A DUAL APPROACH TO PARAMETER ESTIMATION
Classical *vs*. Bayesian Methods in Power Rayleigh Modelling

by

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In this article, we investigated the problem of estimating the parameters of power Rayleigh distribution using a range of classical and Bayesian estimate strategies. For applied statisticians and reliability engineers, parameter estimation provides a guide for choosing the best method of estimating the model parameters. Six frequentist estimation methods, including maximum likelihood estimation, Cramer-von Mises estimation, Anderson-Darling estimation, least square estimation, weighted least square estimation, and maximum product of spacing estimation, were taken into consideration when estimating the parameters of the power Rayleigh model. The expressions for Bayes estimators of the scale parameter are derived under squared error and precautionary loss functions and utilizing extensions of Jeffrey's prior and natural conjugate priors. To investigate the finite sample properties of the parameter estimations, Monte Carlo simulations are also performed. Finally, two applications to real data are used to highlight the versatility of the suggested model and the comparison is made with the Rayleigh and some of its well-known extensions such as exponentiated Rayleigh and weighted Rayleigh distributions.

Key words: *power Rayleigh distribution, Cramer Von-Mises estimation, Anderson-Darling estimation, weighted least square estimation*

Introduction

The Rayleigh distribution (RD) initially pioneered by the physicist Lord Rayleigh [1] while researching the issue of acoustics, specifically the analysis of random vibrations. In many different domains, the RD is frequently employed to simulate specific aspects of wave phenomena, including electronic waves. It is particularly helpful for expressing a wave's amplitude when two random waves with equal powers but random phase angles are combined. The RD offers a probability distribution for the resulting wave amplitude in this situation.

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In addition acoustics and wave modelling, life testing trials and clinical research frequently use the RD. It can be used to simulate the life spans of parts or systems that have a *Weibull-like* failure pattern, in which breakdowns tend to occur more frequently in the beginning and less frequently with time. These applications are ideally matched to the shape and properties of the RD. The RD is a special case of Weibull distribution with the shape parameter equal to 2. The probability density function (PDF) of the RD has the form:

$$f(x;\theta)=\frac{x}{\theta^2}\exp\left(-\frac{x^2}{2\theta^2}\right),\ \ x>0,\ \ \theta>0$$

The RD offers a number of advantageous characteristics and attractive physical explanations. It is frequently used to analyze lifetime data: for additional information, see Johnson *et al.* [2]. Siddiqui [3] and Miller and Sackrowttz [4] discuss the genesis and other features of this distribution. Howlader and Hossain [5] discussed the inferences for RD. Kim and Han [6] estimated its scale parameter under general progressive censoring. However, the use of RD is restricted to the situation of increasing failure rate and hence a number of researchers have made significant contribution extending the practical significance of the RD, by introducing new generalized forms or models. These extensions add more parameters or modify certain characteristics of the present distribution in an attempt to provide greater flexibility and more adaptability to real-world circumstances. For instance, Surles and Padgett [7] have introduced two parameter generalized Rayleigh model. Merovci [8] introduced transmuted RD and studied its mathematical properties. Merovci and Elbatal [9] proposed Weibull RD. Mudasir *et al*. [10] derived weighted Rayleigh model and discuss its informative and non-informative priors. Ramadan *et al*. [11] presented the generalized power Akshaya distribution and its applications, Also, Gomaa *et al*. [12] studied the unit AP-Kum-MSBL-II distribution. Moreover, Bhat *et al*. [13] showed the classical and Bayesian estimation for the extended odd Weibull power Lomax model with applications.

Bhat and Ahmad [14] proposed power RD, an extension that introduces a power parameter to the conventional RD, altering its shape and characteristics. Typically, the PDF of the power RD can be expressed:

$$f(y;\alpha,\theta)=\frac{\alpha}{\theta^2}y^{2\alpha-1}\exp\left(-\frac{y^{2\alpha}}{2\theta^2}\right),\ \ y>0,\ \ \alpha,\theta>0 \tag{1}$$

The Cumulative distribution function corresponding to eq. (1) is given:

$$F(y;\alpha,\theta)=1-\exp\left(-\frac{y^{2\alpha}}{2\theta^2}\right) \tag{2}$$

where Y is the random variable, λ – the scale parameter, and exp – the exponential function.

The pth quantile function of power RD (PRD) is calculated by inverting the cdf given in eq. (2) and is obtained:

$$Q(p)=\left[-2\theta^2\log(1-p)\right]^{1/2\alpha}$$

It is important to note that the RD is defined only for $Y\geq 0$, as the distribution is typically used to model positive quantities such as wave amplitudes or distances.

In this paper, parameter estimation techniques for the power RD using Bayesian and non-Bayesian approaches are presented.

Classical inference methods

Estimating unknown parameters of a distribution using various estimation techniques offers a comprehensive understanding of their performance and suitability for different scenarios, see Bhat *et al.* [13]. This section employs six estimation techniques to estimate the unknown parameters of the PRD. The performance of each of the six estimation procedures on the provided dataset is thoroughly examined in order to provide an extensive assessment of their efficacy in estimating the parameters of the PRD. A comprehensive evaluation of the six estimation strategies, applicability for the provided dataset and their efficacy in estimating the parameters of the PRD is provided by presenting the findings of each one. This comparison facilitates comprehension of the advantages and disadvantages of each approach to model fitting. An outline of the estimating methods discussed is provided below:

Maximum likelihood estimation

Let $y_1, y_2, ..., y_n$ be a random sample of size n having the PDF given in eq. (2). Then, the likelihood function is given:

$$L\left(\theta,\alpha\middle|y\right)=\left(\frac{\alpha}{\theta^2}\right)^n \mu^{2\alpha-1}\exp\left(-\frac{t}{2\theta^2}\right) \tag{3}$$

where

$$\mu=\prod_{k=1}^{n} y_k \quad \text{and} \quad T=\sum_{k=1}^{n} {y_k}^{2\alpha}$$

The logarithmic likelihood function can be written:

$$\ell = n\log\left(\alpha\right)-2n\log\left(\theta\right)+\left(2\alpha-1\right)log\left(\mu\right)-\frac{\sum_{k=1}^{n} {y_k}^{2\alpha}}{2\theta^2} \tag{4}$$

Now, differentiating eq. (3) partially with respect to α and θ, we obtain:

$$\frac{\partial\ell}{\partial\alpha}=\frac{n}{\alpha}+2\log\left(\mu\right)-\frac{\sum_{k=1}^{n} {y_k}^{2\alpha}\log\left(y_k\right)}{\theta^2}$$

$$\frac{\partial\ell}{\partial\theta}=-\frac{2n}{\theta}+\frac{\sum_{k=1}^{n} {y_k}^{2\alpha}}{\theta^3}$$

The maximum likelihood estimators of parameters are obtained by solving the aforementioned system of non-linear equations.

