STATISTICAL PROPERTIES AND APPLICATIONS FOR EXPONENTIATED EXPONENTIAL RAYLEIGH DISTRIBUTION

by

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Modelling in lifetime phenomena is a significant issue in many scientific areas. Sometimes many standard models lack superiority in modelling data set. Designing a new form of probability distribution by using different techniques has received a widely attention in statistical theory in the recent years. In this paper, we formulated exponentiated exponential Rayleigh (EER) distribution. We discussed the reliability and hazard rate functions of the EER distribution and computed the quantile, median, skewness and kurtosis. Moreover, the correlations between the parameters and the median, skewness, and kurtosis are investigated. Bayesian and non-Bayesian approaches are adopted to estimate the unknown parameters of EER distribution. In Bayesian approach, we are used Markov Chen Monte Carlo (MCMC) method to obtain the approximate Bayes point estimate. The proposed distribution is used to analyze, light-emitting diodes data, strength of glass fibers data and Wheaton River data. The estimation results of the EER distribution are compared with exponential Rayleigh, Rayleigh, and exponential distributions.

Key words: *exponentiated and T-X family of distributions, Rayleigh distribution, MCMC technique, Bayes estimation, maximum likelihood estimation*

Introduction

Probability distributions have been used for a long time in various fields of life. One of the interesting points of the statistics is looking for modelling the phenomenon by statistical distributions in many applications in different sciences fields, such as medicine, astronomy, physical, economics and finance. There are various amount of real-life phenomenons can be modelling by different classical probability distributions. But, the underlying trend of the data in different situations may be fail to capture by using standard probability distributions. Therefore, we search of generalizations or extensions to extend any baseline distributions, named as exponentiated family, developed by Gupta *et al.* [1], by adding new shape parameter to existing distribution. This method deals to appear new acceptable and flexible models. This model has the cumulative distribution function (CDF) define:

$$F(x) = \left[H(x)\right]^{\theta}, \ x \in \mathbb{R} \tag{1}$$

where $\theta > 0$ presented the shape parameter and the baseline CDF define by $H(x)$. Different authors discussed the exponentiated extension of several baseline distributions. The exponentiated exponential distribution was introduced by Gupta and Kundu [2], Gamma exponentiated and Gumbel, Weibull distributions were suggested by Nadarajah and Kotz [3] and exponentiated modified Weibull distribution was introduced by Sarhan and Apaloo [4].

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The (*T-X*) family was proposed by Alzaatreh *et al*. [5] which has become progressively used due to its flexibility as any continuous distribution can be choosen as a generator.

The *T-X* family has the CDF given:

$$Z_{T\text{-}X}(x)=\int_{a}^{W[G(x)]} r(t)\,\mathrm{d}t \tag{2}$$

where the function $r(t)$ of the random variable T is a probability density function (PDF) and the function $W[G(x)]$ is differentiable monotonically non-decreasing function of $G(x)$ in $[a, b]$. Also, as $x\to-\infty$ the function $W[G(x)]\to a$ and when $x\to+\infty$ the function $W[G(x)]\to b$.

As given in [5] the *T-X* family of distributions was used to extended the transmuted family of distributions. Different transformed families of distributions such as, transmuted-*G*, Kumaraswamy transmuted-G, beta transmuted-*H* and *T*-transmuted *X* were discussed by Nofal *et al.* [6], Afify *et al.* [7], Afify *et al.* [8], and Jayakumar and Girish Babu [9], respectively.

A well-known Rayleigh distribution proposed by Rayleigh [10], it has significant applications for modelling lifetime data in reliability, medical sciences and engineering, see [11, 12]. The Rayleigh CDF and PDF with scale parameter λ given, respectively:

$$G(x)=1-\mathrm{e}^{-\left(\frac{x}{\lambda}\right)^2},\ x\ge 0 \tag{3}$$

and

$$g(x)=\frac{2}{\lambda^2}x\,\mathrm{e}^{-\left(\frac{x}{\lambda}\right)^2},\ x\ge 0 \tag{4}$$

Our study aims to present generalization for exponential Rayleigh distribution as a member of exponentiated family known by the EER distribution. Also, studying the properties and estimating the parameters of the proposed EER distribution. Finally, how the proposed distribution use to analysis real applicable data.

