GENERALIZED HYPERBOLIC FUNCTION SPACE OF MIXED NORMS WITH SOME APPLICATIONS

by

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This paper presents new definitions for Bloch spaces, hyperbolic derivatives, and general hyperbolic Besov spaces. Furthermore, we present a fresh demonstration of the hyperbolic function through the application of Holder inequality. Through the application of Holder inequality. We also provide attributes for functions within the declared classes in the unit disc. Furthermore, a collection of weighted tent functions is studied, and features of identity operators are investigated for the new tent function spaces.

Key words*: Bloch spaces, hyperbolic Besov-space, mixed norms*

Introduction

Let $\mathbb{D} = \{z \in \mathbb{C}:|z| < 1\}$be the unit disc in the complex plane $\mathbb{C}$, $\partial\mathbb{D}$ it`s boundary. Let $\mathcal{H}(\mathbb{D})$ to denote the space of all hiperbolic function (HF) in $\mathbb{D}$ and let B^*_ω be a subset of $\mathcal{H}(\mathbb{D})$ consisting of these $f \in \mathcal{H}(\mathbb{D})$ for which $|f^*(z)| < 1$ for all $z \in \mathbb{D}$. Also, $\mathrm{d}A(z)$ be the hyperbolic area measure on $\mathbb{D}$ such that $A(\mathbb{D}) \equiv 1$. The usual $\propto$– Bolch spaces $B^*_\propto$ is defined as the set of those $f \in \mathcal{H}(\mathbb{D})$ for which:

$$B^*_\omega = \left\{ f : f \text{ hyperbolic in } \mathbb{D} \text{ and } \sup_{z\in\mathbb{D}} \left(1-|z|^2\right) f^*(z) < \infty \right\},\ \ 0 < \propto < \infty \tag{1}$$

and

$$\lim_{|z|\to 1} \left| f^*(z) \right| \left(1-|z|^2\right)^{\propto} = 0$$

For more information on Bloch-type classes in C, we might turn to [1-6].

The well-known hyperbolic derivatives (HD) is defined:

$$f^*(z) = \frac{\left|f'(z)\right|}{1-\left|f(z)\right|^2}$$

of $f \in \mathcal{H}(\mathbb{D})$ and the hyperbolic distance is given:

$$\rho\left(f(z),0\right) = \frac{1}{2}\log\left(\frac{1+\left|f(z)\right|}{1-\left|f(z)\right|}\right) \tag{2}$$

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Let $2 < p < \infty$, f be HF in $\mathcal{H}(\mathbb{D})$ we define

$$f_p^*(z) = \frac{|f(z)|^{\left(\frac{p}{2}-1\right)}|f'(z)|}{1-|f(z)|^p}$$

So, we acquired that $f_2^*(z) = f^*(z)$.

For more details on Bloch spaces of HF, see references [6, 7]. A class of HF are called Q_p^* – spaces given by [8], where $0 < p < \infty$

$$Q_p^* = \left\{ f : f \text{ hyperbolic in } \mathbb{D} \text{ and } \sup_{a\in\mathbb{D}} \int \left| f^*(z) \right|^2 g^p(z,a) \mathrm{d}\sigma_z < \infty \right\}$$

Analytic Q_p^* – spaces are introduced by Aulaskari and Lappan [12].

Preliminary results

Definition

The $(p, \propto)$ – Bolch spaces $B^*_{P,\omega}$ and $B^*_{P,\omega,O}$ are defined as the set of those $f \in \mathcal{H}(\mathbb{D})$:

$$\|f\|_{B^*_{p,\propto}} = \frac{p}{2} \sup_{z\in\mathbb{D}} |f(z)|^{\left(\frac{p}{2}-1\right)} \left|f^*(z)\right| \left(1-|z|^2\right)^{\propto} < \infty \tag{3}$$

and

$$\lim_{|z|\to 1} |f(z)|^{\left(\frac{p}{2}-1\right)} \left|f^*(z)\right| \left(1-|z|^2\right)^{\alpha} = 0$$

where $2 < p < 4$ and $0 < \propto < \infty$.

