NUMERICAL APPROXIMATION METHOD AND CHAOS FOR A CHAOTIC SYSTEM IN SENSE OF CAPUTO-FABRIZIO OPERATOR

by

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This paper presents a novel numerical method for analvwing chaotic systems, focusing on applications to real-world problems. The Caputo-Fabrizio operator, a fractional derivative without a singular kernel, is used to investigate chaotic behavior. A fractional-order chaotic model is analvwed using numerical solutions derived from this operator, which captures the complexity of chaotic dynamics. In this paper, the uniqueness and boundedness of the solution are established using fixed-point theory. Due to the non-linearity of the system, an appropriate numerical scheme is developed. We further explore the model's dynamical properties through phase portraits, Lyapunov exponents, and bifurcation diagrams. These tools allow us to observe the systemâ€™s sensitivity to varying parameters and derivative orders. Ultimately, this work extends the application of fractional calculus to chaotic systems and provides a robust methodology for obtaining insights into complex behaviors.

Key words: *fractional derivatives, non-linear equations, simulation, numerical results, iterative method, time varying control system, Lyapunov functions*

Introduction

Chaos is widely recognized as a hallmark of complexity, and it has recently garnered the attention of many researchers. Understanding and interpreting this complexity is of great significance, as chaos encompasses non-linear, intricate, and unpredictable behaviors. Chaos appears in various fields, including chemical reactions, astronomy, population dynamics, turbulence, meteorology, linguistics, stock markets, and more. Numerous studies have explored chaotic attractors using classical derivatives. However, with the rise of fractional calculus, there is a growing need to revisit chaotic systems using these advanced mathematical tools, as noted

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in [1-3]. This section highlights key studies from the literature on experimental and numerical outcomes related to chaotic systems. For instance, Basios and Antonopoulos [4] investigated the main features of chaotic systems, focusing on chaotic and hyperchaotic behaviors. Le Berre *et al.* [5] conducted a numerical experiment that revealed a bifurcation from parallel rolls to labyrinthine structures as the most stable configuration in a non-variational model. Le Berre [5] introduced a fractional chaotic model and employed a modified Adams-Bashforth-Moulton numerical method for its solution. Xin *et al* [6] provided numerical results on chaotic systems. Sprott and Chlouverakis [7], analyzed the complexities of this system, covering its route to chaos, attractor dimension, multistability, chaotic diffusion, and symbolic dynamics, as further discussed in [8]. Three primary types of fractional derivative operators, Riemann-Liouville, Caputo-Fabrizio, and Atangana-Baleanu, are frequently applied in various real-world scenarios. These operators are associated with power laws, exponential decay laws, and generalized Mittag-Leffler functions, respectively, and are utilized across disciplines such as engineering, physics, biology, and biomedicine. Fractional derivatives with these operators connect power laws, exponential decay, and extended Mittag-Leffler functions [9-13].

A fractional-order Caputo-Fabrizio derivative has been developed to address singularity and achieve accurate and reliable modeling outcomes. In recent years, this derivative incorporates a non-singular kernel, as proposed by Caputo and Fabrizio. This derivative has been widely used in a variety of applications, including physical systems modeling, signal processing, and financial forecasting [14-18].

In this paper, we introduce a novel chaotic model by utilizing innovative differential and integral operators. Our approach incorporates the newly developed numerical scheme proposed by Atangana and Araz [3], which has demonstrated its effectiveness and utility, as outlined. This method opens up new possibilities for solving novel models, contributing to the application of mathematics in real-world problems. We examine the existence and uniqueness of solutions for the fractional-order chaotic system using the Banach contraction principle. Additionally, the predictor-corrector method and Caputo-Fabrizio numerical approximations are applied to chaotic systems. We calculate Lyapunov exponents, bifurcation diagrams, and various phase portraits for the chaotic system, using Caputo-Fabrizio fractional derivatives, to explore the impact of different derivative orders and parameter values. The proposed numerical method presents a new alternative for generating phase portraits of fractional-order chaotic systems through bifurcation diagrams and Lyapunov exponents. Furthermore, we investigate the stability of equilibrium points in the fractional-order chaotic system using fractional calculus.