The exponentiated exponential Rayleigh distribution

To obtain the CDF of the exponential Rayleigh distribution we first use the exponential distribution with a random variable T in eq. (2) as:

$$Z_{T\text{-}X}(x)=\int_{0}^{-\log[1-G(x)]} c\,\mathrm{e}^{-ct}\,\mathrm{d}t \tag{5}$$

Now, use the CDF of Rayleigh distribution eq. (3) in eq. (5), then the the exponential Rayleigh distribution is formulated:

$$F(x,\lambda,c)=1-\mathrm{e}^{-c\left(\frac{x}{\lambda}\right)^2},\ x\ge 0 \tag{6}$$

The CDF of the *EER* distribution is obtained by using eq. (6) in eq. (1) and is expressed as the form:

$$F(x,\lambda,c,\theta)=\left[1-\mathrm{e}^{-c\left(\frac{x}{\lambda}\right)^2}\right]^{\theta},\ x\ge 0 \tag{7}$$

where ($\theta>0$) is defined as shape parameter and (λ, $c>0$) are scale parameters.

The PDF corresponding to eq. (7):

$$f(x,\lambda,c,\theta)=\frac{2c\theta}{\lambda^2}x\mathrm{e}^{-c\left(\frac{x}{\lambda}\right)^2}\left[1-\mathrm{e}^{-c\left(\frac{x}{\lambda}\right)^2}\right]^{\theta-1}, \quad x\geq 0 \tag{8}$$

Remark. Special cases of the EER distribution are as follows:
- The EER distribution is reduced to Rayleigh distribution when $\theta = c = 1$.
- The EER distribution is reduced to exponentiated Rayleigh distribution when $c = 1$.

The behaviour of the PDF curves for different values of parameters of the EER distribution are provided in fig. 1.

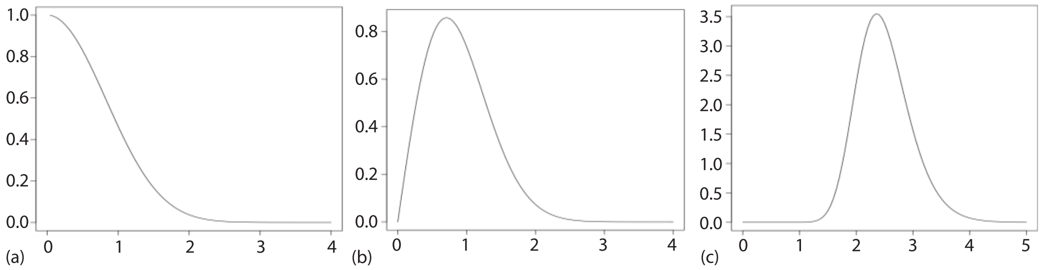

Figure 1. The PDF plot of EER distribution for different choose of parameters; (a) λ = 1, c = 1, θ = 0.5, (b) λ = 1, c = 1, θ = 1, and (c) λ = 0.5, c = 0.5, θ = 13.5

Properties of exponentiated exponential Rayleigh distribution

In this section, we discuss the important properties of the EER distribution. These properties include the reliability and hazard rate functions, quantile and median, kutosis, and skewness.

Reliability and hazard rate functions

In literature, the ability of population units to survive for some time is denoted by the reliability function, see [13, 14]. The reliability function for non-negative random variable X is formulated:

$$R(x)=P(X>x)=1-F(x)$$

Now using eq. (7) in previous equation, the reliability function for the EER distribution:

$$R(x)=1-\left[1-\mathrm{e}^{-c\left(\frac{x}{\lambda}\right)^2}\right]^{\theta} \tag{9}$$

which can be obtained for different choices of parameters.