Anderson Darling estimation

By minimizing the following function with respect to α and θ, respectively, the Anderson and Darling [15] estimators of the parameters denoted by $\hat{\alpha}_{\text{ADE}}$ and $\hat{\theta}_{\text{ADE}}$ of the PRD can be assessed:

$$A=-n-\frac{1}{n}\sum_{k=1}^{n}\left(2k-1\right)\left[\ln\left\{F\left(y_k\right)\right\}+\ln\left\{S\left(y_{n+1-k}\right)\right\}\right]$$

and

$$A(\alpha,\theta)=-n-\frac{1}{n}\sum_{k=1}^{n}(2k-1)\left[\ln\left\{1-\exp\left(-\frac{{y_k}^{2\alpha}}{2\theta^2}\right)\right\}+\ln\left\{\exp\left(-\frac{{y_k}^{2\alpha}}{2\theta^2}\right)\right\}\right]$$

Cramer-von-Mises estimation

The Cramer-von-Mises (CVM) estimation is another significant estimating approach introduced by Macdonald [16]. The CVM estimators $\hat{\alpha}_{\text{CVME}}$ and a $\hat{\theta}_{\text{CVME}}$ are derived by minimizing the value of the function $C(\alpha,\theta)$ with respect to the unknown parameters α and θ as given:

$$C(\alpha,\theta)=\frac{1}{12n}+\sum_{k=1}^{n}\left\{F(y_k)-\frac{2k-1}{2n}\right\}^2$$

$$C(\alpha,\theta)=\frac{1}{12n}+\sum_{k=1}^{n}\left\{1-\exp\left(-\frac{{y_k}^{2\alpha}}{2\theta^2}\right)-\frac{2k-1}{2n}\right\}^2$$

Least square estimation and weighted least square estimation

The least square estimation (LSE) and weighted least square (WLS) estimation approaches are due to Swain *et al*. [17], who suggested these estimation methods while estimating the parameters of the Beta distribution. The LSE of the parameters of the proposed model can be obtained by minimizing the least square function $LS(\alpha,\theta)$ with respect to the unknown parameters:

$$LS(\alpha,\theta)=\sum_{k=1}^{n}n_k\left\{F(y_k)-\frac{k}{n+1}\right\}^2$$

By setting $n_k = 1$, the LSE $\hat{\alpha}_{\text{LSE}}$ and $\hat{\theta}_{\text{LSE}}$ can be obtained:

$$LS(\alpha,\theta)=\sum_{k=1}^{n}\left\{1-\exp\left(-\frac{{y_k}^{2\alpha}}{2\theta^2}\right)-\frac{k}{n+1}\right\}^2$$

While as by setting:

$$n_k=\frac{(n+1)^2(n+2)}{k(n-k+1)}$$

we can obtain the WLS estimators denoted by $\hat{\alpha}_{\text{WLSE}}$ and $\hat{\theta}_{\text{WLSE}}$:

$$WLS(\alpha,\theta)=\sum_{k=1}^{n}\left[\frac{(n+1)^2(n+2)}{k(n-k+1)}\right]\left\{1-\exp\left(-\frac{{y_k}^{2\alpha}}{2\theta^2}\right)-\frac{k}{n+1}\right\}^2$$

Maximum product spacing estimation

The maximum product spacing estimation (MPSE) method was introduced by Cheng and Amin [18] as an alternative to maximum likelihood (ML) estimation methodology for estimating the parameters of continuous univariate distributions. Additionally, Ranneby [19] independently studied the method, considering it as an approximation Kullback-Leibler measure of information and elucidated its consistency property. To motivate our choice of the MPSE method, Cheng and Amin [18] provided compelling evidence by establishing its efficiency and consistency under more general conditions compared to MLE procedure.

We begin by defining the uniform spacings of a random sample from the power RD. Given a random sample of size n from the power RD with order statistics $Y_{(1)}$, $Y_{(2)}$, ..., $Y_{(n)}$ the uniform spacings are defined as the differences between consecutive order statistics as given:

$$D_k = F\left(y_{(k)}\right) - F\left(y_{(k-1)}\right), \qquad k = 1, 2, \ldots, n+1$$

where

$$F\left(y_{(0)}\right) = 0 \qquad \text{and} \qquad F\left(y_{(n+1)}\right) = 1$$

Since, we are sampling from PRD, thus:

$$F\left(y_{(k)}\right) = 1 - \exp\left(-\frac{{y_{(k)}}^{2\alpha}}{2\theta^2}\right)$$

and

$$F\left(y_{(k-1)}\right) = 1 - \exp\left(-\frac{{y_{(k-1)}}^{2\alpha}}{2\theta^2}\right)$$

then

$$D_k = \left\{1 - \exp\left(-\frac{{y_{(k)}}^{2\alpha}}{2\theta^2}\right)\right\} - \left\{1 - \exp\left(-\frac{{y_{(k-1)}}^{2\alpha}}{2\theta^2}\right)\right\}$$

The parameter estimates are obtained by maximizing:

$$MPS(\alpha, \theta) = \frac{1}{n+1} \sum_{k=1}^{n+1} \log D_k$$

$$MPS(\alpha, \theta) = \frac{1}{n+1} \sum_{k=1}^{n+1} \log\left[\left\{1 - \exp\left(-\frac{{y_{(k)}}^{2\alpha}}{2\theta^2}\right)\right\} - \left\{1 - \exp\left(-\frac{{y_{(k-1)}}^{2\alpha}}{2\theta^2}\right)\right\}\right]$$

Bayesian inference method

In this section, we explore the estimation of a scale parameter in the PRD. To enhance our understanding, we consider two distinct prior distributions and employ three different loss functions. This allows us to investigate the effects of different assumptions on estimating process, providing an exhaustive illustration of the parameter estimation procedure in the context of the PRD.