We define the HD by $f \in (\mathbb{D})$:

$$f_p^* = \frac{p}{2} \frac{|f(z)|^{\left(\frac{p}{2}-1\right)} f^*(z)}{\left(1-|f(z)|^p\right)} \tag{4}$$

when $p = 2$ we obtain usual HD as defined A function, $f \in B(\mathbb{D})$ is said to belong to the generalized hyperbolic $(p, \propto)$ – Bolch class $B^*_{p,\alpha}$ if:

$$\|f\|_{B^*_{P,\omega,\alpha}} = \sup_{z\in\mathbb{D}} \frac{f_p^*(z)\left(1-|z|^2\right)^{\alpha}}{\omega^2\left(1-|z|\right)} < \infty \tag{5}$$

The little generalized $(p, \propto, \omega)$ – hyperbolic Bloch type classes $B^*_{p,\alpha,\omega}$ consist of all $f \in B^*_{p,\alpha,\omega}$ such as:

$$\lim_{|z|\to 1} \frac{f_p^*(z)\left(1-|z|^2\right)^{\alpha}}{\omega^2\left(1-|z|\right)} = 0 \tag{6}$$

Let the Green's function of $\mathbb{D}$ be defined:

$$g(z,a) = \log \frac{1}{|\varphi_a(z)|}$$

where

$$\varphi_a(z) = \frac{a - z}{1 - \tilde{a}z}$$

Is the Mobus transformation related to the point $a \in \mathbb{D}$. For $2 < p < 4$ and $0 < \propto < \infty$, the hyperbolic class:

$$\left\| f^*_{(p,\omega)} \right\|^q = \int_{\mathbb{D}} \frac{\left| f^*_{p,\omega}(z) \right|^q \left(1 - |z|^2\right)^q}{\omega^q \left(1 - |z|\right)} \tag{7}$$

Class $Q^*_{(p,\alpha)}$ consist of those function $f \in B(\mathbb{D})$ this is defined:

$$\|f\|^p_{Q^*_{(p,\alpha)}} = \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} \left(f^*(z)\right)^2 g^\alpha(z,a) \mathrm{d}A(z) < \infty \tag{8}$$

Moreover, we say that $f \in Q^*_{(p,\alpha)}$ belong to the classes $Q^*_{(p,\omega,\alpha)}$ if:

$$\lim_{|a| \to 1} \left(f^*(z)\right)^2 g^\alpha(z,a) \mathrm{d}A(z) = 0$$

when $p = 2$ we obtain the hyperbolic Q classes as studied in [1, 9, 10].

Definition

From [7, 11], Let ω:(0, 1] → (0, ∞) be a given reasonable function. Suppose that $0 < q < 2$ and $q < p \ll 2$ a HF f on $\mathbb{D}$ is said to belong to the $B^*_{p,\omega,\alpha}$ spaces if:

$$\|f\|^q_{B^*_{p,q,s;\omega}} = \sup_{\alpha \in \mathbb{D}} \int_{\mathbb{D}} \left| f^*_p(z) \right|^q \frac{\left(1 - \left|\varphi_a(z)\right|^2\right)^s}{\omega^q \left(1 - |z|\right)} \mathrm{d}A(z) < \infty \tag{9}$$

We need the relation eq. (10) in the proofs:

$$\left(1 - \left|\varphi_a(z)\right|^2\right) = \frac{\left(1 - |a|^2\right)\left(1 - |z|^2\right)}{\left|1 - \hat{a}z\right|^2} \tag{10}$$

where

$$1 - |z| \le \left|1 - \hat{a}z\right| \le 1 + |z| \text{ and also } 1 - |a| \le \left|1 - \hat{a}z\right| \le 1 + |a|$$

Remark

The Schwarz-pick lemma implies that $B^*_{P,\alpha} = B(\mathbb{D})$ for all $\propto \ge 1$ with $\|f\|_{B^*_{P,\alpha}} \le 1$ and therefore, the generalized hyperbolic $(p, \propto)$ – class are of interest only, when $0 < \propto < 1$.