In the field of reliability analysis, hazard rate function plays essential role, and it also knowns as the force of mortality function. It gives at any given time a description of the instantaneous rate of failure and it is defined by, see [15]:

$$h(x)=\frac{f(x)}{R(x)}$$

The hazard rate function for the EER distribution is obtained by using eq. (7) and eq. (9) in previous equation and is given:

$$h(x)=\frac{\dfrac{2c\theta}{\lambda^2}x\,\mathrm{e}^{-c\left(\frac{x}{\lambda}\right)^2}\left[1-\mathrm{e}^{-c\left(\frac{x}{\lambda}\right)^2}\right]^{\theta-1}}{1-\left[1-\mathrm{e}^{-c\left(\frac{x}{\lambda}\right)^2}\right]^{\theta}} \tag{10}$$

Quantile and median

The quantile function plays a vital role in many applications, such as engineering, economics and finance. It is obtained by inverse of the CDF:

$$Q(u)=F^{-1}(u)$$

where u is the random variable from uniform distribution $U(0, 1)$. Using the CDF of the EER distribution given in eq. (6) in previous equation, then the quantile function can be written:

$$Q(u)=\lambda\left[\log\frac{1}{\left(1-u^{1/\theta}\right)^{1/c}}\right]^{1/2} \tag{11}$$

The values of Q_1, Q_2 (median), and Q_3 can be obtained for u = 0.25, u = 0.5, and u = 0.75, repectively, for more details see Gilchrist [16]. The median of the EER distribution:

$$Q(0.5)=\lambda\left[\log\frac{1}{\left(1-(0.5)^{1/\theta}\right)^{1/c}}\right]^{1/2}$$

Skewness and Kurtosis

The quantile function can be used to study the effects of the shape parameters on the skewness and kurtosis. As given in Kenny and Keeping [17], the Bowley skewness is the earliest skewness measures which is defined:

$$SK=\frac{Q\left(\frac{3}{4}\right)-2Q\left(\frac{1}{2}\right)+Q\left(\frac{1}{4}\right)}{Q\left(\frac{3}{4}\right)-Q\left(\frac{1}{4}\right)}$$

The alternative measure to obtain kurtosis of the distribution is proposed by Moors [18], which is known as Moors kurtosis and defined:

$$KU=\frac{Q\left(\frac{7}{8}\right)-Q\left(\frac{5}{8}\right)+Q\left(\frac{3}{8}\right)-Q\left(\frac{1}{8}\right)}{Q\left(\frac{6}{8}\right)-Q\left(\frac{2}{8}\right)}$$

where $Q(.)$ is the quantile function, given in eq. (11), the skewness and kurtosis of the EER distribution:

$$SK=\frac{\lambda\left[\log\frac{1}{\left(1-(0.75)^{\frac{1}{\theta}}\right)^{1/c}}\right]^{1/2}-2\lambda\left[\log\frac{1}{\left(1-(0.5)^{\frac{1}{\theta}}\right)^{1/c}}\right]^{1/2}+\lambda\left[\log\frac{1}{\left(1-(0.25)^{\frac{1}{\theta}}\right)^{1/c}}\right]^{1/2}}{\lambda\left[\log\frac{1}{\left(1-(0.75)^{1/\theta}\right)^{1/c}}\right]^{1/c}-\lambda\left[\log\frac{1}{\left(1-(0.25)^{1/\theta}\right)^{1/c}}\right]^{1/2}}$$

and

$$KU=\frac{\lambda\left[\log\frac{1}{\left(1-\left(\frac{7}{8}\right)^{1/\theta}\right)^{1/c}}\right]^{1/2}-\lambda\left[\log\frac{1}{\left(1-\left(\frac{5}{8}\right)^{1/\theta}\right)^{1/c}}\right]^{1/2}+\lambda\left[\log\frac{1}{\left(1-\left(\frac{3}{8}\right)^{1/\theta}\right)^{1/c}}\right]^{1/2}-\lambda\left[\log\frac{1}{\left(1-\left(\frac{1}{8}\right)^{1/\theta}\right)^{1/c}}\right]^{1/2}}{\lambda\left[\log\frac{1}{\left(1-\left(\frac{6}{8}\right)^{1/\theta}\right)^{1/c}}\right]^{1/2}-\lambda\left[\log\frac{1}{\left(1-\left(\frac{2}{8}\right)^{1/\theta}\right)^{1/c}}\right]^{1/2}}$$

The tab. 1 has shown the values of median, skewness and kurtosis of the EER distribution for different choose of the parameters λ, θ, and c. We can see that the median, skewness and kurtosis increase as θ increases.

