ANALYZING FRACTIONAL ORDER VARIABLE COEFFICIENTS HEAT MODEL

by

Mawahib ELAMIN[a], Khadeeja A. A. HELAL[b], Sayed ABDEL-KHALEK[c], Ali MUBARAKI[c], Tahani M. ALBOGAMI[d], and Emad A.-B. ABDEL-SALAM[e*]

[a] Department of Mathematics, College of Science, Qassim University, Buraydah, Saudi Arabia
[b] Department of Mathematics, Faculty of Science, Al-Baha University, Al Bahah, Saudi Arabia
[c] Department of Mathematics and Statistics, College of Science, Taif University, Taif, Saudi Arabia
[d] College of Sciences and Human Studies, Prince Mohammad Bin Fahd University, University, Al-Khobar, Saudi Arabia
[e] Department of Mathematics, Faculty of Science, New Valley University, ElKharja, Egypt

Original scientific paper
https://doi.org/10.2298/TSCI2406143E

The manuscript's primary goal is to utilize the decomposition Adomian approach to approximate solutions for a specific class of space-time fractional order heat model characterized by variable coefficients and appropriate initial values. This method allows for the computation of a power series representation of the solution without the need for linearization, assumptions about weak non-linearity, or reliance on perturbation theory. By employing mathematical software like MATHEMATICA or Maple, the Adomian formulas are employed to evaluate the resulting series solution. Furthermore, this approach shows promise in addressing various types of fractional order non-linear mathematical physics models. The analysis reveals a remarkable convergence between the outcomes derived from the decomposition method utilizing infinite series and the well-established results obtained when the fractional order equals one. This convergence underscores the efficacy and accuracy of the decomposition method in approximating solutions for fractional order equations, particularly when the fractional order approaches unity. Such alignment between the decomposition method's results and those derived from conventional approaches bolsters confidence in its utility and reliability, further solidifying its standing as a valuable tool in the realm of fractional calculus and applied mathematics. Notably, the obtained results reveal that the solution's profile changes based on varying fractional orders. This indicates that the shape of the solution wave can be altered without introducing additional parameters. These findings have far-reaching implications across numerous applications within specific contexts, suggesting the potential for significant advancements in understanding and addressing complex physical phenomena governed by fractional order equations.

Key words: *non-linear fractional models, conformable fractional derivative, variable coefficients wave and heat models*

Introduction

In various domains spanning physics, engineering, mathematics, and beyond, a multitude of physical and natural phenomena exhibit intricate behaviors that find successful representation through the versatile frameworks of integer and fractional calculus. These non-linear

* Corresponding author, e-mail: emad_abdelsalam@yahoo.com

fractional models hold significant allure across an extensive spectrum of disciplines, extending their influence far beyond traditional scientific realms. From the complexities of financial markets to the intricacies of signal processing, from the dynamics of economies to the mysteries of celestial bodies in astronomy, fractional calculus offers invaluable insights and tools for analysis. Moreover, its applications extend into diverse fields such as acoustics, medical processes, biological systems, fluid dynamics, genetics, and even the intricate workings of the human mind. This broad appeal underscores the profound impact and universal relevance of fractional calculus in elucidating and navigating the complexities of the natural and engineered world [1-3].

Exploring non-linear integer differential models has emerged as a prominent and dynamic area of research, capturing considerable attention and interest within the scientific community, as evidenced by numerous recent studies [4-10]. Amidst this landscape, one notable category of mathematical models that has garnered significant focus is that of linear fractional models, among which are the well-studied heat model [3, 6]. This model finds relevance in diverse real-world phenomena, ranging from the stresses induced by earthquakes to the propagation of non-homogeneous elastic waves within soils, both of which can be effectively described by wave equations. Examples include the gradual decay of extended current loops and various observable phenomena in flat superconducting cables exposed to magnetic fields that vary over time [3, 6].

The fractional heat equation finds a wide range of applications across various fields due to its ability to describe the diffusion of heat over time and space. Some prominent applications include: Firstly, in heat conduction science the fractional heat equation is extensively used in engineering and physics to model heat conduction in materials. It describes how heat flows through solids, liquids, and gases, influencing phenomena such as temperature distributions in objects and thermal conductivity. Secondly, in thermal engineering and material science, the fractional heat equation is employed to analyze heat transfer processes, design cooling systems, and predict temperature distributions in various structures and devices. Thirdly, heat transfer in the Earth's crust is a crucial aspect of geothermal exploration. The fractional heat equation is utilized to model the distribution of heat within the Earth's subsurface, aiding in the assessment of geothermal energy potential and the design of geothermal energy extraction systems. Fourthly, the fractional heat equation is utilized in atmospheric science to study temperature distributions in the atmosphere, ocean, and land surfaces. It plays a key role in numerical weather prediction models and climate simulations, helping scientists understand climate dynamics and predict future climate scenarios. Fifthly, in the field of medical imaging, the fractional heat equation is applied in techniques such as magnetic resonance imaging and thermography to reconstruct images of internal body structures based on thermal properties and temperature distributions. These are just a few examples of the myriad applications of the fractional heat equation. Its versatility and effectiveness in modelling heat transfer phenomena have led to its widespread adoption in numerous scientific and engineering disciplines [7-14].

