CONTROLLED CHAOS OF A FRACTAL-FRACTIONAL NEWTON-LEIPNIK SYSTEM

by

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In this study, fractal-fractional derivatives (FFD) with exponential decay laws kernels are applied to explain the chaotic behavior of a Newton-Leipnik system (NLS) with constant and time-varying derivatives. By using Caputo-Fabrizio fractal-fractional derivatives, fixed point theory verifies their existence and uniqueness. Using the implicit finite difference method, the Caputo-Fabrizio (CF) FF NLS is numerically solved. There are several numerical examples presented to illustrate the method's applicability and efficiency. The CF fractal-fractional solutions are more general as compared to classical solutions, as shown in the graphics. Three parameters, three quadratic non-linearity, low complexity time, short iterations per second, a larger step size for the discretized version where chaos is preserved, low cost electronic implementation, and flexibility are some of the unique features that make the suggested chaotic system novel.

Key words: *Newton-Leipnik systems, FFD, chaotic behavior*

Introduction

Chaos control has wide-ranging applications across various disciplines, including communications, electrical systems, computer science, and medicine, as well as numerous physical and circuit-related models of chaotic systems. This broad applicability has captured the attention of researchers, leading to significant interest in the field, as noted in [1-9]. One notable chaotic system is the Newton-Leipnik model, introduced by Newton and Leipnik [1], which serves as a fundamental example of a system exhibiting multiple coexisting attractors. This model is defined by a system of three quadratic differential equations, which generate two strange attractors, with the orbit's initial conditions determining which attractor is followed. Studies of this system, such as numerical analyses and local stability evaluations, have shown that its dynamics are tied to the behavior of an odd-symmetric bimodal map. Subsequent work by Lofaro [2] further explored the bifurcation and dynamics of the NLS, while Wang and Tian

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[3] examined its bifurcation behavior and introduced linear control strategies. Fractional differential equations (FDE) have emerged as powerful tools in addressing non-linear problems across various domains, as highlighted in works such as [3-11]. Their relevance has grown with the need for more precise solutions to variable-order FDE, which require advanced numerical methods [12-14]. These methods have been extended to fractional differential operators with different types of kernels, including power-law, exponential-law, and Mittag-Leffler functions, broadening their applications to a range of complex systems. Recently, attention has shifted towards FFD, given their ability to model real-world phenomena in fields like epidemiology and chaotic systems analysis [15-28]. Building on these advancements, researchers have developed improved numerical techniques for simulating fractional NLS with variable order, particularly using Mittag-Leffler kernels.

In this paper, we propose a generalized numerical scheme for the NLS based on CF FF kernels. This approach utilizes FFD with exponential decay kernels, allowing for more accurate modelling of chaotic systems. A key innovation of this work is the use of the CF derivative, which features a non-singular kernel, offering a highly accurate description of various processes. The implicit solutions are derived and analyzed under different fractional orders, with visual comparisons provided through graphical representations. This extended FF Newton-Leipnik model offers enhanced insight into the chaotic behavior of the system, reflecting the strong memory effects inherent in fractional-order systems. Furthermore, extending the original integer-order model to a variable-order formulation improves its ability to capture memory and hereditary properties. To control the chaotic dynamics, we implement a simple linear controller and perform numerical simulations. Examples are provided to demonstrate the efficiency and accuracy of the proposed method. Additionally, we present a linear state feedback controller designed using Lyapunov stability theory and inverse optimal control principles, illustrating its effectiveness in managing chaos in NLS.

Preliminaries

The classical form of the NLS, which exhibits two strange attractors, is described by the following ODE [3]:

$$\begin{aligned}\dot{u}(t) &= -pu(t)+v(t)+10v(t)w(t)\\ \dot{v}(t) &= -u(t)-0.4v(t)+5u(t)w(t)\\ \dot{w}(t) &= qw(t)-5u(t)v(t)\end{aligned} \tag{1}$$

This system displays chaotic dynamics, with the specific attractor being determined by the initial conditions. By applying fractional calculus, we generalize this system using the CF FF operator (FFO), which results in the form:

$$\begin{aligned}{}^{FF-CF}\mathcal{D}_t^{\gamma,\delta}u(t) &= -pu(t)+v(t)+10v(t)w(t)\\ {}^{FF-CF}\mathcal{D}_t^{\gamma,\delta}v(t) &= -u(t)-0.4v(t)+5u(t)w(t)\\ {}^{FF-CF}\mathcal{D}_t^{\gamma,\delta}w(t) &= qw(t)-5u(t)v(t)\end{aligned} \tag{2}$$