Posterior distribution under the assumption of different priors

The aim of this section is to determine Bayesian estimators for the scale parameter of the PRD using different loss functions and priors. Additionally, a comparison will be conducted to evaluate these estimations.

In the context of estimating the scale parameter of the PRD, an extension of Jeffrey's and natural conjugate priors are utilized. When applying the extension of Jeffrey's prior for parameter estimation, previous knowledge is incorporated into the procedure. Jeffrey's prior is well-known for its invariance quality, which means it remains unaltered even after re-parameterization, making it a popular choice in Bayesian statistics. The extension of Jeffrey's prior proposed by Al-Kutubi [20], takes the form:

$$\pi_1(\theta) \propto \left[I(\theta)\right]^c, \ c \in R^+$$

where

$$I(\theta) = -nE\left[\frac{\partial^2 \log f(y;\alpha,\theta)}{\partial\theta^2}\right]$$

is the Fisher information matrix for the distribution given in eq. (1).

Thus, for scale parameter θ the extension of Jeffrey's prior is given:

$$\pi_1(\theta) = \frac{1}{\theta^{2c}}$$

The posterior density of θ under the assumption of extension of Jeffrey's prior is obtained:

$$P_1(\theta) \propto L(\theta,\lambda|y)\pi_1(\theta)$$

$$P_1(\theta) \propto \frac{\alpha^n}{\theta^{2n}}\prod_{k=1}^{n} {y_k}^{2\alpha-1}\exp\left(-\frac{T}{2\theta^2}\right)\frac{1}{\theta^{2c}}$$

where

$$T = \sum_{k=1}^{n} {y_k}^{2\alpha} P_1(\theta) = K\frac{\exp\left(-\frac{T}{2\theta^2}\right)}{\theta^{2(n+c)}}$$

where K is independent of θ:

$$K^{-1} = \int_0^\infty \frac{\exp\left(-\frac{T}{2\theta^2}\right)}{\theta^{2(n+c)}}\mathrm{d}\theta = \frac{\Gamma\left(n+c-\frac{1}{2}\right)}{2\left(\frac{T}{2}\right)^{n+c-1/2}}$$

therefore

$$K = \frac{2\left(\frac{T}{2}\right)^{n+c-1/2}}{\Gamma\left(n+c-\frac{1}{2}\right)}$$

Hence, the posterior density of θ is obtained:

$$P_1(\theta) = \frac{2\left(\frac{T}{2}\right)^{n+c-1/2}\exp\left(-\frac{T}{2\theta^2}\right)}{\theta^{2(n+c)}\Gamma\left(n+c-\frac{1}{2}\right)} \tag{5}$$

The natural conjugate prior is a term used in Bayesian statistics, notably in relation exponential family distributions. In an exponential family, the probability function can be represented in a precise form that is mathematically convenient to work with. In the context of estimating the scale parameter of the PRD, the natural conjugate prior chosen can be written in the form:

$$\pi_2(\theta)=\frac{1}{\theta^{a+1}}\exp\left(-\frac{b}{2\theta^2}\right),\quad a,b>0$$

Similarly, the posterior distribution of θ using natural conjugate prior is thus obtained:

$$P_2(\theta)=\frac{2\left(\frac{T+b}{2}\right)^{n+a/2}\exp\left(-\frac{T+b}{2\theta^2}\right)}{\Gamma\left(n+\frac{a}{2}\right)\theta^{2n+a+1}} \tag{6}$$

Bayesian estimation under three loss functions

In the fields of statistics and decision theory, a loss function is a mathematical function that assigns a real number to an event, which intuitively represents the associated *cost* of the event. This function is commonly employed in parameter estimation, where the event in consideration relates to the disparity between estimated and actual values for a given dataset. The aforementioned *Lemma* is utilized to derive results in this context.

Parameter estimation under squared error loss function

The squared error loss function (SELF), introduced by Legendre [21] and Gauss [22], is defined as the squared difference between the estimated parameter $\hat{\theta}$ and true parameter θ, with c_1 as constant, the SELF is represented:

$$L(\theta,\hat{\theta})=c_1(\theta-\hat{\theta})^2,\quad c_1>0$$

By utilizing SELF and under the assumption of extension of Jeffrey's prior, the risk function, denoted as $R(\theta, \hat{\theta})$, is given:

$$R(\theta,\hat{\theta})=\int_0^\infty L(\theta,\hat{\theta})P_1(\theta)\mathrm{d}\theta$$

where $R(\theta)$ is the posterior density defined in eq. (4), thus:

$$R(\theta,\hat{\theta})=c_1\left[\frac{T}{2\left(n+c-\frac{3}{2}\right)}+\hat{\theta}^2-2\left(\frac{T}{2}\right)^{1/2}\frac{\Gamma(n+c-1)}{\Gamma\left(n+c-\frac{1}{2}\right)}\hat{\theta}\right]$$

The Bayes estimator is the solution of the equation:

$$\frac{\partial R(\theta,\hat{\theta})}{\partial\hat{\theta}}=0$$

and is given

$$\hat{\theta}=\left(\frac{T}{2}\right)^{1/2}\frac{\Gamma(n+c-1)}{\Gamma\left(n+c-\frac{1}{2}\right)}$$

Similarly, by utilizing SELF and posterior distribution of θ undernatural conjugate prior, the risk function is computed:

$$R\left(\theta,\hat{\theta}\right)=\int_0^{\infty}L\left(\theta,\hat{\theta}\right)P_2(\theta)\mathrm{d}\theta$$

Where $P_2(\theta)$ is the posterior density defined in eq. (5), thus:

$$R\left(\theta,\hat{\theta}\right)=c_1\left[\frac{(T+b)}{2\left(n+\frac{a}{2}-1\right)}+\hat{\theta}^2-2\left(\frac{T+b}{2}\right)^{1/2}\frac{\Gamma\left(n+\frac{a-1}{2}\right)}{\Gamma\left(n+\frac{a}{2}\right)}\hat{\theta}\right]$$