Since its emergence in the 1980's, the Adomian decomposition method has been show cased as a versatile tool for tackling a diverse array of mathematical challenges. Its applicability extends across both linear and non-linear realms, encompassing ordinary and partial differential equations as well as integral equations. One of its notable features is its ability to furnish solutions in the form of infinite series, often converging to highly accurate results. In recent years, researchers have further explored its potential by applying it to the study of vibrations in various structural and mechanical systems. This includes analyses of beams, strings, and other complex configurations operating in two or three spatial dimensions. The method's adaptability and efficacy in addressing such multifaceted problems underscore its significance as a valuable computational technique in the realm of applied mathematics and engineering [15, 16].

Over the past two decades, there has been a notable surge of interest among scientists and researchers in the quest for numerical and analytical solutions to fractional differential equations (FDE). This burgeoning field has attracted the attention of numerous scholars, each contributing unique insights and methodologies to tackle this challenging area of study. Many researchers have collectively contributed to advancing the understanding and techniques for solving FDE, paving the way for applications in various fields such as physics, engineering, biology, and finance. Their efforts have enriched the scientific community's toolkit for tackling complex fractional order phenomena and have spurred further exploration and innovation in this rapidly evolving area of research [16-24].

Utilizing the constraints, Khalil *et al.* [25] introduced the conformable fractional derivative (CFD):

$$D^{\alpha}\psi\left(t\right)=\lim_{\varepsilon\to 0}\frac{\psi\left(t+\varepsilon t^{1-\alpha}\right)-\psi\left(t\right)}{\varepsilon}\,\forall t>0,\ \alpha\in\left(0,1\right] \tag{1}$$

$$\psi^{(\alpha)}\left(0\right)=\lim_{t\to 0^{+}}\psi^{(\alpha)}\left(t\right) \tag{2}$$

When setting $\alpha = 1$ in the preceding equations, the non-integer derivative simplifies to the familiar integer derivative. The CFD exhibits the properties:

$$D^{\alpha}t^{m}=mt^{m-\alpha},\ m\in R,\ D^{\alpha}c=0,\ \forall\,\psi\left(t\right)=c \tag{3}$$

$$D^{\alpha}\left(a\psi+b\varphi\right)=aD^{\alpha}\psi+bD^{\alpha}\varphi,\ \forall a,b\in R \tag{4}$$

$$D^{\alpha}\left(\varphi\psi\right)=\varphi D^{\alpha}\psi+\psi D^{\alpha}\varphi \tag{5}$$

$$D^{\alpha}\left(\frac{\varphi}{\psi}\right)=\frac{\psi D^{\alpha}\varphi-\varphi D^{\alpha}\psi}{\psi^{2}} \tag{6}$$

$$D^{\alpha}\psi\left(\varphi\right)=\frac{\mathrm{d}\psi}{\mathrm{d}\varphi}D^{\alpha}\varphi,\ D^{\alpha}\psi\left(t\right)=t^{1-\alpha}\frac{\mathrm{d}\psi}{\mathrm{d}t} \tag{7}$$

where φ, ψ be two α is the differentiable functions of the dependent variable t and c – the arbitrary constant. Equations (5)-(7) were demonstrated by Khalil *et al.* [25]. The CFD of certain functions:

$$\begin{gathered}D_{s}^{\alpha}\mathrm{e}^{ct}=cs^{1-\alpha}\mathrm{e}^{cs},\ D_{s}^{\alpha}\sin(cs)=cs^{1-\alpha}\cos(cs),\ D_{s}^{\alpha}\cos(cs)=-cs^{1-\alpha}\sin(cs)\\ D_{s}^{\alpha}\mathrm{e}^{cs^{\alpha}}=c\alpha\mathrm{e}^{cs^{\alpha}},\ D_{s}^{\alpha}\sin(cs^{\alpha})=c\alpha\cos(cs^{\alpha}),\ D_{s}^{\alpha}\cos(cs^{\alpha})=-c\alpha\sin(cs^{\alpha})\end{gathered} \tag{8}$$

In this paper, our focus lies in investigating approximate solutions for a specific category of variable coefficients heat equation characterized by space and time-fractional orders, utilizing the concept of CFD. The methodology employed for constructing these solutions predominantly relies on the Adomian decomposition technique. By leveraging this approach, we aim to provide insights into the behavior and characteristics of the solutions to these FDE, shedding light on their dynamics and implications in various scientific and engineering contexts. Through our analysis, we endeavor to contribute to the advancement of understanding and techniques in the realm of fractional calculus and its applications.