Here, the operator ${}^{\text{FF-CF}}D_t^{\gamma,\delta}$ refers to the CF FFD, defined [22]:

$${}^{FF-CF}\mathcal{D}_t^{\gamma,\delta}g(t) = \frac{N(\gamma)}{1-\gamma}\frac{\mathrm{d}}{\mathrm{d}s^{\delta}}\int_0^t \exp\left(-\frac{\gamma(t-s)}{1-\gamma}\right)g(s)\mathrm{d}s$$

where the derivative of $\delta(\tau)$ with respect to $t^{\delta(t)}$ is expressed as:

$$\frac{\mathrm{d}\delta(\tau)}{\mathrm{d}t^{\delta(t)}} = \lim_{t\to\tau}\frac{\delta(t)-\delta(\tau)}{t^{\delta(t)}-\tau^{\delta(\tau)}}$$

The normalization function $N(\gamma)$ for the CF operator is defined by $N(0) = N(1) = 1$.

To control the chaotic behavior of the system, we introduce a linear feedback term. The controlled CF FF system is then given:

$$\begin{gathered}
{}^{\mathrm{FF-CF}}\mathcal{D}_{0,t}^{\gamma,\delta}u(t) = -pu(t)+v(t)+10v(t)w(t)-1.95\big(u(t)+v(t)+w(t)\big)\\
{}^{\mathrm{FF-CF}}\mathcal{D}_{0,t}^{\gamma,\delta}v(t) = -u(t)-0.4v(t)+5u(t)w(t)\\
{}^{\mathrm{FF-CF}}\mathcal{D}_{0,t}^{\gamma,\delta}w(t) = qw(t)-5u(t)v(t)
\end{gathered} \tag{3}$$

This modified system incorporates a linear state feedback controller, which stabilizes the chaotic dynamics while maintaining the influence of the fractional-order terms.

Existence and uniqueness

In this section, we will establish the existence and uniqueness of the solutions for the system eq. (2). The corresponding matrix representation of eq. (2) is formulated:

$${}^{\mathrm{FF-CF}}\mathcal{D}_{t}^{\alpha,\beta}\mathbf{u}(t) = \mathbf{g}(t,\mathbf{u}(t)) \tag{4}$$

where

$$\mathbf{u}(t) = (\xi(t),\eta(t),\zeta(t)),\ \ \mathbf{u}(0) = (\xi(0),\eta(0),\zeta(0)),\ \text{and}$$
$$\mathbf{g}(t,\mathbf{u}(t)) = (\Gamma_1(t,\mathbf{u}(t)),\Gamma_2(t,\mathbf{u}(t)),\Gamma_3(t,\mathbf{u}(t)))$$

defined:

$$\begin{gathered}
\Gamma_1(t,\xi,\eta,\zeta) = -p\xi+\eta+12\eta\zeta\\
\Gamma_2(t,\xi,\eta,\zeta) = -\xi-0.5\eta+6\xi\zeta\\
\Gamma_3(t,\xi,\eta,\zeta) = q\zeta-6\xi\eta
\end{gathered} \tag{5}$$

We assume that the functions $\xi(t)$, $\eta(t)$, and $\zeta(t)$ remain bounded for all $t \in [0, T]$, such that $\|\xi\|_\infty \le M_\xi$, $\|\eta\|_\infty \le M_\eta$, $\|\zeta\|_\infty \le M_\zeta$. The boundedness of the variables ξ, η, and ζ further implies that Γ_1, Γ_2, and Γ_3 are also bounded. For bounded ξ, η, and ζ, constants M_ξ, M_η, and M_ζ can be found:

$$\sup_{t\in D_\xi}|\xi(t)| = \|\xi\|_\infty \le M_\xi,\ \ \sup_{t\in D_\eta}|\eta(t)| = \|\eta\|_\infty \le M_\eta,\ \ \sup_{t\in D_\zeta}|\zeta(t)| = \|\zeta\|_\infty \le M_\zeta$$