Chaotic dynamics of a fractional-order system

In this section, we analyze the chaotic nature of a fractional-order system by investigating key characteristics such as dissipativity, equilibrium points, Lyapunov exponents, and the Kaplan-Yorke dimension.

The chaotic system under consideration is governed by the equations:

$$\begin{aligned} \mathcal{D}_t^{\varrho_1}\varrho(t) &= \eta_1(\gamma(t)-\varrho(t)) \\ \mathcal{D}_t^{\varrho_2}\gamma(t) &= \varrho(t)(\eta_2-\epsilon(t))-\gamma(t) \\ \mathcal{D}_t^{\varrho_3}\epsilon(t) &= \varrho(t)\gamma(t)-\eta_3\epsilon(t) \end{aligned} \tag{1}$$

where η_1, η_2, and η_3 are positive parameters and ϱ_1, ϱ_2, ϱ_3 are the fractional orders.

Equilibrium points and stability

The system has three equilibrium points: the origin, $E_1 = (0,0, 0)$, and two symmetric non-zero points, $E_2 = (X^*, Y^*, Z^*)$ and $E_2 = (–X^*, –Y^*, –Z^*)$. The Jacobian matrix **J** at an equilibrium point $E^* = (X^*, Y^*, Z^*)$ is represented:

$$\mathbf{J} = \begin{bmatrix} -\eta_1 & \eta_1 & 0 \\ \eta_2 - Z^* & -1 & -X^* \\ Y^* & X^* & -\eta_3 \end{bmatrix}$$

The eigenvalues of this matrix determine the stability of the equilibrium points. At the origin E_1, the eigenvalues are: $\lambda_1 \approx -\lambda_a$, $\lambda_2 \approx -\lambda_b$, $\lambda_3 \approx -\eta_3$. For the non-zero points E_2 and E_3, the eigenvalues are: $\lambda_1 \approx -\lambda_c$, λ_2, $\lambda_3 \approx \lambda_d \pm i\lambda_e$, indicating instability in all equilibria. Thus, one obtains the equilibrium points and corresponding eigenvalues in tab. 1.

Table 1. The equilibrium points and corresponding eigenvalues

Equilibrium points	Eigenvalues	Index
$E_1 = (0, 0, 0)$	−22.8277, 11.8277, −8/3	1
$E_2 = (8.4853, 8.4853, 27)$	−13.8546, 0.0940 + 10.1945*i*, 0.0940 − 10.1945*i*	1
$E_3 = (-8.4853,-8.4853, 27)$	−13.8546, 0.0940 + 10.1945*i*, 0.0940 − 10.1945*i*	1

Necessary condition for chaos

A necessary condition for chaotic behavior in the fractional-order system is that the remaining eigenvalues λ in the unstable region satisfy:

$$\gamma > \frac{2}{\pi} \tan^{-1}\left(\frac{|\operatorname{Im}(\lambda)|}{\operatorname{Re}(\lambda)}\right)$$

Applying this condition, we find:

$$\gamma > \frac{2}{\pi} \tan^{-1}\left(\frac{\lambda_e}{\lambda_d}\right) \approx \gamma^*$$

Thus, for $\varrho > \gamma^*$, the fractional-order system meets the conditions for chaotic behavior. For the Caputo fractional system, chaos is ensured if the condition holds:

$$\left(\frac{\pi}{2n}\right) - \min_{1\le j\le 3} |\arg(\lambda_j)| \ge 0$$

Applying this theorem:

$$\gamma > \frac{2}{\pi} \tan^{-1}\left(\frac{10.1945}{0.0940}\right) \approx 0.8676$$

Lyapunov exponents

Using the Danca algorithm [19, 20] with the Adams-Bashforth-Moulton method, the Lyapunov exponents for the fractional-order system are computed. The values of Lyapunov exponents for different fractional orders ϱ presented in tab. 2. The negative sum of the exponents confirms the system's dissipative nature. Thus, one obtains.