Adomian decomposition procedure

Without losing generality, we assume a space and time fractional variable coefficients heat equation given by a (3+1)-D initial boundary value problem (IBVP) with different fractional orders α, β, γ, and δ of the form:

$$D_\tau^\alpha M = f(s,u,v) D_s^{\beta\beta} M + g(s,u,v) D_u^{\gamma\gamma} M + h(s,u,v) D_v^{\delta\delta} M$$
$$0 < s < a,\ 0 < u < b,\ 0 < v < c,\ \tau > 0,\ 0 < \alpha, \beta, \gamma,\ \delta \le 1 \tag{9}$$

the boundary conditions:

$$\begin{aligned} M(0,u,v,\tau) &= f_1(s,u,v),\ M(a,u,v,\tau) = f_2(s,u,v) \\ M(s,0,v,\tau) &= g_1(s,u,v),\ M(s,b,v,\tau) = g_2(s,u,v) \\ M(s,u,0,\tau) &= h_3(s,u,v),\ M(s,u,c,\tau) = h_3(s,u,v) \end{aligned} \tag{10}$$

with the initial condition:

$$M(s,u,v,0) = \varphi(s,u,v) \tag{11}$$

The operator form of eq. (9) can be written:

$$L_{\alpha_\tau} M = f(s,u,v) D_s^{\beta\beta} M + g(s,u,v) D_u^{\gamma\gamma} M + h(s,u,v) D_v^{\delta\delta} M \tag{12}$$

with

$$L_{\alpha_\tau} = \frac{\partial^\alpha}{\partial \tau^\alpha} \tag{13}$$

The inverse fractional operator $L_{\alpha_\tau}^{-1}$ is then written:

$$L_{\alpha_\tau}^{-1}(.) = \int_0^\tau (.)\,\mathrm{d}x^\alpha \tag{14}$$

Applying $L_{\alpha_\tau}^{-1}$ on eq. (12), we obtain:

$$M(s,u,v,\tau) = \varphi(s,u,v) + L_{\alpha_\tau}^{-1}\left(f(s,u,v) D_s^{\beta\beta} M + g(s,u,v) D_u^{\gamma\gamma} M + h(s,u,v) D_v^{\delta\delta} M\right) \tag{15}$$

The Adomian decomposition method stands as a powerful tool for obtaining analytical solutions to integer-order differential equations, playing a pivotal role across various scientific disciplines. Notably, its versatility and effectiveness have been demonstrated in numerous applications, as evidenced by the works of Wazwaz and Goruis [26], among others. Through meticulous decomposition and iterative procedures, this method allows for the systematic approximation of solutions to differential equations, providing valuable insights into the behavior and dynamics of the underlying systems. As such, it remains a cornerstone technique in the arsenal of mathematical tools utilized by researchers and practitioners across a wide array of fields, facilitating advancements in theory and practical applications alike. The decomposition procedure depends on acting the solution $M(s, u, v, \tau)$ by the decomposition series:

$$M(s,u,v,\tau) = \sum_{n=0}^{\infty} M_n(s,u,v,\tau) \tag{16}$$

where the components $M(s, u, v, \tau)$ of the solution $M(s, u, v, \tau)$ will be computed recursively. After substituting eq. (16) into both sides of eq. (15), we have:

$$\sum_{m=10}^{\infty} M_n(s,u,v,\tau) = \varphi(s,u,v) + L_{\alpha_\tau}^{-1}\left(fD_s^{\beta\beta}\left(\sum_{n=10}^{\infty} M_n\right) + gD_u^{\gamma\gamma}\left(\sum_{n=10}^{\infty} M_n\right)_{uu} + hD_v^{\delta\delta}\left(\sum_{n=10}^{\infty} M_n\right)_{vv}\right) \tag{17}$$

The decomposition procedure determines the zeroth component $M_0(s, u, v, \tau)$ through all terms stemming from the initial condition and integrating the source term, the Adomian decomposition procedure formally introduces the use of the recursive relation compute the components $M_n(s, u, v, \tau)$, $n \geq 0$ of the solution $M(s, u, v, \tau)$:

$$\begin{gathered} M_0(s,u,v,\tau) = \varphi(s,u,v,\tau) \\ M_{k+1}(s,u,v,\tau) = L_{\alpha_\tau}^{-1}\left(fD_s^{\beta\beta}M_k + gD_u^{\gamma\gamma}M_k + hD_v^{\delta\delta}M_k\right), \quad k \geq 0 \end{gathered} \tag{18}$$

subsequently, we can frequently calculate every component of $\sum_{n=0}^{\infty} M_n$. Thus, the series representation of M_s solution can be easily articulated. Notably, it has been noted that the iterative solution rapidly converges to the exact solution, assuming it exists.