The Bayes estimator for parameter θ is obtained by minimizing the risk function and given:

$$\hat{\theta}=\left(\frac{T+b}{2}\right)^{1/2}\frac{\Gamma\left(n+\frac{a-1}{2}\right)}{\Gamma\left(n+\frac{a}{2}\right)},\quad R\left(\theta,\hat{\theta}\right)=\int_0^{\infty}L\left(\theta,\hat{\theta}\right)P_2(\theta)\mathrm{d}\theta$$

Parameter estimation under precautionary loss function

The precautionary loss function (PLF) introduced by Norstrom [23] is an asymmetric loss function that penalizes the squared difference between the true parameter θ and the estimator $\hat{\theta}$to the estimator $\hat{\theta}$. Mathematically, PLF is represented:

$$L\left(\theta,\hat{\theta}\right)=\frac{\left(\theta-\hat{\theta}\right)^2}{\hat{\theta}}$$

The asymmetric structure of this loss function is reflected in the denominator, which involves the estimator $\hat{\theta}$. The structure penalizes underestimation more severely when the estimator $\hat{\theta}$ is smaller than the true parameter θ, as the denominator decreases.

By using PLF and posterior distribution of θ under the assumption of extension of Jeffrey's prior, the risk function is given:

$$R\left(\theta,\hat{\theta}\right)=\frac{T}{2\left(n+c-\frac{3}{2}\right)\hat{\theta}}+\hat{\theta}-2\left(\frac{T}{2}\right)^{1/2}\frac{\Gamma(n+c-1)}{\Gamma\left(n+c-\frac{1}{2}\right)}$$

The Bayes estimator for scale parameter θ is obtained by minimizing the risk function and given:

$$\hat{\theta}=\left[\frac{T}{2\left(n+c-\frac{3}{2}\right)}\right]^{1/2}$$

Similarly, by using PLF and posterior distribution of θ under the natural conjugate prior, the risk function is given:

$$R\left(\theta,\hat{\theta}\right)=\frac{\left(T+b\right)}{2\left(n+\frac{a}{2}-1\right)\hat{\theta}}+\hat{\theta}-2\left(\frac{T+b}{2}\right)^{1/2}\frac{\Gamma\left(n+\frac{a-1}{2}\right)}{\Gamma\left(n+\frac{a}{2}\right)}$$

The Bayes estimator is the solution the equation:

$$\frac{\partial R\left(\theta,\hat{\theta}\right)}{\partial\hat{\theta}}=0$$

which results

$$\hat{\theta}=\left[\frac{T+b}{2\left(n+\frac{a}{2}-1\right)}\right]^{1/2}$$

Parameter estimation under entropy loss function

In numerous practical scenarios, it seems to be more plausible to articulate the loss using the ratio $\hat{\theta}/\theta$. When considering this circumstance, Calabria and Pulcini [24] highlight the significance of an advantageous asymmetrical loss function, namely the entropy loss function (ELF) given:

$$L\left(\delta\right)\propto\left[\delta-\log\left(\delta\right)-1\right]$$

By utilizing ELF for some constant c_2 and posterior distribution of θ under the assumption of extension of Jeffrey's prior, the risk function is obtained:

$$R\left(\theta,\hat{\theta}\right)=c_2\int_0^\infty\left[\frac{\hat{\theta}}{\theta}-\log\left(\frac{\hat{\theta}}{\theta}\right)-1\right]\frac{2\left(\frac{T}{2}\right)^{n+c-1/2}\exp\left(-\frac{T}{2\theta^2}\right)}{\theta^{2(n+c)}\Gamma\left(n+c-\frac{1}{2}\right)}\mathrm{d}\theta$$

$$R\left(\theta,\hat{\theta}\right)=c_2\left[\left(\frac{2}{T}\right)^{1/2}\frac{\Gamma\left(n+c\right)}{\Gamma\left(n+c-\frac{1}{2}\right)}\hat{\theta}-\log\left(\hat{\theta}\right)-\frac{\Gamma'\left(n+c-\frac{1}{2}\right)}{2\Gamma\left(n+c-\frac{1}{2}\right)}-1\right]$$

Now solving

$$\frac{\partial R\left(\theta,\hat{\theta}\right)}{\partial\hat{\theta}}=0$$

we obtain the Bayes estimator:

$$\hat{\theta}=\left(\frac{T}{2}\right)^{1/2}\frac{\Gamma\left(n+c-\frac{1}{2}\right)}{\Gamma\left(n+c\right)}$$

Similarly, by utilizing ELF and posterior distribution of θ under the assumption of natural conjugate prior, the risk function is obtained:

$$R\left(\theta,\hat{\theta}\right)=c_2\left[\left(\frac{2}{T+b}\right)^{1/2}\frac{\Gamma\left(n+\frac{a+1}{2}\right)}{\Gamma\left(n+\frac{a}{2}\right)}\hat{\theta}-\log\left(\hat{\theta}\right)-\frac{\Gamma'\left(n+\frac{a}{2}\right)}{2\Gamma\left(n+\frac{a}{2}\right)}-1\right]$$

The Bayes estimator for scale parameter θ is obtained by minimizing the risk function and given:

$$\hat{\theta}=\left(\frac{T+b}{2}\right)^{1/2}\frac{\Gamma\left(n+\frac{a}{2}\right)}{\Gamma\left(n+\frac{a+1}{2}\right)}$$

Simulation results

In this section, we present an overview of a Monte Carlo simulation study that encompasses the assessment of various estimation methods, including MLE, ADE, CVME, MPSE, LSE, WLSE, and Bayesian methods. The evaluation is performed using performance metrics such as bias, mean square error (MSE), and mean relative error (MRE) with corresponding formulas:

$$Bias=\frac{1}{N}\sum_{i=1}^{N}\left(\widehat{\vartheta}_i-\vartheta_i\right),\ MSE=\frac{1}{N}\sum_{i=1}^{N}\left(\widehat{\vartheta}_i-\vartheta_i\right)^2,\ MRE=\frac{1}{N}\sum_{i=1}^{N}\left(\frac{\widehat{\vartheta}_i-\vartheta_i}{\vartheta_i}\right)$$

The goal of the Monte Carlo simulation is to assess the performance of different estimation methods in the context of the PRD. The simulation is conducted using the R programming language, exploring different sample sizes and parameter values for the PRD. Random samples of sizes 25, 50, 75, 100, and 150 are generated from the PRD with three sets of real parameter values: Set 1 (α = 1.25, θ = 0.25), Set 2 (α = 0.75, θ = 0.75), and Set 3 (α = 1.75, θ = 1.25). The quantile function (QF) of PRD is utilized to generate 1000 random samples (N = 1000) for each scenario. The simulation results for non-Bayesian methods were obtained using R software and are presented in tabs. 1-3. Similarly, the results for Bayesian methods can be found in tab. 4.