Table 1. Median, skewness and kurtosis of the EER distribution

λ	θ	c	Median	Skewness	Kurtosis
2	4.5	0.5	3.94628	0.05744	1.23934
2	8.5	0.5	4.51405	0.06179	1.24471
2	13.5	0.5	4.89470	0.06571	1.24746
2	20.5	0.5	5.21827	0.06932	1.24946

Parameters estimation

In this section, we formulate the parameters estimators of EER distribution under maximum likelihood (ML) method. Also, Bayes estimators of the unknown parameters are formulated with respected to squared error loss function.

Maximum likelihood estimation

Let $X = \{X_1, X_2, ..., X_n\}$ be a random sample of size n from EER distribution. The joint likelihood function of observed sample $x = \{x_1, x_2, ..., x_n\}$ with the density function from eq. (8) is given:

$$L=\left(\frac{2c\theta}{\lambda^2}\right)^n \mathrm{e}^{-c\sum_{i=1}^{n}\left(\frac{x_i}{\lambda}\right)^2}\prod_{i=1}^{n}x_i\left[1-\mathrm{e}^{-c\left(\frac{x_i}{\lambda}\right)^2}\right]^{\theta-1} \tag{12}$$

The log-likelihood function:

$$\ell = n\left(\log[c]+\log[\theta]-2\log[\lambda]\right)+\sum_{i=1}^{n}\left[\log(x_i)-c\left(\frac{x_i}{\lambda}\right)^2\right]+(\theta-1)\sum_{i=1}^{n}\log\left[1-\mathrm{e}^{-c\left(\frac{x_i}{\lambda}\right)^2}\right] \tag{13}$$

The estimate values of parameters vector Θ = (λ, c, θ) are computed under maximization of the log-likelihood function (13). Likelihod equations are obtained by the first partially derivatives with respect to unknown parameters and equating to zero value:

$$\frac{\partial\ell}{\partial\lambda}=-\frac{2n}{\lambda}+\frac{2c}{\lambda}\sum_{i=1}^{n}\left(\frac{x_i}{\lambda}\right)^2-\frac{2c(\theta-1)}{\lambda}\sum_{i=1}^{n}\frac{\left(\frac{x_i}{\lambda}\right)^2\mathrm{e}^{-c\left(\frac{x_i}{\lambda}\right)^2}}{1-\mathrm{e}^{-c\left(\frac{x_i}{\lambda}\right)^2}}=0 \tag{14}$$

$$\frac{\partial\ell}{\partial c}=\frac{n}{c}-\sum_{i=1}^{n}\left(\frac{x_i}{\lambda}\right)^2+(\theta-1)\sum_{i=1}^{n}\frac{\left(\frac{x_i}{\lambda}\right)^2\mathrm{e}^{-c\left(\frac{x_i}{\lambda}\right)^2}}{1-\mathrm{e}^{-c\left(\frac{x_i}{\lambda}\right)^2}}=0 \tag{15}$$

and

$$\frac{\partial\ell}{\partial\theta}=\frac{2}{\theta}+\sum_{i=1}^{n}\log\left[1-\mathrm{e}^{-c\left(\frac{x_i}{\lambda}\right)^2}\right]=0 \tag{16}$$

The likelihood equations are reduced:

$$\theta=\frac{-2}{\sum_{i=1}^{n}\log\left[1-\mathrm{e}^{-c\left(\frac{x_i}{\lambda}\right)^2}\right]} \tag{17}$$

$$-\frac{2n}{\lambda}+\frac{2c}{\lambda}\sum_{i=1}^{n}\left(\frac{x_i}{\lambda}\right)^2-2c\left(\frac{-2}{\lambda\sum_{i=}^{n}\log\left[1-\mathrm{e}^{-c\left(\frac{x_i}{\lambda}\right)^2}\right]}-\frac{1}{\lambda}\right)\sum_{i=1}^{n}\frac{\left(\frac{x_i}{\lambda}\right)^2\mathrm{e}^{-c\left(\frac{x_i}{\lambda}\right)^2}}{1-\mathrm{e}^{-c\left(\frac{x_i}{\lambda}\right)^2}}=0 \tag{18}$$