It is worth noting that the arbitrary selection of fractional orders α, β, γ, and δ offers a considerable degree of flexibility and may lead to the discovery of more intricate structures within the models. This flexibility allows us to tailor the models to better match the physical phenomena under consideration, potentially leading to more accurate and insightful explanations. Additionally, the series solution obtained by incorporating only the initial conditions highlights an intriguing aspect of the problem-solving process. While the given boundary conditions are instrumental for validating the solution, the fact that the series solution emerges primarily from the initial conditions underscores their importance in shaping the solution trajectory. This approach underscores the significance of carefully considering initial conditions in modelling and analyzing complex systems, as they can profoundly influence the behavior and outcomes of the solutions obtained.

In the subsequent sections, we employ the outlined procedure to address particular instances of time-fractional heat equation. By applying the methodology described earlier, we aim to obtain solutions tailored to these specific equations. Through this process, we seek to elucidate the behavior and characteristics of these FDE, shedding light on their dynamics and implications within the context of heat phenomena. This targeted approach enables us to explore the unique features of time-fractional equations and their significance in various scientific and engineering domains.

Variable coefficients heat equations

We utilize the aforementioned procedure to tackle the heat equations characterized by four space and time fractional variable coefficients. By applying the methodology outlined earlier, we aim to derive solutions tailored specifically to these equations. This implementation allows us to explore the behavior and properties of heat propagation in systems characterized by such fractional variable coefficients. Through this analysis, we seek to gain insights into the intricate dynamics of heat transfer phenomena, particularly in contexts where the coefficients exhibit spatial and temporal variability.

First application

We first consider the (1+1)-D heat time fractional IBVP equation:

$$D_\tau^\alpha M = \frac{1}{2}s^2 D_s^{\beta\beta} M, \; 0 < s < 1, \; \tau > 0 \tag{19}$$

with the boundary conditions:

$$M\left(0,\tau^\alpha\right) = 0, \; M\left(1,\tau^\alpha\right) = \mathrm{e}^{\frac{\beta^2\tau^\alpha}{\alpha}} \tag{20}$$

at the initial value:

$$M(s,0) = s^{2\beta} \tag{21}$$

Equation (19) has the following form:

$$L_{\alpha_\tau} M = \frac{1}{2} s^{2\beta} D_s^{\beta\beta} M, \ \ 0 < s < 1 \tag{22}$$

Applying the inverse operator $L_{\alpha_\tau}^{-1}$ to eq. (22) while considering the initial value yields:

$$M\left(s,\tau\right) = s^{2\beta} + L_{\alpha_\tau}^{-1}\left(\frac{1}{2} s^{2\beta} D_s^{\beta\beta} M\right) \tag{23}$$

Substituting eq. (16) for M into eq. (23) yields:

$$\sum_{n=0}^{\infty} M_n\left(s,\tau\right) = s^{2\beta} + L_{\alpha_\tau}^{-1}\left(\frac{1}{2} s^{2\beta} D_s^{\beta\beta}\left(\sum_{n=0}^{\infty} M_n\right)\right) \tag{24}$$

so, the recurrence relation is given:

$$M_0 = s^{2\beta}, M_{k+1} = L_{\alpha_\tau}^{-1}\left(\frac{1}{2} s^{2\beta} D_s^{\beta\beta}\left(M_k\right)\right), \qquad k \geq 0 \tag{25}$$

the series form of the solution is given

$$M\left(s,\tau\right) = s^{2\beta}\left(1 + \frac{\beta^2}{\alpha}\tau^{\alpha} + \frac{\beta^4}{\alpha^2 2!}\tau^{2\alpha} + \frac{\beta^6}{\alpha^3 3!}\tau^{3\alpha} + \frac{\beta^8}{\alpha^4 4!}\tau^{4\alpha} + \ldots\right) = s^{2\beta} \mathrm{e}^{\frac{\beta^2 \tau^{\alpha}}{\alpha}} \tag{26}$$

When $\alpha = \beta = 1$, eq. (26) has the form:

$$M\left(s,\tau\right) = s^2\left(1 + \tau + \frac{1}{2!}\tau^2 + \frac{1}{3!}\tau^3 + \frac{1}{4!}\tau^4 + \ldots\right) = s^2 \mathrm{e}^{\tau}$$

Wazwaz and Goruis [28] give the same results.