Next, we will demonstrate the linear growth property of the functions as stated in eq. (5):

$$\left|\Gamma_1(t,\xi,\eta,\zeta)\right| \le |p|\sup_{t\in D_\xi}|\xi| + \sup_{t\in D_\eta}|\eta| + 12\|\eta\|_\infty\|\zeta\|_\infty \le |p|M_\xi + M_\eta + 12M_\eta M_\zeta = M_{\Gamma_1} < \infty$$

$$\left|\Gamma_2(t,\xi,\eta,\zeta)\right| \le 6\sup_{t\in D_\xi}|\xi|\sup_{t\in D_\zeta}|\zeta| + 0.5\sup_{t\in D_\eta}|\eta| + \sup_{t\in D_\xi}|\xi| \le M_\xi + 0.5M_\eta + 6M_\xi M_\zeta = M_{\Gamma_2} < \infty$$

$$\left|\Gamma_3(t,\xi,\eta,\zeta)\right| \le 6\sup_{t\in D_\xi}|\xi|\sup_{t\in D_\eta}|\eta| + q\sup_{t\in D_\zeta}|\zeta| \le 6\|\xi\|_\infty\|\eta\|_\infty + q\|\zeta\|_\infty \le 6M_\xi M_\eta + qM_\zeta = M_{\Gamma_3} < \infty$$

Thus, the functions Γ_1, Γ_2, and Γ_3 satisfy the Lipschitz condition with corresponding constants p, 0.5, and q. Contraction occurs if $V < 1$. We will verify the linear growth and Lipschitz conditions for Γ_1, Γ_2, and Γ_3:

$$\left|\Gamma_1(t,\xi,\eta,\zeta)\right|^2 \le 3p^2\left|\xi\right|^2 + 3\left|\eta\right|^2 + 120M_\eta^2 M_\zeta^2 \le K_\xi(1+\left|\xi\right|^2),\ K_\xi = \frac{V^2}{3M_\eta^2} < 1$$

$$\left|\Gamma_2(t,\xi,\eta,\zeta)\right|^2 \le 90M_\xi^2 M_\zeta^2 + 0.5\left|\eta\right|^2 + \left|\xi\right|^2 \le K_\eta(1+\left|\eta\right|^2),\ K_\eta = \frac{0.5}{6M_\xi^2 M_\zeta^2 + M_\xi^2} < 1$$

$$\left|\Gamma_3(t,\xi,\eta,\zeta)\right|^2 \le 90\left|\xi\right|^2\left|\eta\right|^2 + 3q\left|\zeta\right|^2 \le K_\zeta(1+\left|\zeta\right|^2),\ K_\zeta = \frac{3q}{6M_\xi^2 M_\eta^2} < 1$$

Thus, we have:

$$\left|\Gamma_1(t,f_1,\eta,\zeta) - \Gamma_1(t,f_2,\eta,\zeta)\right|^2 \le \bar{K}_\xi\left|f_1 - f_2\right|^2,\ \text{with}\ \bar{K}_\xi = \frac{3}{2}p^2$$

$$\left|\Gamma_2(t,\xi,\eta_1,\zeta) - \Gamma_2(t,\xi,\eta_2,\zeta)\right|^2 \le \bar{K}_\eta\left|\eta_1 - \eta_2\right|^2,\ \text{with}\ \bar{K}_\eta = \frac{3}{2}(0.5)^2$$

$$\left|\Gamma_3(t,\xi,\eta,\zeta_1) - \Gamma_3(t,\xi,\eta,\zeta_2)\right|^2 \le \bar{K}_\zeta\left|\zeta_1 - \zeta_2\right|^2,\ \text{with}\ \bar{K}_\zeta = \frac{3}{2}q^2$$

Fractal-fractional integral with exponential decay kernel

Based on the works of [22], let $f(t)$ be a differentiable function, γ a constant fractional order satisfying $0 < \gamma \le 1$, and $\delta(t)$ a positive continuous function, $\delta(t) > 0$. By consider eq. (4), a novel fractional integral with an exponential decay kernel is defined:

$$^{FF-CF}\mathcal{I}_t^{\gamma,\delta(t)} f(t) = \\ = \frac{1-\gamma}{N(\gamma)} t^{\delta(t)}\left[\delta'(t)\ln(t) + \frac{\delta(t)}{t}\right] f(t) + \frac{\gamma}{N(\gamma)}\int_0^t f(\tau)\left[\delta'(\tau)\ln(\tau) + \frac{\delta(\tau)}{\tau}\right]\tau^{\delta(\tau)}\mathrm{d}\tau \tag{6}$$

