INVESTIGATION OF THE UNIQUENESS END OCCURRENCE INVARIANT POINT OUTCOMES BASED ON BI-G METRIC SPACE

by

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Recent years have seen a significant increase in the application of bi-b metric (BB-m) spaces. In this study, we aim to expand the scope of invariant point theory by extending it from BB-m spaces to bi-G metric (BG-m) spaces. The goal is to explore both the existence and uniqueness of invariant point solutions in these spaces.

Key words: *fixed point, continuous mapping, G metric (G-m) space, complete space*

Introduction

Fixed-point and invariant-point theories are fundamental tools in non-linear analysis, playing a vital role in the study of non-linear functional analysis in metric spaces. Numerous important studies have explored the properties of invariant points in b-metric and E-metric spaces [1-4]. Kumar and Bhardwaj extended certain aspects of invariant point theory to BB-m spaces by introducing the concept of a novel metric space known as the G-m space. Further developments in invariant point theory have been explored in these areas [4-10]. This study aims to extend invariant point theory to BG-m spaces, contributing to the ongoing research in this field.

Fixed points play a crucial role in various areas of mathematics, including analysis, topology, and dynamical systems, as they often represent equilibrium states or solutions to specific types of equations [11, 12]. The study of invariant points has led to foundational results such as Banach's fixed-point theorem (also known as the *Contraction Mapping Theorem*), which guarantees the existence and uniqueness of invariant points under certain conditions in complete metric spaces. Beyond simple functions, the theory of invariant points extends to operators in more complex structures, including metric spaces, topological spaces, and normed spaces. Invariant point theory has wide-ranging applications in fields such as differential equations, optimization, game theory, and the study of dynamical systems [13, 14].

The G-metric spaces are a generalization of traditional metric spaces and have found valuable applications across various areas of mathematics and its interdisciplinary domains [1]. These spaces are particularly important in the study of invariant point theory, as they offer a broader framework for examining the existence and uniqueness of invariant points in non-linear operators [4-6]. The flexibility of G-metrics, which allow for the comparison of distances in more complex ways than standard metrics, makes them ideal for addressing problems in functional analysis, differential equations, and optimization theory. For example, G-metric spaces have been successfully applied in the study of dynamical systems, where they help characterize

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the convergence behavior of iterative processes. In computer science, G-metric spaces are used in the analysis of algorithms, particularly in machine learning and data clustering, where similarity measures must be generalized beyond traditional Euclidean distances. Their application also extends to the study of topological spaces, providing a framework for investigating continuity and compactness under more generalized conditions [9-12].

The Banach contraction mapping principle is a foundational and highly influential theory in mathematical analysis, making it a crucial tool for solving fixed-point problems across various branches of mathematics. Its applications extend far beyond mathematics, influencing many areas of science and engineering. Bakhtin [13] introduced the concept of generalized b-metric spaces, which expanded the potential of fixed-point theory in non-standard settings. Later, Czerwik [14] extended these results to B-m spaces, further broadening the scope of the approach. Over the years, numerous researchers have worked to generalize Banach's invariant point theory within the framework of B-m spaces. Notably, Czerwik [15] presented a generalization of the Banach fixed-point theorem specifically in b-metric spaces. The existence and uniqueness of fixed points in b-metric spaces were rigorously developed by Agrawal in 1968. Additionally, Maia's [16] generalization of the Banach Contraction Principle, which considers two metrics on a set, marked an important advancement in the field. Mishra [17] later extended Maia's invariant point theory to B-m spaces, and Soni [18] further contributed by providing an invariant point theory for mappings in B-m spaces. Furthermore, Chopade *et al*. [19] obtained common fixed-point theorems for contractive-type mappings in metric spaces. Roshan *et al.* [20] established the common fixed point for four maps in b-metric spaces. Suzuki *et al*. [21] derived basic inequalities in b-metric spaces and explored their applications. These developments highlight the growing interest in, and utility of, B-m and BB-m spaces in invariant point theory.

The BB-m spaces offer significant potential for modelling and analyzing complex systems where interactions between different components or sets must be considered using multiple measures of distance. In engineering, for example, BB-m spaces could be used to model multi-component systems where interactions occur not only within individual components but also between different types of components. Applications include electrical circuits, multi-phase fluid dynamics, and control systems involving multiple distinct physical or abstract components. In the context of social networks, BB-m spaces could model relationships by considering both individual friendships (using one metric) and the interactions between groups (using another metric). This dual-metric approach would allow the study of how networks evolve over time, helping to identify stable communities or conflict zones where members may be moving toward or away from each other, analogous to fixed-point convergence in the network's structure. In bioinformatics, particularly in the analysis of protein-protein interaction networks, BB-m spaces could capture both functional distances between proteins and structural distances between different biological complexes. This dual-metric perspective could provide deeper insights into network evolution and disease progression. These innovative applications span a wide range of fields, from biological sciences to network theory, offering new perspectives on convergence, optimization, and system behavior in both theoretical and applied mathematics.