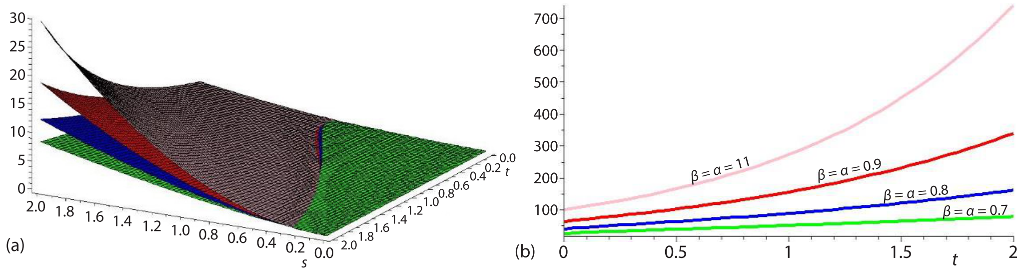

Figure 1. Represented the solution eq. (26) when $\alpha = \beta = 1$, 0.9, 0.8, 0.7 and (b) the cross-section at $s = 10$

Figure 1 illustrates the evolutionary behavior of the solution eq. (26) when $\alpha = \beta$ at different values of the fractional orders, the pink layer when $\alpha = \beta = 1$, the red layer when $\alpha = \beta = 0.9$, the blue layer when $\alpha = \beta = 0.8$, and the green layer when $\alpha = \beta = 0.7$. Figure 1(b) is the cross-section at $s = 10$. In fig. 1, we observe a distinct shift in solution shape as we alter the fractional order. This dynamic variation in solution shape, driven by changes in fractional order, holds significant implications for the development of robust signal processing methodologies tailored to handle the intricate nature of real-world signals. Signals encountered in various domains, including biomedical, seismic, and financial sectors, often exhibit non-stationary behavior, wherein their statistical attributes evolve over time. Conventional techniques relying on integer-order differential equations may fall short in capturing the intricate dynamics of such signals adequately. Embracing fractional-order dynamics equips signal processing methodologies with enhanced capabilities to effectively model and analyze non-stationary signals. Fractional calculus serves as a potent tool for characterizing signal memory and long-range dependence characteristics, paving the way for the creation of more precise models and algorithms. These advancements contribute significantly to signal denoising, feature extraction, and classification endeavors, enabling more robust and accurate signal processing solutions.

Second application

We consider the (2+1)-D space and time fractional IBVP:

$$D_\tau^\alpha M = \frac{1}{2}\left(s^{2\beta} D_s^{\beta\beta} M + u^{2\gamma} D_u^{\gamma\gamma} M\right),\ 0<s<1,\quad 0<u<1,\ \ \tau>0 \tag{27}$$

with boundary values:

$$\begin{gathered} M(0,u,\tau) = u^{2\gamma} \mathrm{e}^{\frac{\gamma^2}{\alpha}\tau^\alpha},\ \ M(1,u,\tau) = u^{2\gamma}\mathrm{e}^{\frac{\gamma^2}{\alpha}\tau^\alpha} + \mathrm{e}^{\frac{\beta^2}{\alpha}\tau^\alpha} \\ M(s,0,\tau) = s^{2\beta}\mathrm{e}^{\frac{\beta^2}{\alpha}\tau^\alpha},\ \ M(s,1,\tau) = s^{2\beta}\mathrm{e}^{\frac{\beta^2}{\alpha}\tau^\alpha} + \mathrm{e}^{\frac{\gamma^2}{\alpha}\tau^\alpha} \end{gathered} \tag{28}$$

at the initial value:

$$M(s,u,0) = s^{2\beta} + u^{2\gamma} \tag{29}$$

so we have:

$$L_{\alpha_\tau} M = \frac{1}{2}\left(s^{2\beta} D_s^{\beta\beta} M + u^{2\gamma} D_u^{\gamma\gamma} M\right),\ 0<s<1,\ 0<u<1,\ \tau<0 \tag{30}$$