Table 1. Simulation results (α = 1.25, θ = 0.25)

Sample size	Maximum likelihood estimates							
	Average estimate		Bias		MSE		MRE	
	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\theta}$
25	1.31830	0.23914	0.17756	0.03331	0.05336	0.00177	0.14205	0.13324
50	1.28286	0.24542	0.11454	0.02316	0.02214	0.00086	0.09163	0.09266
75	1.27317	0.24594	0.09161	0.01941	0.01410	0.00060	0.07329	0.07766
100	1.26902	0.24672	0.08327	0.01731	0.01113	0.00047	0.06661	0.06923
150	1.26145	0.24778	0.06374	0.01367	0.00676	0.00030	0.05099	0.05467
Anderson Darling estimates								
25	1.27377	0.24879	0.17880	0.03709	0.05303	0.00211	0.14304	0.14838
50	1.26370	0.24840	0.12096	0.02566	0.02486	0.00106	0.09677	0.10265
75	1.26073	0.24965	0.09763	0.02090	0.01552	0.00070	0.07810	0.08361
100	1.25091	0.25133	0.08554	0.01838	0.01189	0.00053	0.06843	0.07350
150	1.25299	0.25049	0.07153	0.01492	0.00806	0.00035	0.05723	0.05969
Cramer-Von Mises estimates								
25	1.32540	0.23970	0.21312	0.04243	0.08580	0.00297	0.17050	0.16971
50	1.29295	0.24461	0.14321	0.03026	0.03431	0.00144	0.11457	0.12104
75	1.27820	0.24570	0.11960	0.02514	0.02275	0.00098	0.09568	0.10057
100	1.26953	0.24680	0.09623	0.02052	0.01474	0.00066	0.07699	0.08209
150	1.25996	0.24858	0.08203	0.01794	0.01057	0.00050	0.06562	0.07174
Maximum product spacing estimates								
25	1.17062	0.27357	0.17651	0.04021	0.04661	0.00243	0.14120	0.16083
50	1.19548	0.26442	0.12257	0.02760	0.02229	0.00114	0.09806	0.11039
75	1.20916	0.26006	0.09874	0.02115	0.01492	0.00070	0.07899	0.08462
100	1.21683	0.25803	0.08145	0.01804	0.01044	0.00051	0.06516	0.07215
150	1.22122	0.25676	0.06685	0.01429	0.00687	0.00031	0.05348	0.05717
Least square estimates								
25	1.24654	0.25746	0.20050	0.04413	0.06558	0.00296	0.16040	0.17652
50	1.24978	0.25336	0.13882	0.02946	0.03018	0.00134	0.11105	0.11786
75	1.24887	0.25218	0.11618	0.02516	0.02116	0.00097	0.09294	0.10063
100	1.24809	0.25163	0.09824	0.02173	0.01548	0.00073	0.07859	0.08692
150	1.25031	0.25069	0.07596	0.01681	0.00934	0.00044	0.06077	0.06725
Weighted least square estimates								
25	1.24883	0.25618	0.18518	0.04001	0.05641	0.00245	0.14815	0.16003
50	1.25736	0.25116	0.13014	0.02736	0.02786	0.00117	0.10411	0.10945
75	1.24896	0.25245	0.10054	0.02129	0.01577	0.00070	0.08043	0.08517
100	1.25242	0.25044	0.08767	0.01894	0.01242	0.00056	0.07013	0.07576
150	1.25781	0.24989	0.07031	0.01473	0.00794	0.00033	0.05625	0.05893

Table 2. Simulationr (α = 0.75, θ = 0.75)

Sample size	Maximum likelihood estimates							
	Average estimate		Bias		MSE		MRE	
	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\theta}$
25	0.79098	0.75517	0.10653	0.06971	0.01921	0.00814	0.14204	0.09295
50	0.76971	0.75452	0.06872	0.04535	0.00797	0.00340	0.009163	0.06047
75	0.76390	0.75087	0.05497	0.03808	0.00508	0.00235	0.07329	0.05077
100	0.76141	0.75089	0.04996	0.03314	0.00401	0.00168	0.06662	0.04419
150	0.75687	0.74984	0.03825	0.02725	0.00243	0.00116	0.05100	0.03634
Anderson Darling estimates								
25	0.76426	0.75306	0.10728	0.06744	0.01909	0.00750	0.14304	0.08992
50	0.75822	0.74989	0.07258	0.04766	0.00895	0.00365	0.09677	0.06354
75	0.75644	0.75332	0.05858	0.03912	0.00559	0.00238	0.07810	0.05217
100	0.75054	0.75259	0.05133	0.03335	0.00428	0.00179	0.06843	0.04447
150	0.75180	0.75221	0.04292	0.02818	0.00290	0.00128	0.05723	0.03758
Cramer-Von Mises estimates								
25	0.79524	0.75170	0.12787	0.07086	0.03089	0.00842	0.17050	0.09448
50	0.77577	0.75535	0.08593	0.04819	0.01235	0.00380	0.11457	0.06426
75	0.76692	0.75100	0.07176	0.03910	0.00819	0.00243	0.09568	0.05214
100	0.76172	0.75041	0.05774	0.03478	0.00531	0.00189	0.07699	0.04637
150	0.75597	0.75005	0.04922	0.02786	0.00381	0.00125	0.06562	0.03715
Maximum product spacing estimates								
25	0.70238	0.75949	0.10591	0.06383	0.01678	0.00668	0.14122	0.08511
50	0.71729	0.75309	0.07356	0.04426	0.00802	0.00322	0.09808	0.05901
75	0.72550	0.75064	0.05925	0.03677	0.00537	0.00210	0.07900	0.04903
100	0.73011	0.75038	0.04887	0.03208	0.00376	0.00163	0.06516	0.04277
150	0.73274	0.75018	0.04012	0.02554	0.00248	0.00101	0.05349	0.03406
Least square estimates								
25	0.74792	0.75706	0.12303	0.07133	0.02361	0.00865	0.16040	0.09511
50	0.74987	0.75421	0.08329	0.04882	0.01087	0.00378	0.11105	0.06509
75	0.74932	0.75154	0.06971	0.03961	0.00762	0.00250	0.09294	0.05282
100	0.74885	0.75028	0.05894	0.03308	0.00557	0.00171	0.07859	0.04411
150	0.75019	0.75037	0.04558	0.02752	0.00336	0.00119	0.06077	0.03669
Weighted least square estimates								
25	0.74930	0.75679	0.11111	0.06473	0.02031	0.00692	0.14815	0.08630
50	0.75442	0.75334	0.07808	0.04643	0.01003	0.00356	0.10411	0.06190
75	0.74938	0.75400	0.06032	0.03879	0.00568	0.00243	0.08043	0.05171
100	0.75145	0.75067	0.05260	0.03314	0.00447	0.00182	0.07013	0.04419
150	0.75469	0.75354	0.04218	0.02749	0.00286	0.00120	0.05625	0.03665