Preliminaries

The following section offers the definitions of G-m spaces and presents extra necessary results that will be used in the following section.

Definition 1. [10, 11, 16, 18] Let it be the case that A is not null set and M: $A\times A\times A$ $[0, +\infty)$ is a function such that:

– $M(\alpha, \beta, \gamma) = 0$, for $\alpha = \beta = \gamma$.

- $M(\alpha, \beta, \gamma) > 0$, $\forall \alpha, \beta \in A$, and $\alpha \neq \beta$.
- $M(\alpha, \alpha, \beta) \leq M(\alpha, \beta, \gamma)$, $\forall \alpha, \beta, \gamma \in A$, and $\beta \neq \gamma$.
- $M(\alpha, \beta, \gamma) = G(\beta, \alpha, \gamma) = G(\gamma, \alpha, \beta) = ...$ (Symmetry with reverence to every one of the three variables).
- $M(\alpha, \alpha, \beta) \leq M(\alpha, \rho, \rho), + M(\rho, \beta, \gamma)$, $\forall \alpha, \beta, \gamma, \rho \in A$.

Thus, the pair (M, A) is well-defined to be a G-m space.

For Example. [10, 17] Let it be the case that $A = \mathbb{R}\backslash\{0\}$, we can redefine:

$$M: A \times A \times A\ [0, +\infty)$$

by

$$M(\alpha,\beta,\gamma) = \begin{cases} |\alpha-\beta| + |\beta-\gamma| + |\alpha-\gamma|; \text{ if } \alpha,\beta,\gamma \text{ all have the same sign} \\ 1 + |\alpha-\beta| + |\beta-\gamma| + |\alpha-\gamma|; \text{ otherwise} \end{cases}$$

therefore, (M, A) is considered to be a G-m space.

Clarification 1. [10, 11, 16] Suppose that (M, A) be a G-m space and $\{a_n\}$ be a points Chain of A a point $a \in A$ is said to be the limit of the Chain $\{a_n\}$ if $\lim_{s,r\to\infty} M(a, a_s, a_r) = 0$ and the Chain $\{a_n\}$ is referred to be G-convergent to a. Therefore, if $a_s \to a$ in a G-m universe (M, A), $\forall \varepsilon > 0\ \exists\ N > 0$, satisfy that $M(a, a_n, a_m) < \varepsilon\ \forall n, m \geq N$.

Clarification 2. [10, 11, 16, 18] Suppose that (M, A), be a G-m universe. A Chain $\{a_n\}$ in A is referred to be a G-Cauchy if $\forall \varepsilon > 0\ \exists\ N > 0$ such that $M(a_s, a_r, a_l) < \varepsilon\ \forall s, r, l \geq N$, therefore, if $M(a_s, a_r, a_l) \to 0$ with $s, r, l \to \infty$.

Clarification 3. [10, 11, 16, 18] A G-m universe at (M, A), is referred to be G-complete if each G-Cauchy chain inat (M, A), is chain in A.

Key findings

In this section, we explore and approve the uniqueness and existence of invariant point solutions under the preview of a BG-m space.

Result 1. Suppose, for the purpose of discussion, that (Ψ_1, A) and (Ψ_2, A) are BG-m space, therefore:

- $\Psi_1(a_1, a_2, \rho) \leq \Psi_2(a_1, a_2, \rho)\ \forall\ a_1, a_2, \rho \in A,$
- the pair (Ψ_1, A). Is a space that satisfies completeness, and
- $\varphi : A \to A$ and $\psi : A \to A$ be any two self-maps on A satisfying:

$$\begin{gathered} \alpha \min\{\Psi_2(\varphi a,\psi b,\rho), \Psi_2(\varphi a,\psi b,\rho), \Psi_2(a,b,\rho)\} + \Psi_2(\varphi a,\psi b,\rho) \leq \\ \leq \frac{\beta\Psi_2(b,\psi b,\rho)\Psi_2(a,\varphi a,\rho)}{\Psi_2(b,\psi b,\rho)+\Psi_2(b,\varphi a,\rho)} + \\ + \frac{\gamma[\Psi_2(b,\varphi b,\rho)]^q[\Psi_2(b,\varphi a,\rho)]^r[p+\Psi_2(a,\varphi a,\rho)]}{\Psi_2(a,b,\rho)+\lambda\Psi_2(b,\varphi a,\rho)+\mu\Psi_2(a,\psi b,\rho)+1} + \\ + \frac{\delta\Psi_2(a,\psi b,\rho)\left[\sqrt{\Psi_2(a,b,\rho)\Psi_2(b,\varphi a,\rho)} + \sqrt{\Psi_2(a,b,\rho)\Psi_2(a,\varphi a,\rho)} + 1\right]}{\Psi_2(a,\psi a,\rho)\Psi_2(a,\psi b,\rho)\Psi_2(b,\varphi b,\rho)+\Psi_2(a,b,\rho)+1} + \\ + \eta \max\left\{\Psi_2(a,b,\rho), \frac{\Psi_2(b,\psi b,\rho)\Psi_2(a,\varphi a,\rho)}{\Psi_2(\varphi a,\psi b,\rho)+1}\right\} \end{gathered} \tag{1}$$

where $p, q, r, \lambda, \mu, \in R^+$and $\alpha, \beta, \gamma, \delta, \eta \in [0, 1)$ are such that $\beta + 2\delta + \eta - \alpha < 1$.

- If φ and ψ are continuous in (A, Ψ_1). After that φ and ψ possess A one-of-a-kind common invariant point in (A, Ψ_1). Then we can write:

$$\alpha\min\{\Psi_2(\varphi a_{2s},\psi a_{2s+1},\rho),\Psi_2(\varphi a_{2s+1},\psi a_{2s},\rho),\Psi_2(a_{2s},a_{2s+1},\rho)\}+\Psi_2(\varphi a_{2s},\psi a_{2s+1},\rho)\le$$
$$\le\frac{\beta\Psi_2(a_{2s+1},\psi a_{2s+1},\rho)\Psi_2(a_{2s},\varphi a_{2s},\rho)}{\Psi_2(a_{2s+1},\varphi a_{2s},\rho)+\Psi_2(a_{2s+1},\psi a_{2s+1},\rho)}+$$
$$+\frac{\gamma[p+\Psi_2(a_{2s},\varphi a_{2s},\rho)][\Psi_2(a_{2s+1},\varphi a_{2s},\rho)]^r[\Psi_2(a_{2s+1},\varphi a_{2s+1},\rho)]^q}{\mu\Psi_2(x_{2s},\psi a_{2s+1},\rho)+\Psi_2(a_{2s},a_{2s+1},\rho)+\lambda\Psi_2(a_{2s+1},\varphi a_{2s},\rho)+1}+$$
$$+\frac{\delta\left[1+\sqrt{\Psi_2(a_{2s},a_{2s+1},\rho)\Psi_2(a_{2s},\varphi a_{2s},\rho)}+\sqrt{\Psi_2(a_{2s},a_{2s+1},\rho)\Psi_2(a_{2s+1},\varphi a_{2s},\rho)}\right]\Psi_2(a_{2s},a_{2s+1},\rho)}{\Psi_2(a_{2s},\psi a_{2s},\rho)\Psi_2(a_{2s},\psi a_{2s+1},\rho)\Psi_2(a_{sn+1},\varphi a_{2s},\rho)\Psi_2(a_{2s+1},\varphi a_{2s+1},\rho)+1+\Psi_2(a_{2s},a_{2s+1},\rho)}+$$
$$+\eta\max\left\{\Psi_2(a_{2s},a_{2s+1},\rho),\frac{\Psi_2(a_{2s},\varphi a_{2s},\rho)\Psi_2(a_{2s+1},\psi a_{2s+1},\rho)}{\Psi_2(\varphi a_{2s},\psi a_{2s+1},\rho)+1}\right\}$$