Applying the inverse operator $L_{\alpha_\tau}^{-1}$ to eq. (30) while considering the initial value yields:

$$M(s,u,\tau) = s^{2\beta} + u^{2\gamma} + L_{\alpha_\tau}^{-1}\left(\frac{1}{2}\left(s^{2\beta} D_s^{\beta\beta} M + u^{2\gamma} D_u^{\gamma\gamma} M\right)\right) \tag{31}$$

the decomposition series form of eq. (31) is given

$$\sum_{n=0}^{\infty} M_n(s,u,\tau) = s^{2\beta} + u^{2\gamma} + \frac{1}{2} L_{\alpha_\tau}^{-1}\left(s^{2\beta} D_s^{\beta\beta}\left(\sum_{n=10}^{\infty} M_n\right) + u^{2\gamma} D_u^{\gamma\gamma}\left(\sum_{n=10}^{\infty} M_n\right)\right) \tag{32}$$

thus the recurrence relation is written:

$$M_0 = s^{2\beta} + u^{2\gamma}, M_{k+1} = \frac{1}{2} L_{\alpha_\tau}^{-1}\left(s^{2\beta} D_s^{\beta\beta}(M_k) + u^{2\gamma} D_u^{\gamma\gamma}(M_k)\right),\quad k \ge 0 \tag{33}$$

Hence the series representation of the solution is expressed:

$$\begin{gathered} M = s^{2\gamma}\left(1 + \frac{\beta^2}{\alpha}\tau^\alpha + \frac{\beta^4}{\alpha^2 2!}\tau^{2\alpha} + ...\right) + u^{2\beta}\left(1 + \frac{\gamma^2}{\alpha}\tau^\alpha + \frac{\gamma^4}{\alpha^2 2!}\tau^{2\alpha} + ...\right) = \\ = s^{2\beta} e^{\frac{\beta^2}{\alpha}\tau^\alpha} + u^{2\gamma}\mathrm{e}^{\frac{\gamma^2}{\alpha}\tau^\alpha} \end{gathered} \tag{34}$$

when $\alpha = \beta = 1$, eq. (34) turns to the integer version

$$M = s^2\left(1 + \tau + \frac{1}{2!}\tau^2 + ...\right) + u^2\left(1 + \tau^\alpha + \frac{1}{2!}\tau^2 + ...\right) = \left(s^2 + u^2\right)\mathrm{e}^\tau$$

Third application

We assume the (2+1)-D space and time fractional IBVP as the third application with different boundary and initial conditions:

$$D_\tau^\alpha M = D_s^{\beta\beta} M + D_u^{\gamma\gamma} M,\ 0 < s < 2\pi,\ 0 < u < 2\pi,\ \tau > 0 \tag{35}$$

with boundary conditions:

$$M(0,u,\tau) = 0,\ M(2\pi,u,\tau) = 0,\ M(s,0,\tau) = 0,\ M(s,2\pi,\tau) = 0 \tag{36}$$

and the initial condition:

$$M(s,u,0) = \sin s^\beta \sin u^\gamma \tag{37}$$

so we have:

$$L_{\alpha_\tau} M = D_s^{\beta\beta} M + D_u^{\gamma\gamma} M,\ 0 < s < 2\pi,\quad 0 < u < 2\pi,\ \tau > 0 \tag{38}$$

Applying the inverse operator $L_{\alpha_\tau}^{-1}$ to eq. (35) while considering the initial value yields:

$$M(s,u,\tau) = \sin s^\beta \sin u^\gamma + L_{\alpha_\tau}^{-1}\left(D_s^{\beta\beta} M + D_u^{\gamma\gamma} M\right) \tag{39}$$

the decomposition series form of eq. (35):

$$\sum_{n=0}^\infty M_n(s,u,\tau) = \sin s^\beta \sin u^\gamma + L_{\alpha_\tau}^{-1}\left(D_s^{\beta\beta}\left(\sum_{n=0}^\infty M_n\right) + D_u^{\gamma\gamma}\left(\sum_{n=0}^\infty M_n\right)\right) \tag{40}$$

thus, the recurrence relation will be provided

$$M_0 = \sin s^\beta \sin u^\gamma, M_{k+1} = L_{\alpha_\tau}^{-1}\left(D_s^{\beta\beta}\left(\sum_{n=0}^\infty M_n\right) + D_u^{\gamma\gamma}\left(\sum_{n=0}^\infty M_n\right)\right), \qquad k \geq 0, \tag{41}$$

Therefore, the solution in series form is expressed:

$$M = \sin s^\beta \sin u^\gamma \left(1 - \frac{\left(\beta^2+\gamma^2\right)}{\alpha}\tau^\alpha + \frac{(\beta^2+\gamma^2)^2}{\alpha^2 2!}\tau^{2\alpha} - \frac{(\beta^2+\gamma^2)^3}{\alpha^3 3!}\tau^{3\alpha} + \frac{(\beta^2+\gamma^2)^4}{\alpha^4 4!}\tau^{4\alpha} - \ldots\right) = \tag{42}$$

$$= \mathrm{e}^{-\frac{\left(\beta^2+\gamma^2\right)\tau^\alpha}{\alpha}} \sin s^\beta \sin u^\gamma$$