and

$$\frac{n}{c}-\sum_{i=1}^{n}\left(\frac{x_i}{\lambda}\right)^2+\left(\frac{-2}{\sum_{i=1}^{n}\log\left[1-\mathrm{e}^{-c\left(\frac{x_i}{\lambda}\right)^2}\right]}-1\right)\sum_{i=1}^{n}\frac{\left(\frac{x_i}{\lambda}\right)^2\mathrm{e}^{-c\left(\frac{x_i}{\lambda}\right)^2}}{1-\mathrm{e}^{-c\left(\frac{x_i}{\lambda}\right)^2}}=0 \tag{19}$$

Likelihood equations are reduced to two non-linear eqs (18) and (19) of the parameters λ and c can be solved by Newton Rahson or fixed-point iterations to obtain $\hat{\lambda}$ and $\hat{c}$. Also, the ML estimate of parameter θ is obtained by replacing λ and c by $\hat{\lambda}$ and $\hat{c}$ in eq. (17).

Bayesian estimation

Under Bayesian approach, the parameters are considered as a random variable. Hence, to estimate of the parameters vector $\Theta = (\lambda, c, \theta)$ need to formulate the joint distribution which is known as posterior distribution. The past experiences of the parameters are formulated in the form of prior distribution. Suppose that, the prior information about the parameters are independent gamma prior distributions:

$$\Theta_k \propto \Theta_k^{a_k-1} \mathrm{e}^{-b_k\Theta_k}, \quad a_k, b_k > 0, \quad k = 1, 2, 3 \tag{20}$$

The joint prior distribution of parameters vector $\Theta = (\lambda, c, \theta)$ is given:

$$\pi^*\left(\lambda, c, \theta\right) \propto \lambda^{a_1-1} c^{a_2-1} \theta^{a_3-1} \mathrm{e}^{-\left(b_1\lambda + b_2 c + b_3\theta\right)} \tag{21}$$

The joint posterior distribution can be formulated from eqs. (12) and (21):

$$\pi\left(\lambda, c, \theta \mid x\right) \propto \lambda^{a_1-2n-1} c^{n+a_2-1} \theta^{n+a_3-1} \, \mathrm{e}^{-\left(b_1\lambda + b_2 c + b_3\theta\right) - c\sum_{i=1}^{n}\left(\frac{x_i}{\lambda}\right)^2 + (\theta-1)\sum_{i=1}^{n}\log\left[1-\mathrm{e}^{-c\left(\frac{x_i}{\lambda}\right)^2}\right]} \tag{22}$$

The posterior distribution eq. (22) show that, the closed form of posterior distribution or posterior estimate under squared error loss function need to 3-D integral which is more complicated. Therefore, we adopted the important numerical method known by MCMC method as follows.

The posterior full conditional distributions can be obtained from eq. (22):

$$\pi\left(\lambda \mid c, \theta, x\right) \propto \lambda^{a_1-2n-1} \, \mathrm{e}^{-b_1\lambda - c\sum_{i=1}^{n}\left(\frac{x_i}{\lambda}\right)^2 + (\theta-1)\sum_{i=1}^{n}\log\left[1-\mathrm{e}^{-c\left(\frac{x_i}{\lambda}\right)^2}\right]} \tag{23}$$

$$\pi\left(c \middle| \lambda, \theta, x\right) \propto c^{n+a_2-1} \, \mathrm{e}^{-b_2 c - c\sum_{i=1}^{n}\left(\frac{x_i}{\lambda}\right)^2 + (\theta-1)\sum_{i=1}^{n}\log\left[1-\mathrm{e}^{-c\left(\frac{x_i}{\lambda}\right)^2}\right]} \tag{24}$$

and

$$\pi\left(\theta \middle| \lambda, c, x\right) \propto \theta^{n+a_3-1} \, \mathrm{e}^{-b_3\theta + \theta\sum_{i=1}^{n}\log\left[1-\mathrm{e}^{-c\left(\frac{x_i}{\lambda}\right)^2}\right]} \tag{25}$$

The posterior distribution reduced to gamma distribution function eq. (25) and two function general functions similar to normal distribution. The empirical posterior distribution and the corresponding parameters estimate are built as *Algorithm 1*.