Table 2. Lyapunov exponents for different values of ϱ

ϱ	LE1	LE2	LE3
0.7	11.4658	–0.8648	–66.8788
0.8	6.7932	–0.0014	–45.6108
0.9	4.2983	0.0007	–28.7688
0.95	3.3276	–0.0070	–22.4533
0.98	3.1797	–0.0005	–19.6501
1	2.7530	0.0003	–17.6510

Kaplan-Yorke dimension

The Kaplan-Yorke dimension dim(*LE*) is computed from the Lyapunov exponents for different fractional orders:

$$\dim(LE) = 2 + \frac{LE_1 + LE_2}{|LE_3|}$$

For $\varrho = 0.70$:

$$\dim(LE) = 2 + \frac{11.4658 + (-0.8648)}{66.8788} \approx 2.158$$

Similarly, for $\varrho = 1$, we obtain: $\dim(LE) \approx 2.157$. The dimension decreases as ϱ increases, indicating weaker chaotic behavior at higher values of ϱ. This confirms that the fractional-order system exhibits chaotic behavior for $\varrho > \gamma^*$. The values in tab. 2 show that chaos persists in the fractional-order system, with the Kaplan-Yorke dimension providing an upper estimate of the dimensionality of the attractor.

Caputo-Fabrizio model for the given system

Existence and uniqueness

This section describes a Caputo-Fabrizio fractional-order model for the interaction of three variables in a dynamic system. Through the use of the Banach fixed-point theorem, we ensure the existence and uniqueness of solutions, allowing for an exploration of the system's dynamics under fractional differentiation. Future studies may involve numerical simulations to analyze the model's behavior under various initial conditions and parameters. To establish the existence and uniqueness of solutions for this Caputo-Fabrizio fractional-order system, we define the following functions based on the original model equations:

$$g_1(t,\Psi(t)) = \eta_1(\gamma(t) - \varrho(t))$$
$$g_2(t,\Psi(t)) = \varrho(t)(\eta_2 - \epsilon(t)) - \gamma(t)$$
$$g_3(t,\Psi(t)) = \varrho(t)\gamma(t) - \eta_3\epsilon(t)$$

The vector form of the system under initial conditions can be expressed:

$${}_0^{\mathrm{CF}}\mathcal{D}_t^{\delta}\Psi(t) = G(t,\Psi(t)), \ \ \Psi(0) = \Psi_0$$

where

$$\Psi(t) = (\varrho(t), \gamma(t), \epsilon(t)) \ \text{ and } \ \Psi_0 = (\varrho_0, \gamma_0, \epsilon_0)$$

We define the set:

$$[0,T]\times D(\Psi_0,\rho)=\left\{(t,\Psi(t))\in[0,T]\times\mathbb{R}_+^3\,\middle|\,\sup_{t\in T}\|\Psi(t)-\Psi(0)\|_2\le\rho\right\}$$

Using the Caputo derivative properties, the system can be reformulated:

$$\varrho(t)-\varrho_0=\frac{1-\varrho_1}{C(\varrho_1)}(\eta_1(\gamma(t)-\varrho(t)))+\frac{\varrho_1}{\Theta(\varrho_1)C(\varrho_1)}\int_0^t\eta_1(\gamma(\tau)-\varrho(\tau))(t-\tau)^{\varrho_1-1}\mathrm{d}\tau$$