When $\alpha = \beta = \gamma = 1$, eq. (42) has the form:

$$M = \sin s \sin u\left(1 - 2\tau + \frac{4}{2!}\tau^2 - \frac{8}{3!}\tau^3 + \frac{16}{4!}\tau^4 + \ldots\right) = \mathrm{e}^{-2\tau} \sin s\ \sin u$$

Fourth application

We contemplate the IBVP within the (3+1)-D space and time fractional framework:

$$D_\tau^\alpha M = \left(\beta^2+\gamma^2+\delta^2\right)s^{4\beta}u^{4\gamma}v^{4\delta} + \frac{1}{12}\left(s^{2\beta}D_s^{\beta\beta}M + u^{2\gamma}D_u^{\gamma\gamma}M + v^{2\delta}D_v^{\delta\delta}M\right),$$
$$0 < s,u,\ v < 1,\ \tau > 0 \tag{43}$$

with the boundary values:

$$M(0,u,v,\tau)=0,\ M(1,u,v,\tau)=u^{4\gamma}v^{4\delta}\left(\mathrm{e}^{\left(\beta^2+\gamma^2+\delta^2\right)\frac{\tau^\alpha}{\alpha}}-1\right)$$
$$M(s,0,v,\tau)=0,\ M(s,1,v,\tau)=s^{4\beta}v^{4\delta}\left(\mathrm{e}^{\left(\beta^2+\gamma^2+\delta^2\right)\frac{\tau^\alpha}{\alpha}}-1\right) \tag{44}$$
$$M(s,u,0,\tau)=0,\ M(s,u,1,\tau)=s^{4\beta}u^{4\gamma}\left(\mathrm{e}^{\left(\beta^2+\gamma^2+\delta^2\right)\frac{\tau^\alpha}{\alpha}}-1\right)$$

and the initial value:

$$M(s,u,v,0)=0 \tag{45}$$

the solution in series form is expressed

$$M=s^{4\beta}u^{4\gamma}v^{4\delta}\left(1-1+\frac{\left(\beta^2+\gamma^2+\delta^2\right)}{\alpha}\tau^\alpha+\frac{(\beta^2+\gamma^2+\delta^2)^2}{2!\alpha^2}\tau^{2\alpha}+\frac{(\beta^2+\gamma^2+\delta^2)^3}{3!\alpha^3}\tau^{3\alpha}+...\right)=$$
$$=s^{4\beta}u^{4\gamma}v^{4\delta}\left(\mathrm{e}^{\left(\beta^2+\gamma^2+\delta^2\right)\frac{\tau^\alpha}{\alpha}}-1\right) \tag{46}$$

When $\alpha = 1$, eq. (46) has the form:

$$M=s^4u^4v^4\left(1-1+3\tau+\frac{9}{2!}\tau^2+\frac{27}{3!}\tau^3+...\right)=s^4u^4v^4\left(e^{3\tau}-1\right)$$

Conclusions

This work show cased the versatility of the Adomian decomposition technique for analytically solving space and time FDE featuring variable coefficients. We introduced a straightforward and effective scheme for tackling such equations, demonstrating its applicability across a range of scenarios. By leveraging the CFD, we were able to address functions exhibiting differentiability as well as those that are non-differentiable. This flexibility enables the method to be applied to systems characterized by both continuous and non-continuous media, expanding its utility in modelling and analyzing complex phenomena across various domains.

The arbitrariness of fractional orders introduces a wide range of possibilities and enables the representation of more complex structures. Each fractional order value corresponds to a distinct surface, thereby influencing the behavior of the solution significantly. Consequently, even small changes in the non-integer derivative order can lead to noticeable variations in the solution's form. Remarkably, these alterations occur without necessitating modifications to the underlying properties of the medium. This observation underscores the sensitivity of FDE to changes in fractional order values and highlights the potential for tailoring solution behaviors to suit specific modelling needs without altering the intrinsic characteristics of the medium being studied.

Acknowledgment

The authors extend their appreciation Taif University, Saudi Arabia, for supporting this work through Project No. (TU-DSPP-2024-08)

Fundings

This research was funded by Taif University, Taif, Saudi Arabia (TU-DSPP-2024-231).