Utilizing the defined fractional integral with the exponential decay kernel, we can express eq. (6) in the form:

$$f(t) = \frac{1-\gamma}{N(\gamma)} t^{\delta(t)}\left[\delta'(t)\ln(t) + \frac{\delta(t)}{t}\right] g(t, f(t)) + \\ + \frac{\gamma}{N(\gamma)}\int_0^t g(\tau, f(\tau))\left[\delta'(\tau)\ln(\tau) + \frac{\delta(\tau)}{\tau}\right]\tau^{\delta(\tau)}\mathrm{d}\tau \tag{7}$$

At the point $t_{n+1} = (n+1)\Delta t$, eq. (7) simplifies:

$$f(t_{n+1}) = \frac{1-\gamma}{N(\gamma)} t_n^{\delta(t_n)}\left[\frac{\delta(t_{n+1}) - \delta(t_n)}{\Delta t}\ln t_n + \frac{\delta(t_n)}{t_n}\right] g(t_n, f^n) + \\ + \frac{\gamma}{N(\gamma)}\int_0^{t_{n+1}} g(\tau, f(\tau))\left[\delta'(\tau)\ln(\tau) + \frac{\delta(\tau)}{\tau}\right]\tau^{\delta(\tau)}\mathrm{d}\tau \tag{8}$$

To streamline eq. (8), we define:

$$h(\tau, f(\tau)) = g(\tau, f(\tau))\left[\delta'(\tau)\ln(\tau) + \frac{\delta(\tau)}{\tau}\right]\tau^{\delta(\tau)} \tag{9}$$

Taking the difference of eq. (8) at the points $t_{n+1} = (n + 1)\Delta t$ and $t_n = n\Delta t$ we have:

$$f(t_{n+1}) = f(t_n) + \frac{1-\gamma}{N(\gamma)} t_n^{\delta(t_n)} \left[\frac{\delta(t_{n+1}) - \delta(t_n)}{\Delta t} \ln t_n + \frac{\delta(t_n)}{t_n} \right] g(t_n, f^n) - \\ - \frac{1-\gamma}{N(\gamma)} t_{n-1}^{\delta(t_{n-1})} \left[\frac{\delta(t_n) - \delta(t_{n-1})}{\Delta t} \ln t_{n-1} + \frac{\delta(t_{n-1})}{t_{n-1}} \right] g(t_{n-1}, f^{n-1}) + \frac{\gamma}{N(\gamma)} \int_{t_n}^{t_{n+1}} h(\tau, f(\tau)) \mathrm{d}\tau \tag{10}$$

Incorporating the Lagrange polynomial into eq. (9), we can reframe eq. (10):

$$\tilde{f}_{n+1} = \tilde{f}_n + \frac{1-\gamma}{N(\gamma)} t_n^{\delta(t_n)} \left[\frac{\delta(t_{n+1}) - \delta(t_n)}{\Delta t} \ln t_n + \frac{\delta(t_n)}{t_n} \right] g(t_n, f^n) - \frac{1-\gamma}{N(\gamma)} t_{n-1}^{\delta(t_{n-1})} \cdot \\ \cdot \left[\frac{\delta(t_n) - \delta(t_{n-1})}{\Delta t} \ln t_{n-1} + \frac{\delta(t_{n-1})}{t_{n-1}} \right] g(t_{n-1}, f^{n-1}) + \frac{\gamma}{N(\gamma)} \int_{t_n}^{t_{n+1}} \left\{ \begin{array}{l} \frac{h(t_n, f^n)}{\Delta t} (\tau - t_{n-1}) \\ - \frac{h(t_{n-1}, f^{n-1})}{\Delta t} (\tau - t_n) \end{array} \right\} \mathrm{d}\tau \tag{11}$$

We can rearrange eq. (11):

$$\tilde{f}_{n+1} = \tilde{f}_n + \frac{1-\gamma}{N(\gamma)} t_n^{\delta(t_n)} \left[\frac{\delta(t_{n+1}) - \delta(t_n)}{\Delta t} \ln t_n + \frac{\delta(t_n)}{t_n} \right] g(t_n, f^n) - \frac{1-\gamma}{N(\gamma)} t_{n-1}^{\delta(t_{n-1})} \cdot \\ \cdot \left[\frac{\delta(t_n) - \delta(t_{n-1})}{\Delta t} \ln t_{n-1} + \frac{\delta(t_{n-1})}{t_{n-1}} \right] g(t_{n-1}, f^{n-1}) + \frac{\gamma}{N(\gamma)} \frac{h(t_n, f^n)}{\Delta t} \int_{t_n}^{t_{n+1}} (\tau - t_{n-1}) \mathrm{d}\tau - \\ - \frac{\gamma h(t_{n-1}, f^{n-1})}{N(\gamma)\Delta t} \int_{t_n}^{t_{n+1}} (\tau - t_n) \mathrm{d}\tau \tag{12}$$