$$\gamma(t)-\gamma_0=\frac{1-\varrho_2}{C(\varrho_2)}(\varrho(t)(\eta_2-\epsilon(t))-\gamma(t))+\frac{\varrho_2}{\Theta(\varrho_2)C(\varrho_2)}\int_0^t(\varrho(\tau)(\eta_2-\epsilon(\tau))-\gamma(\tau))(t-\tau)^{\varrho_2-1}\mathrm{d}\tau$$

$$\epsilon(t)-\epsilon_0=\frac{1-\varrho_3}{C(\varrho_3)}(\varrho(t)\gamma(t)-\eta_3\epsilon(t))+\frac{\varrho_3}{\Theta(\varrho_3)C(\varrho_3)}\int_0^t(\varrho(\tau)\gamma(\tau)-\eta_3\epsilon(\tau))(t-\tau)\varrho^{3-1}\mathrm{d}\tau$$

Taking the limit as n approaches infinity yields the actual solution of the system. The function G is Lipschitz continuous on [0, T] × D(Ψ_0, ρ), meaning there exists a constant $M\in\mathbb{R}_+$ such that for all:

$$(t,\Psi_1(t)),(t,\Psi_2(t))\in[0,T]\times D(\Psi_0,\rho),\ \ \|G(t,\Psi_1(t))-G(t,\Psi_2(t))\|\le M\|\Psi_1(t)-\Psi_2(t)\|$$

Proof.

To show that G satisfies the Lipschitz condition, we evaluate:

$$\|G(t,\Psi_1(t))-G(t,\Psi_2(t))\|=\left|g_1(t,\Psi_1(t))-g_1(t,\Psi_2(t))\right|+\left|g_2(t,\Psi_1(t))-g_2(t,\Psi_2(t))\right|+$$
$$+\left|g_3(t,\Psi_1(t))-g_3(t,\Psi_2(t))\right|\le\eta_1\left|\gamma_2-\gamma_1\right|+\eta_2\left|\varrho_2-\varrho_1\right|+\eta_3\left|\epsilon_2-\epsilon_1\right|=L\|\Psi_1-\Psi_2\|_2$$

where L is a Lipschitz constant. Thus, the Lipschitz condition holds, which implies the existence and uniqueness of solutions for the Caputo-Fabrizio fractional-order system.

Numerical method for the modified Riemann-Liouville derivative applied to a dynamical system

To apply the numerical method for the modified Riemann-Liouville derivative to the given model of fractional differential equations, we start with the system defined The modified Riemann-Liouville derivative for a time-varying order can be applied to each of the equations. Thus, the system can be written:

$${}_0^{RLM}\mathcal{D}_t^{\varrho_1}\varrho(t)=\eta_1(\gamma(t)-\varrho(t))$$
$${}_0^{RLM}\mathcal{D}_t^{\varrho_2}\gamma(t)=\varrho(t)(\eta_2-\epsilon(t))-\gamma(t)$$
$${}_0^{RLM}\mathcal{D}_t^{\varrho_3}\epsilon(t)=\varrho(t)\gamma(t)-\eta_3\epsilon(t)$$

Applying the fundamental theorem of fractional calculus, we obtain:

$$u(t)-u(0)=\frac{1-\varrho(t)}{N(\varrho(t))}h(t,u(t))+\frac{\varrho(t)}{N(\varrho(t))}\int_0^t h(\phi,u(\phi))\mathrm{d}\phi$$

where $N(\varrho(t))$ is a normalization function.

From the equations, we can derive the following expressions for the updates for each variable:

$$\varrho(t_{m+1})-\varrho(0)=\frac{(3-\varrho(t))(1-\varrho(t))}{3}\eta_1(\gamma(t_m)-\varrho(t_m))+\frac{\varrho(t)(3-\varrho(t))}{3}\int_0^{t_{m+1}}\eta_1(\gamma(t)-\varrho(t))\mathrm{d}t$$