Also, we define the classes of general hyperbolic Besov-space $B^*_{p,q,\omega}$ and $f \in B(\mathbb{D})$ for which:

$$\|f\|_{B^*_{(p,q,\omega)}} = \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} \left| f^*_p(z) \right|^q \frac{\left(1 - |z|^2\right)^{(q-p)} \left(1 - \left|\varphi_a(z)\right|^2\right)^p}{\omega^q \left(1 - |z|\right)} \mathrm{d}A(z) < \infty \tag{11}$$

when $2 < p < 4$ and $p < q < \infty$.

Preposition

Preposition

Let f be HF belong to $B^*_{(p,\,\omega)}$, then we have:

$$\int_{\mathbb{D}} \left| f_p^*(z) \right| \frac{\left(1-|z|^2\right)^{(q-p)} \left(1-\left|\varphi_a(z)\right|^2\right)^p}{\omega^q \left(1-|z|\right)} \mathrm{d}A(Z) < 2^{(P+1)} \pi \left\| f_p^*(z) \right\|_{B^*_{(P,\omega)}}$$

Proof [12]:

$$I = \int_{\mathbb{D}} \left| f_p^*(z) \right|^q \frac{\left(1-|z|^2\right)^{(q-p)} \left(1-\left|\varphi_a(z)\right|^2\right)^p}{\omega^q \left(1-|z|\right)} \mathrm{d}A(Z) =$$

$$= \int_{\mathbb{D}} \frac{\left| f_p^*(z) \right|^q}{\omega^q \left(1-|z|\right)} \frac{\left(1-\left|\varphi_a(z)\right|^2\right)^p}{\left(1-|z|^2\right)^p} \mathrm{d}A(z) \le \left\| f^*_{(p,\omega)}(z) \right\|^q \int_{\mathbb{D}} \frac{\left(1-\left|\varphi_a(z)\right|^2\right)^p}{\left(1-|z|^2\right)^p} \mathrm{d}A(z) \ \therefore$$

$$\therefore \left(1-\left|\varphi_a(z)\right|^2\right)^p = \frac{\left(1-|z|^2\right)^p \left(1-|a|^2\right)^p}{\left|1-\tilde{a}z\right|^{2p}} \ \therefore \tag{12}$$

$$\therefore \ \mathrm{I} \ \le \ \left\| f^*_{(p,\omega)}(z) \right\|^q \int_{\mathbb{D}} \frac{\left(1-|a|^2\right)^p}{\left|1-\tilde{a}z\right|^{2p}} \mathrm{d}A(z) =$$

$$= \left(1-|a|^2\right)^p \left\| f^*_{(p,\omega)}(z) \right\|^q \int_{\mathbb{D}} \frac{1}{\left|1-\tilde{a}z\right|^{2p}} \mathrm{d}A(z) =$$

$$= \left\| f^*_{(p,\omega)}(z) \right\|^q \left(1-|a|\right)^p \left(1+|a|\right)^p \int_0^{2\pi} \frac{1}{\left|1-\hat{a}z\right|^{2p}} \mathrm{d}\Gamma r \mathrm{d}r \mathrm{d}\theta$$