Proof: Assume for the sake of argument that $a_0\in A$ defined a sequence $\{a_s\}$in A therefore:

$$\varphi(a_{2s})=a_{2s+1}\ \text{ and }\ \psi(a_{2s-1})=a_{2s},\ \ s=1,2 \tag{2}$$

therefore

$$\alpha\min\{\Psi_2(a_{2s+1},a_{2s+2},\rho),\Psi_2(a_{2s+2},a_{2s+1},\rho),\Psi_2(a_{2s},a_{2s+1},\rho)\}+\Psi_2(a_{2s+1},a_{2s+2},\rho)\le$$
$$\le\frac{\beta\Psi_2(a_{2s},a_{2s+1},\rho)\Psi_2(a_{2s+1},a_{2s+2},\rho)}{\Psi_2(a_{2s+1},a_{2s+1},\rho)+\Psi_2(a_{2s+1},a_{2s+2},\rho)}+$$
$$+\frac{\gamma[\Psi_2(a_{2s+1},a_{2s+2},\rho)]^q[\Psi_2(a_{2s+1},a_{2s+1},\rho)]^r[p+\Psi_2(a_{2s},a_{2s+1},\rho)]}{\mu\Psi_2(a_{2s},a_{2s+2},\rho)+\Psi_2(a_{2s},a_{2s+1},\rho)+1+\lambda\Psi_2(a_{2s+1},a_{2s+1},\rho)}+$$
$$+\delta\frac{\Psi_2(a_{2s},a_{2s+2},\rho)[1+\sqrt{\Psi_2(a_{2s},a_{2s+1},\rho)\Psi_2(a_{2s},a_{2s+1},\rho)}+\sqrt{\Psi_2(a_{2s},a_{2s+1},\rho)\Psi_2(a_{2s+1},a_{2s+1},\rho)}]}{1+\Psi_2(a_{2s},a_{2s+1},\rho)+\Psi_2(a_{2s},a_{2s+1},\rho)\Psi_2(a_{2s},a_{2s+2},\rho)\Psi_2(a_{2s+1},a_{2s+1},\rho)\Psi_2(a_{2s+1},a_{2s+2},\rho)}+$$
$$+\eta\max\left\{\Psi_2(a_{2s},a_{2s+1},\rho),\frac{\Psi_2(a_{2s+1},a_{2s+2},\rho)\Psi_2(a_{2s},a_{2s+1},\rho)}{\Psi_2(a_{2s+1},a_{2s+2},\rho)+1}\right\}$$

This gives:

$$\begin{gathered}\alpha\min\{\Psi_2(a_{2s+1},a_{2s+2},\rho),\Psi_2(a_{2s},a_{2s+1},\rho)\}+\Psi_2(a_{2s+1},a_{2s+2},\rho)\le\\ \le\beta\Psi_2(a_{2s},a_{2s+1},\rho)+\delta\Psi_2(a_{2s},a_{2s+2},\rho)+\eta\Psi_2(a_{2s},a_{2s+1},\rho)\le\\ \le\eta\Psi_2(a_{2s},a_{2s+1},\rho)+\beta\Psi_2(a_{2s},a_{2s+1},\rho)+\delta t\{\Psi_2(a_{2s+1},a_{2s+2},\rho)+\Psi_2(a_{2s},a_{2s+1},\rho)\}\end{gathered} \tag{3}$$

Case 1:

$$\Psi_2(a_{2s},a_{2s+1})=\min\{\Psi_2(a_{2s+1},a_{2s+2},\rho),\Psi_2(a_{2s},a_{2s+1},\rho)\}$$

Hence from the expression eq. (3):

$$\alpha\Psi_2(a_{2s},a_{2s+1},\rho)+\Psi_2(a_{2s+1},a_{2s+2},\rho)\le\delta t\Psi_2(a_{2s+1},a_{2s+2},\rho)+(\beta+\delta t+\eta)\Psi_2(a_{2s},a_{2s+1},\rho)$$
$$(1-\delta t)\Psi_2(a_{2s+1},a_{2s+2},\rho)\le(\beta+\delta t+\eta-\alpha)\Psi_2(a_{2s},a_{2s+1},\rho)$$
$$\Psi_2(a_{2s+1},a_{2s+2},\rho)\le\frac{(\beta+\delta t+\eta-\alpha)}{(1-\delta t)}\Psi_2(a_{2s},a_{2s+1},\rho)$$
$$\Psi_2(a_{2s+1},a_{2s+2},\rho)\le k\Psi_2(a_{2s},a_{2s+1},\rho)$$

with

$$1>\frac{(\eta-\alpha+\beta+\delta t)}{(1-\delta t)}$$

In general, $\forall n \in N$:

$$\Psi(a_{s+1}, a_{s+2}, \rho) \le k\,\Psi(a_s, a_{s+1}, \rho) \tag{4}$$

Case 2:

$$\Psi_2(a_{2s+1}, a_{2s+2}) = \min\{\Psi_2(a_{2s+1}, a_{2s+2}, \rho), \Psi_2(a_{2s}, a_{2s+1}, \rho)\}$$

