving N-Level Atomic System, *J. Quantum Inf*., *9* (2019), 22
[37] Milburn, G. J., Intrinsic Decoherence In Quantum Mechanics, *Phys. Rev. A*, *44* (1991), Nov., pp. 5401-5406
[38] Shor, P. W., Scheme for Reducing Decoherence in Quantum Computer Memory, *Physical Review A*, *48* (1995), 2493
[39] Isaac L., Chuang, Yamamoto, Y., Creation of a Persistent Quantum Bit Using Error Correction, *Phys. Rev. A*, *55* (1997), 114
[40] Moya-Cessa, H., *et al*., Intrinsic Decoherence in the Atom-Field Interaction, *Phys. Rev. A*, *48* (1993), 3900
[41] Plenio, M. B., Knight P. L, Decoherence Limits to Quantum Computation Using Trapped Ions, *Proc. R. Soc. Lond. A*., *453* (1997), Oct., pp. 2017-2041
[42] Zurek, W. H., Pointer Basis of Quantum Apparatus: Into what Mixture Does the Wave Packet Collapse, *Phys. Rev. D*, *24* (1981), Sept., pp. 1516-1525
[43] W. H. Zurek, Environment-Induced Superselection Rules, *Phys. Rev. D*, *26* (1982), Oct., pp. 1862-1880
[44] Leggett, A. J., *et al*., Dynamics of the Dissipative Two-State System, *Rev. Mod. Phys*., *59* (1987), Jan., pp. 1-85

[45] Zheng, L., Zhang, G.-F., Intrinsic Decoherence In Jaynes-Cummings Model with Heisenberg Exchange Interaction, *The European Physical Journal D*, *71* (2017), 288
[46] Abdel-Khalek, S., Quantum Fisher Information for Moving Three-Level Atom, *Quantum Inf. Process*, *12* (2013), 3761
[47] Enaki, N. A., Ciobanu, N. J., Quantum Trapping Conditions for Three-Level Atom Flying through Bimodal Cavity Field, *Mod. Opt*., 55 (2008), 10, pp. 589-598
[48] Guo J. L., Song H. S., Entanglement between Two Tavis-Cummings Atoms With Phase Decoherence, *Journal of Modern Optics*, *56* (2009), 4, pp. 496-501
[49] Lu, X., *et al*., Quantum Fisher Information Flow and Non-Markovian Processes of Open Systems, *Phys. Rev. A*, *82* (2010), 042103
[50] Almalki, S., *et al*., Dynamics of Quantum Correlations And Fisher Information Of A Multipartite System under the Stark Effect, *Alexandria Engineering Journal*, *104* (2024), Oct., pp. 378-385
[51] Barndorff-Nielsen, *et al*., On Quantum Statistical Inference, *J. R. Stat. Soc. B*, 65 (2003), 4, pp. 775-816

Paper submitted: June 20, 2024
Paper revised: October 3, 2024
Paper accepted: October 29, 2024