$$\gamma(t_{m+1})-\gamma(0)=\frac{(3-\varrho(t))(1-\varrho(t))}{3}\left[\varrho(t_m)(\eta_2-\epsilon(t_m))-\gamma(t_m)\right]+$$
$$+\frac{\varrho(t)(3-\varrho(t))}{3}\int_0^{t_{m+1}}\left[\varrho(t)(\eta_2-\epsilon(t))-\gamma(t)\right]\mathrm{d}t$$

$$\epsilon(t_{m+1})-\epsilon(0)=\frac{(3-\varrho(t))(1-\varrho(t))}{3}\left[\varrho(t_m)\gamma(t_m)-\eta_3\epsilon(t_m)\right]+$$
$$+\frac{\varrho(t)(3-\varrho(t))}{3}\int_0^{t_{m+1}}\left[\varrho(t)\gamma(t)-\eta_3\epsilon(t)\right]\mathrm{d}t$$

Combining these updates, we can express the numerical solution for the system in a more compact form:

$$\varrho_{m+1}=\varrho_m+\left[\frac{(3-\varrho(t))(1-\varrho(t))}{3}+\frac{4h}{6}\varrho(t)(3-\varrho(t))\right]\eta_1(\gamma_m-\varrho_m)-$$
$$-\left[\frac{(3-\varrho(t))(1-\varrho(t))}{3}+\frac{h}{6}\varrho(t)(3-\varrho(t))\right]\eta_1(\gamma_{m-1}-\varrho_{m-1})$$

$$\gamma_{m+1}=\gamma_m+\left[\frac{(3-\varrho(t))(1-\varrho(t))}{3}+\frac{4h}{6}\varrho(t)(3-\varrho(t))\right]\left[\varrho_m(\eta_2-\epsilon_m)-\gamma_m\right]-$$
$$-\left[\frac{(3-\varrho(t))(1-\varrho(t))}{3}+\frac{h}{6}\varrho(t)(3-\varrho(t))\right]\left[\varrho_{m-1}(\eta_2-\epsilon_{m-1})-\gamma_{m-1}\right]$$

$$\epsilon_{m+1}=\epsilon_m+\left[\frac{(3-\varrho(t))(1-\varrho(t))}{3}+\frac{4h}{6}\varrho(t)(3-\varrho(t))\right]\left[\varrho_m\gamma_m-\eta_3\epsilon_m\right]-$$
$$-\left[\frac{(3-\varrho(t))(1-\varrho(t))}{3}+\frac{h}{6}\varrho(t)(3-\varrho(t))\right]\left[\varrho_{m-1}\gamma_{m-1}-\eta_3\epsilon_{m-1}\right]$$

This provides a complete numerical method for solving the system of fractional differential equations with dynamic orders ϱ_1, ϱ_2, ϱ_3 using the modified Riemann-Liouville derivative.

Numerical simulations

To demonstrate the system's dynamics, we will conduct numerical simulations using suitable initial conditions and parameter values. The outcomes will highlight the chaotic nature of the Caputo-Fabrizio Lorenz-type model. Numerical simulations of the Caputo-Fabrizio-Caputo (CFC) fractional chaotic system (1) are illustrated in figs. 1-4 for the following cases: $\varrho(t) = 1$, $\varrho(t) = 0.98$, $\varrho(t) = 0.97 - 0.03 \times \cos(t/10)$, and $\varrho(t) = 0.97 + 0.03 \times \tanh(t/10)$.

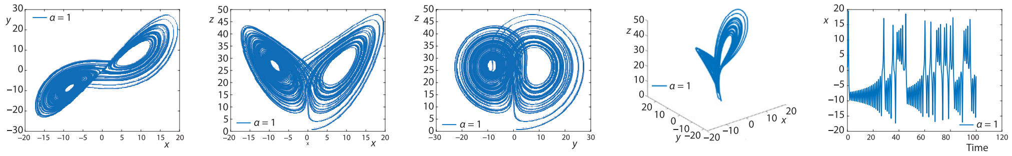

Figure 1. Numerical simulation for chaotic system using Caputo-Fabrizio at $\varrho(t) = 1$