Algorithm 1. (Bayes estimate under MCMC method):

- *Step 1*. Begin with initial gausses values

$$\Theta^{(0)} = \left(\lambda^{(0)}, c^{(0)}, \theta^{(0)}\right) = \left(\hat{\lambda}, \hat{c}, \hat{\theta}\right) \quad \text{and} \quad \kappa = 0$$

- *Step 2*. Put the indicator $\kappa = \kappa + 1$.
- *Step 3*. Generate $\theta^{(\kappa)}$ from gamma distributions eq. (25).
- *Step 4*. Generate $\lambda^{(\kappa)}$ and $c^{(\kappa)}$ from eqs. (23) and (24) with normal proposal distributions with Metropolis-Hastings (MH) algorithm.

– *Step 5*. Repeat Steps from 2-4 MC times.
– Step 6. The Bayes estimate of the parameters vector computed from:

$$\hat{\Theta}_k = \frac{1}{\left(MC - MC^*\right)} \sum_{i=MC^*+1}^{MC} \hat{\Theta}_k^{(i)}, \quad k = 1,2,3 \tag{26}$$

where MC^* is the number need to reach stationary posterior distribution.

Data analysis

In this section, three real datasets are considered as applicant on the EER distribution. For each dataset, we calculate the descriptive statistics by using the new form of EER distribution. The results compared with three other life-time distributions, exponentiated Rayleigh distribution with ($\lambda > 0$) scale parameter and ($\theta > 0$) shape parameter, exponential distribution with ($\lambda > 0$) scale parameter and Rayleigh distribution with ($\lambda > 0$) scale parameter. For each distribution, we computed the ML estimates and Bayes estimate with the help of MCMC method. The R-package are used to obtain the estimate values. The performance of the developed distribution with respected to the competing distributions are assessed under the values of log-likelihood function and Akaike's information criterion (AIC).

Light-emitting diodes

The first data set that is about the case study on the light-emitting diodes manufacturing process that focusses on the luminous intensities of light-emitting diodes sources, given in [19] as: 2.163, 5.972, 1.032, 0.628, 2.995, 3.766, 0.974, 4.352, 3.920, 1.375, 0.618, 4.575, 1.027, 6.279, 2.821, 7.125, 5.443, 1.766, 7.155, 0.830, 3.590, 5.965, 3.177, 4.634, 7.261, 2.247, 6.032, 4.065, 5.434, and 1.336. We fit the data to the EER distribution under consideration the distances between the fit distribution and the empirical function (Kolmogorov-Smirnov distance) and corresponding p-values. Kolmogorov-Smirnov distance is given by 0.1100 p-values is given by 0.8925. Figure 2 has shown the draw of survival and empirical functions. The results showed that, EER distribution fits quite well to the given real data. The maximum likelihood results and Bayes estimates are given in tab. 2. In Bayesian approach, we used non-informative prior information (mean $a_k = b_k = 0.0001$, $k = 1, 2, 3$). The computed values of the log-likelihood function and AIC are given in tab. 3.

Also, the summary statistics for this data set is given in tab. 4.

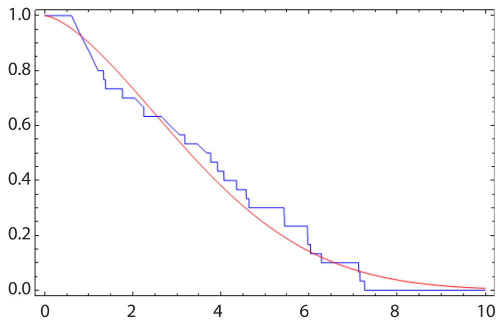

Figure 2. Diagnostic plots for the fitted EER survival functions for the data

Table 2. The MLE and Bayes estimate of the parameters under light-emitting diodes data