As

$$\int_{\partial\mathbb{D}} \frac{1}{\left|1-\hat{a}z\right|^{2p}} \, \mathrm{d}\Gamma \le \frac{1}{\left|1-\hat{a}z\right|^p}, \ 2 \le p < \infty$$

$$\therefore 1+|a| < 2$$

$$\therefore \mathrm{I} \le 2^P \left\| f^*_{(p,\omega)}(z) \right\|^q 2\pi = 2^{(P+1)} \pi \left\| f^*_{(p,\omega)}(z) \right\|^q$$

$$\mathrm{I} \le 2^{(P+1)} \pi \left\| f^*_{(p,\omega)}(z) \right\|^q$$

Corollary

From proposition 1.4, we get for $0 \le p < \infty$ and $0 < q < \infty$ that:

$$B^*_{p,\omega} \subset B^*_{(p,q,\,\omega)}.$$

Theorem

Let $0 < q < \infty$ and $2 < p_1 < 4 - p$ we have that:

$$\cup_{p_1} Q^*_{(p_1,\omega)} \subset \cap_{(p,q)} B^*_{(p,q,\omega)}$$

Proof: Let $f \in Q^*_{p_1,\omega}$ for fixed $2 < p_1 < 4 - q$ and $0 < q < 2$. Then by using Holder in eq. (1) we acquire that:

$$I = \int_{\mathbb{D}} \left| f^*_{(p,\omega)}(z) \right|^q \left(1-|z|^2\right)^{(q-p)} \frac{\left(1-|\varphi_a(z)|^2\right)^P}{\omega^q\left(1-|z|\right)} \mathrm{d}A(z) \le \left\{ \int_{\mathbb{D}} \left[\frac{\left| f^*_{(p,\omega)}(z) \right|^q}{\omega^q\left(1-|z|\right)} \left(1-|\varphi_a(z)|^2\right)^{\frac{qp_1}{2}} \right]^{2/q} \mathrm{d}A(z) \right\}^{q/2} \times$$

$$\times \left\{ \int_{\mathbb{D}} \left[\left(1-|z|^2\right)^{(q-p)} \left(1-|\varphi_a(z)|^2\right)^{\left(p-\frac{qp_1}{2}\right)} \right]^{\left(\frac{2}{2-q}\right)} \mathrm{d}A(z) \right\}^{\left(\frac{2-q}{2}\right)} =$$

$$= \left\| f^*(z) \right\|^2_{B^*_{(p,\omega)}} \int_{\mathbb{D}} \left[\left(1-|\varphi_a(z)|^2\right)^{p_1} \mathrm{d}A(z) \right]^{(2/q)} \times \left\{ \int_{\mathbb{D}} \left(1-|z|^2\right)^{\left(\frac{2(q-p)}{2-q}\right)} \left(1-|\varphi_a(z)|\right)^{\left(\frac{2p-qp_1}{2-q}\right)} \mathrm{d}A(z) \right\}^{\left(\frac{2-q}{2}\right)} =$$

$$= \left\| f^*(z) \right\|^2_{B^*_{(p,\omega)}} \int_{\mathbb{D}} \left[\left(\frac{\left(1-|a|^2\right)\left(1-|z|\right)}{\left|1-\tilde{a}z\right|^2} \right)^{\left(\frac{p_1 q}{2}\right)} \mathrm{d}A(z) \right]^{(2/q)} \times \tag{13}$$

$$\times \int_{\mathbb{D}} \left\{ \left(1-|a|^2\right)^{2\left(\frac{q-p}{2-q}+1\right)} \left(1-|z|^2\right)^{2\left(\frac{q-p}{2-q}\right)} \mathrm{d}A(z) \right\}^{\left(\frac{2-q}{2}\right)} = \left\| f^*(z) \right\|^2_{B^*_{(p,\omega)}} 2^p \left(\int_0^{2\pi} \frac{1}{\left|1-\tilde{a}z\right|} \mathrm{d}\Gamma r \mathrm{d}r \mathrm{d}\theta \right)^{(q/2)} \times$$

$$\times \int_{\mathbb{D}} \left\{ \left(1-|a|^2\right)^{\left(\frac{4-2p}{2-q}\right)} \sum_{n=0}^{\infty} \frac{\Gamma\left(n+\dfrac{4-2p}{2-p}\right)}{n!\,\Gamma\left(\dfrac{4-2p}{2-q}\right)} |a|^{2n} \right\}^{\left(\frac{2-q}{2}\right)} = \left\| f^*(z) \right\|^2_{B^*_{(p,\omega)}} 2^{\left(p+\frac{1}{2}\right)} (\pi)^{\left(\frac{q}{2}\right)} \left\{ \frac{2-q}{q-2p+1} \right\}^{\left(\frac{2-q}{2}\right)}$$

Remark

Some authors discussed the investigation of particular differential equations that evolve into specific analytic function spaces, see [10, 12]. We can formulate the following question in terms of the defined generic hyperbolic classes: How can we use certain hyperbolic types to solve certain differential equations?