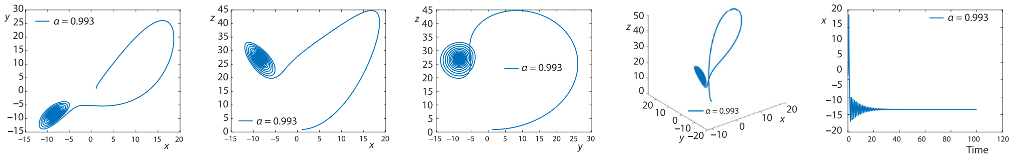

Figure 2. Numerical simulation for chaotic system using Caputo-Fabrizio at $\varrho(t) = 0.98$

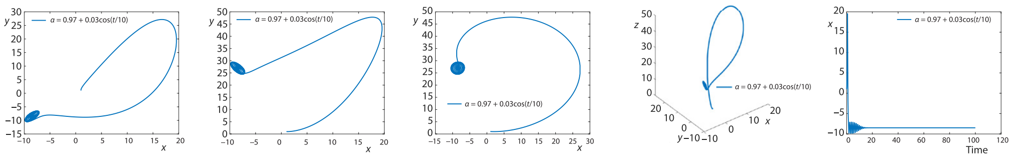

Figure 3. Numerical simulation for chaotic system using Caputo-Fabrizio at $\varrho(t) = 0.97 - 0.03 \times \cos(t/10)$

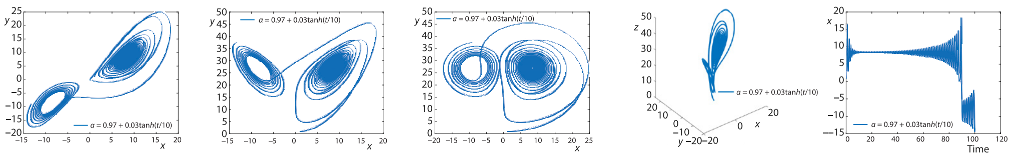

Figure 4. Numerical simulation for chaotic system using Caputo-Fabrizio at $\varrho(t) = 0.97 + 0.03 \times \tanh(t/10)$

Discussion

This study emphasizes the effectiveness of fractional-order calculus, particularly the Caputo-Fabrizio operator, in modelling chaotic systems, which are challenging due to their sensitivity to initial conditions and non-linear dynamics. By utilizing the Caputo-Fabrizio operator, the researchers ensured that the solutions remained bounded and stable, validating the numerical method for real-world chaotic systems. The analysis included Lyapunov exponents, indicating the system's sensitivity, as positive values reflect the exponential divergence of nearby trajectories, a key feature of chaos. The study also investigated equilibrium points, revealing several unstable points that underscore the model's chaotic nature. The findings align with established theories in dynamical systems, particularly through the Kaplan-Yorke dimension, which, being a fractional value, underscores the fractal nature of the system. Given the high non-linearity of chaotic systems, the research calls for the development of more efficient algorithms to reduce computational costs while maintaining accuracy, especially for larger or more complex systems. Additionally, the study notes that real-world chaotic systems often introduce complexities like noise and stochastic behavior. Future research should incorporate these stochastic elements into fractional-order chaotic models to enhance their applicability and understanding in various fields, including meteorology, biology, and engineering.

Conclusion

In this paper, we investigated the Caputo-Fabrizio Lorenz chaotic system, defined through a fractional derivative, and established the existence and uniqueness of its solutions using the Banach fixed-point theorem. We applied a numerical method based on the modified Riemann-Liouville derivative to solve fractional differential equations with dynamic order. The iterative scheme presented provides a practical approach for obtaining approximate solutions in fractional-order systems. Future work could focus on improving the accuracy of the numerical approximation and extending the method to other classes of fractional systems.

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