Distribution	$\hat{\lambda}$	$\hat{\theta}$	$\hat{c}$	$\hat{\lambda}_B$	$\hat{\theta}_B$	$\hat{c}_B$
Exponentiated exponential Rayleigh	1.2865	0.1789	0.0008	1.3365	0.1870	0.0014
Exponential Rayleigh	301.7786	0.1072	–	302.009	0.2074	–
Exponential	408.2188	–	–	413.6662	–	–
Rayleigh	259.688	–	–	261.6774	–	–

Table 3. The values of criteria under model selection

Distribution	LogLik	AIC
Exponentiated exponential Rayleigh	–118.3249	242.6497
Exponential Rayleigh	–128.4738	260.9476
Exponential	–180.6200	363.2400
Rayleigh	–302.3912	606.7823

Table 4. Summary statistics for light-emitting diodes dataset

Sample Size	Minimum	Q_1	Median	Mean	Q_3	Maximum
30	0.618	1.473	3.678	3.619	5.441	7.261

Wheaton river data

The second data set is the exceedances of the floor peaks in m/s of Wheaton River, Yukon Territory, Canada, for the years from 1958 to 1984, rounded to one decimal place, given in [20] as: 1.7, 1.4, 0.6, 9.0, 5.6, 1.5, 2.2, 18.7, 2.2, 1.7, 30.8, 2.5, 14.4, 8.5, 39.0, 7.0, 13.3, 27.4, 1.1, 25.5, 0.3, 20.1, 4.2, 1.0, 0.4, 11.6, 15.0, 0.4, 25.5, 27.1, 20.6, 14.1, 11.0, 2.8, 3.4, 20.2, 5.3, 22.1, 7.3, 14.1, 11.9, 16.8, 0.7, 1.1, 22.9, 9.9, 21.5, 5.3, 1.9, 2.5, 1.7, 10.4, 27.6, 9.7, 13.0, 14.4, 0.1, 10.7, 36.4, 27.5, 12.0, 1.7, 1.1, 30.0, 2.7, 2.5, 9.3, 37.6, 0.6, 3.6, 64.0, and 27.0. Table 5 includes the descriptive statistics for the data set. The Kolmogorov-Smirnov distance and corresponding p-values of fit data to the EER distribution is given by 0.1088 and 0.9232, respectively. Figure 3 has shown the draw of survival and empirical functions. The results showed that, EER distribution fits quite well to the given real data. The maximum likelihood results and Bayes estimates are given in tab. 6. In Bayesian approach, we used non-informative prior. The computed values of the log-likelihood function and AIC are given in tab. 7.

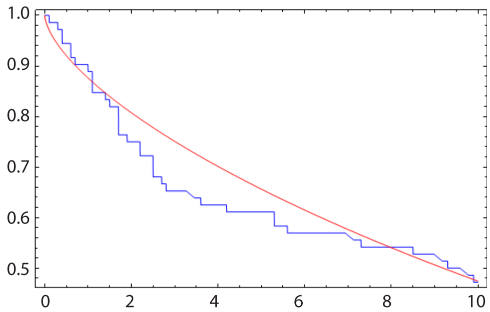

Figure 3. Diagnostic plots for the fitted EER survival functions for the data

Table 5. Summary statistics for Wheaton River dataset

Sample size	Minimum	Q_1	Median	Mean	Q_3	Maximum
72	0.100	2.125	9.500	12.204	20.125	64

Table 6. The MLE and Bayes estimate of the parameters (Wheaton River data)

Distribution	$\hat{\lambda}$	$\hat{\theta}$	$\hat{c}$	$\hat{\lambda}_B$	$\hat{\theta}_B$	$\hat{c}_B$
Exponentiated exponential Rayleigh	$4.206 \cdot 10^{00}$	$1.29 \cdot 10^{-01}$	$2.17 \cdot 10^{-04}$	$4.741 \cdot 10^{00}$	$1.33 \cdot 10^{-01}$	$7 \cdot 10^{-04}$
Exponential Rayleigh	791.1266	0.1026	–	794.4521	0.1144	–
Exponential	677.0260	–	–	679.3214	–	–
Rayleigh	103.5271	–	–	105.4178	–	–

Table 7. The values of criteria under model selection

Distribution	LogLik	AIC
Exponentiated exponential Rayleigh	–357.9027	721.8054
Exponential Rayleigh	–366.6486	737.2973
Exponential	–470.573	943.1459
Rayleigh	–610.7144	1223.429