References

[1] Kilbas, A. A., *et al.*, *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, The Netherlands, 2006
[2] Hilfer, R., *Applications of fractional Calculus in Physics*, World Science, Singapore, Singapore, 2000
[3] Manolis, G. D., Rangelov, T. V., Non-Homogeneous Elastic Waves in Soils: Notes on the Vector Decomposition Technique, *Soil Dynamics and Earthquake Engineering*, *26* (2006), 10, pp. 952-959
[4] Rizvi, S. T., *et al.*, Study of Multiple Lump and Rogue Waves to the Generalized Unstable Space Time Fractional Non-Linear Schrodinger Equation, *Chaos, Solitons and Fractals*, *151* (2021), 111251
[5] Rizvi, S. T., *et al.*, Rogue, Multi-Wave, Homoclinic Breather, M-Shaped Rational and Periodic-Kink Solutions for a Non-Linear Model Describing Vibrations, *Results in Physics*, *29* (2021), 104654
[6] Akhmetov, A., Long Current Loops as Regular Solutions of The Equation for Coupling Currents in a Flat Two-Layer Superconducting Cable, *Cryogenics*, *43* (2003), 3-5, pp. 317-322
[7] Carslaw, H. S., Jaeger, J. C., *Conduction of Heat in Solids*, Oxford University Press, Oxford, UK, 1959
[8] Ozisik, M, N., *Heat Conduction*, 2nd ed., John Wiley and Sons, New York, USA, 1993
[9] Bejan, A., *Convection Heat Transfer*, 4th ed., John Wiley and Sons, New York, USA, 2013
[10] Turcotte, D. L., *Schubert, G., Geodynamics*, 2nd ed., Cambridge University Press, Cambridge, UK, 2002
[11] Peck, L., *Heat as a Tool for Studying the Earth: Thermal Modelling in Geosciences*, Cambridge University Press, Cambridge, UK, 2016
[12] Hartmann, D. L., *Global Physical Climatology*, 2nd ed., Academic Press, Cambridge, Mass., USA, 2016
[13] Abdel-Salam, E. A.-B., Periodic Structures Based on the Symmetrical Lucas Function of the (2+1)-Dimensional Dispersive Long-Wave System, *Zeitschrift für Naturforschung A*, *63* (2008), Apr., pp. 671-678
[14] Arridge, S. R., Optical Tomography In Medical Imaging, *Inverse Problems*, *15* (1999), 2, pp. R41-R93
[15] He, J. H., Approximate Analytical Solution for Seepage Flow with Fractional Derivatives in Porous Media, *Comput. Methods Appl. Mech. Eng*. *167* (1998), 1-2, pp. 57-68
[16] Momani, S., Odibat, Z., Analytical Approach to Linear Fractional Partial Differential Equations Arising in Fluid Mechanics, *Phys. Lett. A*, *355* (2006), 4-5, pp. 271-279
[17] Yousif, E. A., *et al*., On the Solution of the Space-Time Fractional Cubic Non-Linear Schrodinger Equation, *Results in Physics*, *8* (2018), Mar., pp. 702-708
[18] Abdel-Salam, E. A.-B., Mourad, M. F., Fractional Quasi AKNS-Technique for Non-Linear Space – Time Fractional Evolution Equations, *Math. Meth. Appl. Sci.*, *42* (2018), 18, pp. 5953-5968
[19] Abdel-Salam, E. A.-B., *et al*., Geometrical Study and Solutions for Family of Burgers-Like Equation with Fractional Order Space Time, A*lexandria Engineering Journal*, *61* (2022), 1, pp. 511-521
[20] Nouh, M. I., *et al*., Modelling Fractional Polytropic Gas Spheres Using Artificial Neural Network, *Neural Comput and Applic*, *33* (2021), Aug., pp. 4533-4546
[21] Nouh, M. I., Abdel-Salam, E. A.-B., Analytical Solution the Fractional Polytropic Gas Spheres, *Eur.* P*hys. J. Plus*, *133* (2018), 149
[22] Abdel-Salam, E. A.-B., Nouh, M. I., Conformable Fractional Polytropic Gas Spheres, *New Astronomy*, *76* (2020), 101322
[23] Abdel-Salam, E. A.-B., *et al*., Analytical Solution of the Space-Time Fractional Non-Linear Schrodinger Equation, *Reports on Mathematical Physics*, *77* (2016), 1, pp. 19-3
[24] Kilany, A. A, *et al.,* Non-Integer Order Analysis of Electro-Magneto-Thermoelastic with Diffusion and Voids Considering Lord-Shulman and Dual-Phase-Lag Models with Rotation and Gravity, *Waves in Random and Complex Media*, *11* (2022), July, pp. 1-31
[25] Khalil, R., *et al*., A New Definition of Fractional Derivative, J. *Comput. Appl. Math.*, *264* (2014), 65
[26] Wazwaz, A. M., Goruis, A., Exact Solutions for Heat-Like and Wave-Like Equations with Variable Coefficients, *Appl. Math. Comput.*, *149* (2004), 1, pp. 51-59

Paper submitted: July 11, 2024
Paper revised: October 5, 2024
Paper accepted: October 28, 2024