Calculating the integrals in eq. (12) gives:

$$\int_{t_n}^{t_{n+1}} (\tau - t_{n-1}) \mathrm{d}\tau = \frac{3(\Delta t)^2}{2}, \quad \int_{t_n}^{t_{n+1}} (\tau - t_n) \mathrm{d}\tau = \frac{(\Delta t)^2}{2} \tag{13}$$

Substituting eq. (13) back into eq. (12) yields the approximation:

$$\tilde{f}_{n+1} = \tilde{f}_n + \frac{1-\gamma}{N(\gamma)} t_n^{\delta(t_n)} \left[\frac{\delta(t_{n+1}) - \delta(t_n)}{\Delta t} \ln t_n + \frac{\delta(t_n)}{t_n} \right] g(t_n, f^n) - \frac{1-\gamma}{N(\gamma)} t_{n-1}^{\delta(t_{n-1})} \cdot \\ \cdot \left[\frac{\delta(t_n) - \delta(t_{n-1})}{\Delta t} \ln t_{n-1} + \frac{\delta(t_{n-1})}{t_{n-1}} \right] g(t_{n-1}, f^{n-1}) + \frac{3\gamma}{4N(\gamma)} h(t_n, f^n)\Delta t + \frac{\gamma}{4N(\gamma)} h(t_{n-1}, f^{n-1})\Delta t \tag{14}$$

Thus, we arrive at the final approximation:

$$\tilde{f}_{n+1} = \tilde{f}_n + \frac{1-\gamma}{N(\gamma)} t_n^{\delta(t_n)} \left[\frac{\delta(t_{n+1}) - \delta(t_n)}{\Delta t} \ln t_n + \frac{\delta(t_n)}{t_n} \right] g(t_n, f^n) - \frac{1-\gamma}{N(\gamma)} t_{n-1}^{\delta(t_{n-1})} \cdot \\ \cdot \left[\frac{\delta(t_n) - \delta(t_{n-1})}{\Delta t} \ln t_{n-1} + \frac{\delta(t_{n-1})}{t_{n-1}} \right] g(t_{n-1}, f^{n-1}) + \frac{3\gamma}{4N(\gamma)} h(t_n, f^n)\Delta t + \frac{\gamma}{4N(\gamma)} h(t_{n-1}, f^{n-1})\Delta t$$

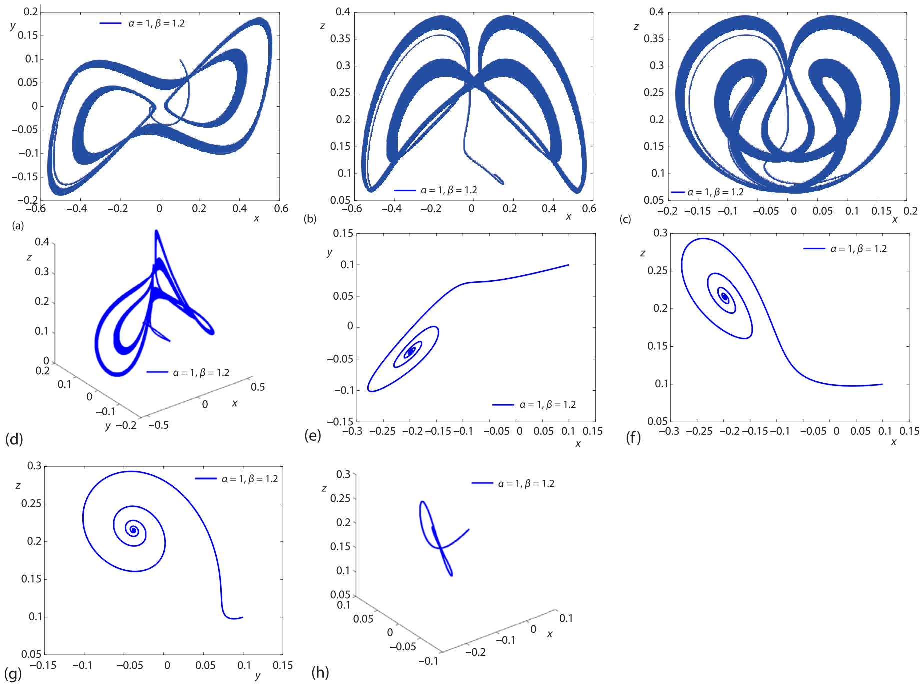

Figure 1. At $\alpha = 1$ and $\beta = 1.2$, figs. 1(a)-1(d) illustrate the synchronization of the original system, while figs. 1(e)-1(h) depict the synchronization of the controlled system