Table 3. Simulation Results (α = 1.75, θ = 1.25)

Sample size	Maximum likelihood estimates							
	Average estimate		Bias		MSE		MRE	
	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\theta}$
25	1.84561	1.30568	0.24858	0.18097	0.10459	0.06203	0.14205	0.14478
50	1.79600	1.27899	0.16036	0.11346	0.04340	0.02256	0.09163	0.09077
75	1.78244	1.26622	0.12825	0.09280	0.02764	0.01460	0.07329	0.07424
100	1.77663	1.26347	0.11657	0.08330	0.02181	0.01061	0.06661	0.06664
150	1.76602	1.25701	0.08924	0.06534	0.01325	0.00693	0.05099	0.05227
Anderson Darling estimates								
25	1.78328	1.27785	0.25032	0.16886	0.10395	0.05376	0.14304	0.13509
50	1.76918	1.26163	0.16935	0.11606	0.04872	0.02360	0.09677	0.09284
75	1.76503	1.26404	0.13668	0.09569	0.03042	0.01463	0.07810	0.07655
100	1.75127	1.25715	0.11976	0.08142	0.02331	0.01115	0.06843	0.06514
150	1.75419	1.25691	0.10015	0.07087	0.01580	0.0796	0.05723	0.05670
Cramer-Von Mises estimates								
25	1.85556	1.30515	0.29838	0.18791	0.16818	0.06937	0.17050	0.15033
50	1.81013	1.28692	0.20050	0.12657	0.06726	0.02820	0.11457	0.10125
75	1.78949	1.26982	0.16744	0.10247	0.04459	0.01688	0.09568	0.08198
100	1.77734	1.26329	0.13473	0.08659	0.02889	0.01223	0.07699	0.06927
150	1.76394	1.25689	0.11484	0.07017	0.02072	0.00801	0.06562	0.05614
Maximum product spacing estimates								
25	1.63890	1.23416	0.24712	0.15918	0.09137	0.04249	0.14121	0.12735
50	1.67368	1.23143	0.17163	0.11031	0.04369	0.01936	0.09807	0.08825
75	1.69283	1.23304	0.13826	0.09046	0.02924	0.01285	0.07900	0.07237
100	1.70359	1.23568	0.11403	0.07738	0.02047	0.00947	0.06516	0.06190
150	1.70972	1.23694	0.09360	0.06314	0.01348	0.00617	0.05348	0.05051
Least square estimates								
25	1.74516	1.27170	0.28071	0.17783	0.12853	0.06060	0.16040	0.14226
50	1.74969	1.26244	0.19435	0.12422	0.05916	0.02570	0.11106	0.09937
75	1.74841	1.25567	0.16265	0.10115	0.04147	0.01687	0.09294	0.08092
100	1.74732	1.25197	0.13754	0.08410	0.03035	0.01122	0.07859	0.06728
150	1.75044	1.25236	0.10635	0.06739	0.01831	0.00739	0.06077	0.05391
Weighted least square estimates								
25	1.74836	1.27084	0.25926	0.16468	0.11056	0.04938	0.14815	0.13174
50	1.76031	1.26469	0.18219	0.11921	0.05460	0.02506	0.10411	0.09537
75	1.74855	1.25929	0.14076	0.09575	0.03092	0.01515	0.08043	0.07660
100	1.75339	1.25468	0.12274	0.08193	0.02435	0.01115	0.07013	0.06555
150	1.76093	1.26153	0.09843	0.06840	0.01556	0.00762	0.05625	0.05472