Strength of glass fibers data

The third dataset is obtained by workers at the UK National Physical Laboratory about the braking strength of glass fibers of length 1.5 cm, the obtained originally given by [21] as: 0.55, 1.04, 1.28, 1.48, 1.51, 1.58, 1.61, 1.66, 2.00, 1.70, 1.78, 0.74, 1.11, 1.29, 1.48, 1.52, 1.59, 1.62, 1.66, 2.01, 1.70, 1.81, 0.77, 1.13, 1.30, 1.49, 1.53, 1.60, 1.62, 1.67, 2.24, 1.73, 1.82, 0.81, 1.24, 1.36, 1.49, 1.54, 1.61, 1.63, 1.68, 1.76, 1.84, 0.84, 1.25, 1.39, 1.50, 1.55, 1.61, 1.64, 1.68, 1.76, 1.84, 0.93, 1.27, 1.42, 1.50, 1.55, 1.61, 1.66, 1.69, 1.77, and 1.89. Some summaries statistics for the data appear in tab. 8. The Kolmogorov-Smirnov distance and corresponding p-values of fit data to the EER distribution is given by 0.2151 and 0.9578, respectively. Figure 4 has shown the draw of survival and empirical functions. The results showed that, EER distribution fits quite well to the given real data. The maximum likelihood results and Bayes estimates are given in tab. 9. In Bayesian approach, we used non-informative prior information. The computed values of the log-likelihood function and AIC are given in tab. 10.

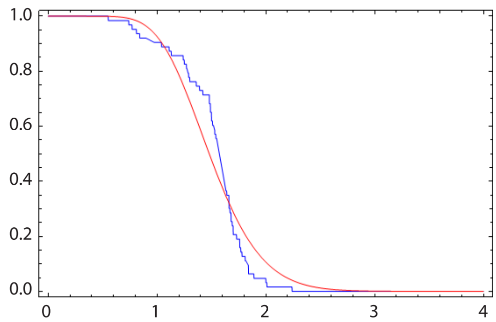

Figure 4. Diagnostic plots for the fitted EER survival functions for the data

Table 8. Summary statistics for strength of glass fibers dataset

Sample size	Minimum	Q_1	Median	Mean	Q_3	Maximum
63	0.550	1.375	1.590	1.507	1.685	2.240

Table 9. The MLE and Bayes estimate of the parameters (strength of glass fibers data)

Distribution	$\hat{\lambda}$	$\hat{\theta}$	$\hat{c}$	$\hat{\lambda}_B$	$\hat{\theta}_B$	$\hat{c}_B$
Exponentiated exponential Rayleigh	4.7121	0.1671	0.0261	4.8472	0.15473	0.0287
Exponential Rayleigh	$1.13 \cdot 10^{03}$	$7.516 \cdot 10^{-02}$	–	$1.31 \cdot 10^{03}$	$7.871 \cdot 10^{-02}$	–
Exponential	317.8779	–	–	319.4217	–	–
Rayleigh	62.4761	–	–	63.5574	–	–

Table 10. The values of criteria under model selection

Distribution	LogLik	AIC
Exponentiated exponential Rayleigh	–210.2389	426.4778
Exponential Rayleigh	–250.0626	504.1253
Exponential	–363.2837	728.5674
Rayleigh	–499.2847	1000.569

According to tabs. 3, 6, and 9, we observe the results of point estimate under non-informative prior information of ML are closed to Bayes estimate. Also, according to tabs. 4, 7, and 10, we can see that the EER has smallest value of AIC and largest value of log-likelihhod hence is considered as the best fit for these data.

Conclusion

We introduced in this paper a new form of distribution called the EER distribution. Additionally, we discussed various statistical properties of the EER distribution. Also, the unknown parameters are estimated with classical ML estimation. We adopt Bayesian approach with the help of MCMC method to obtain the estimate values of the unknown parameters. The proposed distribution is applied on three real data sets. It has been observed that the EER distribution was the most suitable for modelling the data used. This version of EER distribution can be useful in modelling of complex data.

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