Conclusion

We use a variety of tools to investigate this topic. Several scholars have investigated HF, see [13-16], among others. This research focuses on certain assumptions about specific classes of weighted HF spaces when they are applied to a general HD. In order to achieve this goal, we present intriguing definitions and introduce a novel class of uniquely private general HF. These functions generate hyperbolic-type functions that release the general derivative; hence, decreasing scales and providing obvious benefits. We also suggested a chordal metric.

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References

[1] Levine, N., Semi-Open Sets and Semi-Continuity in Topological Spaces, *Amer. Math. Monthly*, *70* (1963), 1, pp. 36-41
[2] Namiq, S. F., The λ∗ − R0 and λ∗ − R1 spaces, *Journal of Garmyan University*, *4* (2014)
[3] Ogata, H., Operations on Topological Spaces and Associated Topology, *Math. Japon*., *36* (1991), Sept., pp. 175-184
[4] Siskakis, A. G., Zhao, R., *A Volterra Type Operator on Spaces of Analytic Functions, Function Spaces* (Edwardsville IL, 1998), 299-311, Contemp. Math. 232, Amer. Math. Soc. Providerce, R.I., USA, 1999
[5] Xiao, J., Riemann-Stieltjes Operators on Weighted Bloch and Bergman Spacesof the Unit Ball, *J. London. Math. Soc*., *70* (2004), 199214
[6] Xiao, J., Geometric Qp functions, in: *Frontiers in Mathematics*, Birkhäuser Verlarg Publisher, Basel, Switzerland, 2006
[7] Rashwan, A., *et al*., Integral Characterizations of Weighted Bloch Spaces and QK, ω (p, q) Spaces, *Mathematica Cluj*, *51* (2009), 1, pp. 63-76
[8] Zhao, R., On a General Family of Function Spaces, in: *Annales Academiae Scientiarum Fennicae*, Series A I, Mathematica, Dissertationes, 105, Suomalainen, Helsinki, Finland, 1996
[9] Oda, N., Nakaoka, F., Some Applications of Minimal Open Sets, *Int. J. Math. Math. Sci*., *27* (2001), 8, pp. 471-476
[10] Santhi, P., *et al.*, Contra Quotient Functions on Generalized Topological Spaces, *Fasciculi Mathematica,* (2014), 53
[11] Rashwan, R. A, *et al.*, Some Characterizations of Weighted Holomorphic Bloch Space, *European Journal of Pure and Applied Mathematics*, *2* (2009), 2, pp. 250-267
[12] Glaisa, T. C., *et al.*, On β-Open Sets and Ideals in Topological Spaces, *European Journal of Pure and Applied Mathematics*, *3* (2019), 12, pp. 893-905
[13] Namiq, S. F., The λ_{sc}-Open Sets and Topological Properties, *Journal of Garmyan University*, 2014
[14] Rosas, E., Namiq, S. F., On Minimal λrc-Open Sets, *Italian Journal of Pure and Applied Mathematics*, *43* (2020), Feb., pp. 868-877
[15] Darwesh, H. M., *et al*., Maximal λc-Open Sets, *Proceedings*, 1st Int. Cont. of Natural Science, Chamchamal, Iraq, ICNS-2016, 2016
[16] Ekici, E., *et al.*, Weakley λ-Continuous Functions, *Novi Sad J. Math*, *2* (2008), 38, pp. 47-56

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