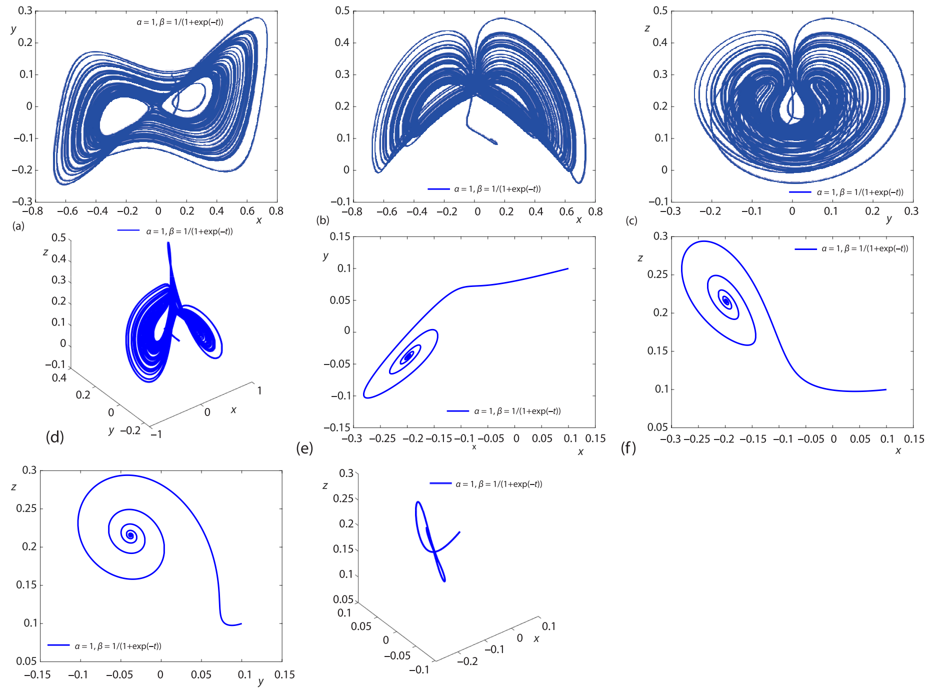

Figure 2. Comparison between the synchronization of the CF FF system (2), resp. control system (3), illustrated in figs. 2(a)-2(d), resp. figs. 2(e)-2(h), for $\alpha = 1$ and $\beta = \tanh(1 + t)$

Numerical simulation

To generate the numerical results for each compartment of the model, we created a numerical scheme. The MATLAB was used to analyze the results for various fractional orders and fractal dimensions. Comparison between the synchronization of the CF fractal-fractional system (2), resp. control system (3), illustrated in figs. 1(a)-1(d), resp. figs. 1(e)-1(h), for $\alpha = 1$, $\beta = 1.2$, see fig. 1. Comparison between the synchronization of the CF fractal-fractional system (2), resp. control system (3), illustrated in figs. 2(a)-2(c), resp. figs. 2(d)-2(f), for $\alpha = 1$, $\beta = \tanh(1 + t)$, see fig. 2.

Discussion

This paper applies CF FFO to analyze chaotic behaviors in the NLS, utilizing a FF calculus approach to enhance the analysis of chaotic systems. It explores chaos theory, focusing on systems highly sensitive to initial conditions and strange attractors. The study highlights the advantage of CF operators in modelling complex behaviors and memory effects, which are limitations of classical calculus. Using fixed-point theorems (Schauder and Banach), the authors establish the existence and uniqueness of solutions, ensuring the observed chaos is intrinsic to the system. Numerical simulations conducted in MATLAB explore system dynamics across different fractional orders, visually validating the theoretical findings and demonstrating how parameter variations affect the system. Additionally, the paper discusses control techniques for managing Newton-Leipnik chaos.

Conclusion

This study models the time-varying NLS using CF FFD with non-singular kernels. It combines qualitative analysis (existence, stability) and quantitative analysis (numerical solutions, simulations), proving solution uniqueness with fixed-point theorems. By using FF calculus, the research offers a refined approach to modelling and controlling chaotic systems, advancing chaos theory.

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