Table 4. Bayesian estimate and posterior risk in parenthesis for simulated data

Sample size, n	Parameter (true value)				Methods of estimation			
	α	c	C_1	C_2	$\hat{\theta}_{\text{MLE}}$	$\hat{\theta}_{\text{SELF}}$	$\hat{\theta}_{\text{PLF}}$	$\hat{\theta}_{\text{ELF}}$
25	1.31830	0.25	0.5	0.4	0.601872	0.614266 (0.072356)	0.617507 (0.605136)	0.607966 (0.091926)
		1.25	1.5	0.8		0.601857 (0.202861)	0.604904 (0.565529)	0.595928 (0.174041)
		2.25	2.5	1.2		0.590170 (0.316550)	0.593042 (0.529481)	0.584577 (0.247110)
		3.25	3.5	1.6		0.579139 (0.415619)	0.581851 (0.496564)	0.573850 (0.311808)
50	1.28286	0.25	0.5	0.4	0.409336	0.413489 (0.014563)	0.414551 (0.118679)	0.411395 (0.021150)
		1.25	1.5	0.8		0.409333 (0.041567)	0.410363 (0.112915)	0.407302 (0.039057)
		2.25	2.5	1.2		0.405301 (0.065932)	0.406299 (0.107460)	0.403328 (0.053887)
		3.25	3.5	1.6		0.401385 (0.087868)	0.402355 (0.102294)	0.399469 (0.065796)
75	1.27317	0.25	0.5	0.4	0.330682	0.332908 (0.003969)	0.333473 (0.032283)	0.331789 (-0.00392)
		1.25	1.5	0.8		0.330681 (0.011321)	0.331235 (0.030688)	0.329585 (-0.00923)
		2.25	2.5	1.2		0.328499 (0.017932)	0.329041 (0.029164)	0.327423 (-0.01585)
		3.25	3.5	1.6		0.326359 (0.023851)	0.326891 (0.027709)	0.325304 (-0.02373)
100	1.26902	0.25	0.5	0.4	0.285077	0.286512 (0.000895)	0.286875 (0.007261)	0.285791 (-0.01441)
		1.25	1.5	0.8		0.285076 (0.002512)	0.285433 (0.006789)	0.284366 (-0.02938)
		2.25	2.5	1.2		0.283661 (0.003911)	0.284013 (0.006341)	0.282961 (-0.04485)
		3.25	3.5	1.6		0.282267 (0.005109)	0.282614 (0.005916)	0.281578 (-0.06081)
150	1.26145	0.25	0.5	0.4	0.230836	0.231609 (0.000176)	0.231804 (0.001419)	0.231223 (-0.01439)
		1.25	1.5	0.8		0.230836 (0.000564)	0.231029 (0.001518)	0.230452 (-0.02859)
		2.25	2.5	1.2		0.230071 (0.001003)	0.230261 (0.001620)	0.229691 (-0.04259)
		3.25	3.5	1.6		0.229312 (0.001497)	0.229501 (0.001726)	0.228936 (-0.05638)

Conclusions at the end of simulation results

The relative biases of $\hat{\alpha}$ and $\hat{\theta}$ decrease as the sample size n increases across all estimation methodologies.

- With the increase in sample size *n*, the MSE diminishes for all estimation approaches, meeting the criteria of consistency.
- Across all estimation techniques, the gap between the estimated values and the true parameters shrinks as n increases.
- The method of MPSE demonstrates superior performance in terms of MSE compared to other methods in most scenarios.
- With the increase in sample size n, posterior risk under SELF, PLF, and ELF decreases.
- The general deduction derived from the simulation outcomes indicates that with an increase in sample size, the bias, MSE and MRE for all parameters exhibit a consistent decrease, eventually converging towards zero. This trend highlights the accuracy and precision of both the numerical computations related to the power RD parameters and the employed estimation methodologies.

Model validation and application

In this section, the utilization of the PRD distribution is illustrated through the application on two real datasets. The presentation of the datasets employed for the implementation of the proposed distribution is provided:

Dataset I: The tensile strength, expressed in GPa, of 69 carbon fibres put under stress at gauge lengths of 20 mm is illustrated in this dataset, which was first reported by Bader and Priest [25]. The recorded data is presented in the following manner:

0.312, 0.314, 0.479, 0.552, 0.700, 0.803, 0.861, 0.865, 0.944, 0.958, 0.966, 0.977, 1.006, 1.021, 1.027, 1.055, 1.063, 1.098, 1.140, 1.179, 1.224, 1.240, 1.253, 1.270, 1.272, 1.274, 1.301, 1.301, 1.359, 1.382, 1.382, 1.426, 1.434, 1.435, 1.478, 1.490, 1.511, 1.514, 1.535, 1.554, 1.566, 1.570, 1.586, 1.629, 1.633, 1.642, 1.648, 1.684, 1.697,1.726, 1.770, 1.773, 1.800, 1.809, 1.818, 1.821, 1.848, 1.880, 1.954, 2.012, 2.067, 2.084,2.090, 2.096, 2.128, 2.233, 2.433, 2.585, 2.585.

Dataset II: This dataset illustrates the duration until malfunction of 40 turbocharger units in diesel engines. Akarawak *et al.* [26] have previously employed this dataset for analysis. The recorded data consists of the following observations:

1.6,2.0, 2.6, 3.0, 3.5, 3.9, 4.5, 4.6, 4.8, 5.0, 5.1, 5.3, 5.4, 5.6, 5.8, 6.0, 6.0, 6.1, 6.3, 6.5, 6.5, 6.7, 7.0, 7.1, 7.3, 7.3, 7.3, 7.7, 7.7, 7.8, 7.9, 8.0, 8.1, 8.3, 8.4, 8.4, 8.5, 8.7, 8.8, 9.0.

The evaluation of PRD performance is conducted through the assessment of goodness of fit criteria (GoF), which encompasses metrics such as the Akaike information criterion (AIC), corrected AIC (CAIC), Schawrz information criterion (SIC), Hannan-Quinn information criterion (HQIC), Cramer-von Mises, (W^*), Anderson-Darling (AD^*) and Kolmogorov-Smirnov (KS) test statistics along with their respective p-values. Typically, a model that exhibits the lowest AIC, CAIC, BIC and KS values coupled with the highest p-value is deemed to offer a superior fit for the data. tabs. 5-9 provide the maximum likelihood estimators (MLE) and GoF criteria for PRD as well as other distributions in all datasets. The PRD model shows excellent fits in fig. 1 for the two data sets when it is compared with RD, ERD and WRD models.

Table 5. Model selection and GoF statistics for the *Dataset-I*

Model	*AIC*	*SIC*	*CAIC*	*HQIC*	A^*	W^*	*KS*	*p*-value
PRD	102.0657	106.5339	102.2475	103.8384	0.2275	0.0267	0.0440	0.9993
RD	120.8367	123.0708	120.8964	121.7230	0.4291	0.0572	0.1999	0.0081
ERD	105.8098	110.2780	105.9916	107.5825	0.5052	0.0691	0.0752	0.8293
WRD	104.6399	109.1081	104.8217	106.4126	0.4191	0.0558	0.0664	0.9205

Table 6. Model selection and GoF statistics for the *Dataset-II*

Model	*AIC*	*SIC*	*CAIC*	*HQIC*	A^*	W^*	*KS*	*p*-value
PRD	168.9510	172.3288	169.2754	170.1723	0.5742	0.0771	0.1079	0.7398
RD	185.7655	187.4544	185.8707	186.3761	1.6317	0.2527	0.9978	2.2e-16
ERD	175.5926	178.9704	175.9169	176.8139	1.0921	0.1595	0.1173	0.6399
WRD	174.0418	177.4196	174.3661	175.2631	0.9817	0.1413	0.1157	0.6578

Table 7. The ML Estimates of different models using *Dataset-I* and *Dataset-II*

Model	Dataset I			Dataset II		
	Parameter estimates					
PRD	$\hat{\alpha} = 1.62324$	$\hat{\theta} = 1.54249$	–	$\hat{\alpha} = 1.93623$	$\hat{\theta} = 29.93145$	–
RD	–	$\hat{\theta} = 1.08334$	–	–	$\hat{\theta} = 4.62721$	–
ERD	$\hat{\alpha} = 2.17464$	$\hat{\theta} = 0.66219$	–	$\hat{\alpha} = 2.38607$	$\hat{\theta} = 0.03778$	–
WRD	–	$\hat{\theta} = 0.74574$	$\hat{\beta} = 2.22097$	–	$\hat{\theta} = 2.78369$	$\hat{\beta} = 2.99194$

Table 8: Model estimates and GoF statistics using different estimation approaches for *dataset-I*

Method	$\hat{\alpha}$	$\hat{\theta}$	*LL*	A^*	W^*	*KS*	*p*-value
MLE	1.62323	1.54249	49.03285	0.22755	0.02670	0.04400	0.99933
ADE	1.66180	1.57740	49.06434	0.22363	0.02610	0.04079	0.99984
CVME	1.71848	1.61718	49.23791	0.21840	0.02529	0.03627	0.99999
LSE	1.68127	1.59363	49.10448	0.22182	0.02582	0.03983	0.99990
WLSE	1.65419	1.57458	49.05349	0.22472	0.02626	0.04266	0.99962
MPSE	1.53532	1.44160	49.23232	0.23664	0.02813	0.07094	0.87813

Table 9. Model estimates and GoF statistics using different estimation approaches for *Dataset-II*

Method	$\hat{\alpha}$	$\hat{\theta}$	*LL*	A^*	W^*	*KS*	*p*-value
MLE	1.93623	29.92215	82.47551	0.57309	0.07700	0.10772	0.74214
ADE	1.83076	25.29793	82.71921	0.62181	0.08434	0.08746	0.91964
CVME	1.90089	29.53607	82.74596	0.60319	0.08151	0.08780	0.91742
LSE	1.83520	26.07100	82.89007	0.62731	0.08517	0.09638	0.85135
WLSE	1.94835	31.67120	82.55653	0.57966	0.07797	0.08425	0.93898
MPSE	1.57307	13.76369	83.75375	0.68757	0.09447	0.14974	0.53117

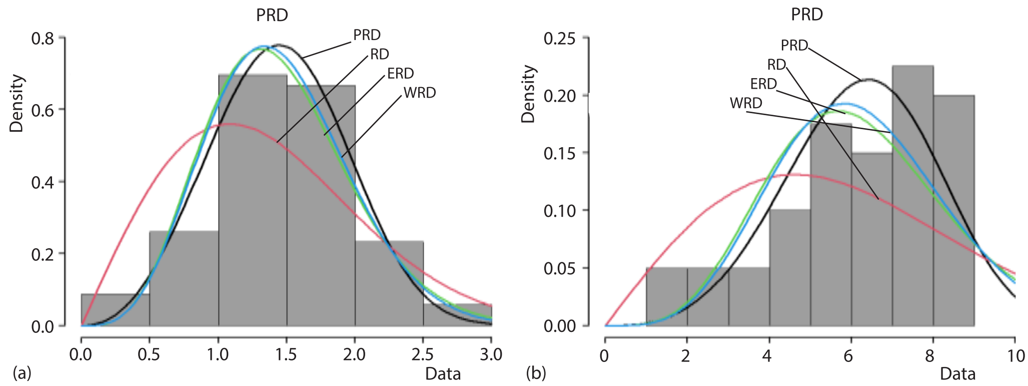

Figure 1. Histogram plots

Discussion and conclusion

This work significantly advances the field of statistical methodology by examining parameter estimation techniques for the PRD. The research yields insightful information that may be used by experts in a variety of fields, especially applied statistics and reliability engineering, through the evaluation of classical and Bayesian methodologies. In order to estimate PRD parameters, a thorough overview of the advantages and disadvantages of six frequentist estimation techniques, including MLE, W*, AD*, least square, weighted least square and maximum product of spacing estimations, is provided. Based on data properties and analytical goals, this study provides a useful road map for selecting the best estimation technique. Additionally, by offering different frameworks for parameter estimation, extending the research to Bayesian estimation broadens the scope of methodology. The range of instruments available to researchers for trustworthy statistical inference is expanded by the introduction of Bayes estimators under various loss functions and the inclusion of priors, such as Jeffrey's prior and natural conjugate priors. The Monte Carlo simulations are used to assess the finite sample qualities of the estimation techniques in order to improve the accuracy and relevance of the results. The study gives practitioners assurance about the efficacy of the suggested approaches in a range of sample scenarios by putting the estimation methods through empirical evaluation. The applicability of the PRD model to actual data demonstrates how flexible and effective it is at identifying complex patterns in data. By contrasting it with well-known distributions such as the RD and its extensions, one may highlight the unique advantages of the PRD in simulating a variety of events, hence highlighting its potential utility in real-world contexts.

In conclusion, the investigation of parameter estimation for the PRD provides an extensive analysis that unites classical and Bayesian approaches. By means of the evaluation of several estimating methods, empirical analyses, and real-world examples, the research provides experts with vital tools for solid statistical inference in a variety of scenarios. This study provides a solid foundation for future developments in statistical methodology and its use in real-world settings.

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