From the expression in eq. (3) goes:

$$\alpha\Psi_2(a_{2s+1}, a_{2s+2}, \rho) + \Psi_2(a_{2s+1}, a_{2s+2}, \rho) \le \delta t\Psi_2(a_{2s+1}, a_{2s+2}, \rho) + (\beta + \eta + \delta t)\Psi_2(a_{2s}, a_{2s+1}, \rho)$$

$$(\alpha - \delta t + 1)\Psi_2(a_{2s+1}, a_{2s+2}, \rho) \le (\beta + \eta + \delta t)\Psi_2(a_{2s}, a_{2s+1}, \rho)$$

$$\Psi_2(a_{2s+1}, a_{2s+2}, \rho) \le \frac{(\beta + \delta t + \eta)}{(1 - \delta t + \alpha)}\Psi_2(a_{2s}, a_{2s+1}, \rho)$$

$$\Psi_2(a_{2s+1}, a_{2s+2}, \rho) \le k\Psi_2(a_{2s}, a_{2s+1}, \rho)$$

with

$$1 > \frac{(\beta + \delta t + \eta)}{(1 - \delta t + \alpha)}$$

In general, $\forall n \in N$:

$$\Psi(a_{s+1}, a_{s+2}, \rho) \le k\Psi(a_s, a_{s+1}, \rho) \tag{5}$$

Hence, the sequence $\{a_s\}$ satisfies the conditions of a Cauchy sequence in A. Meanwhile the Cauchy sequence $\{a_s\}$ definite by eq. (2) has convergent subsequence $\{a_{sk}\}$ in (Ψ_1, A) converging to a^* in (Ψ_1, A), the sequence $\{a_n\}$ also converges to a^* in (Ψ_1, A). Therefore:

$$\lim_{s\to\infty} a_s = \lim_{s\to\infty} a_{2s} = \lim_{s\to\infty} a_{2s-1} = \lim_{s\to\infty} a_{2s+1} = a^*$$

We are now approving this a^* is an invariant point of both mappings φ and ψ since φ and ψ are continuous in (Ψ_1, A), thus:

$$\varphi(a^*) = \psi(\lim_{s\to\infty} a_{2s}) = \lim_{s\to\infty} \varphi a_{2s} = a^*$$

Similarly:

$$\psi(a^*) = \psi(\lim_{s\to\infty} a_{2s-1}) = \lim_{s\to\infty} \psi a_{2s-1} = a^*$$

Hence, a^* serves as a common invariant point of the mapping φ and ψ.

Assume that a^* and b^* are invariant points of φ and ψ. Therefore, $\varphi b^* = \psi b^* = b^*$ and $\varphi a^* = \psi a^* = a^*$. Suppose that:

$$\alpha \min\{\Psi_2(\varphi a^*, \psi b^*, \rho), \Psi_2(\varphi b^*, \psi a^*, \rho), \Psi_2(a^*, b^*, \rho)\} + \Psi_2(\varphi a^*, \psi b^*, \rho) \le$$

$$\le \frac{\beta\Psi_2(b^*, \psi b^*, \rho)\Psi_2(a^*, \varphi a^*, \rho)}{\Psi_2(b^*, \varphi a^*, \rho) + \Psi_2(b^*, \psi b^*, \rho)} + \frac{\gamma[\Psi_2(b^*, \varphi b^*, \rho)]^q[\Psi_2(b^*, \varphi a^*, \rho)]^r[p + \Psi_2(a^*, \varphi a^*, \rho)]}{\Psi_2(a^*, b^*, \rho) + \mu\Psi_2(a^*, \psi b^*, \rho) + \lambda\Psi_2(b^*, \varphi a^*, \rho) + 1} +$$

$$+ \frac{\delta\Psi_2(a^*, \psi b^*, \rho)\left[\sqrt{\Psi_2(a^*, b^*, \rho)\Psi_2(b^*, \varphi a^*, \rho)} + \sqrt{\Psi_2(a^*, b^*, \rho)\Psi_2(a^*, \varphi a^*, \rho)} + 1\right]}{\Psi_2(b^*, \varphi a^*, \rho)\Psi_2(b^*, \varphi b^*, \rho)\Psi_2(a^*, \psi a^*, \rho)\Psi_2(a^*, \psi b^*, \rho) + \Psi_2(a^*, b^*, \rho) + 1} +$$

$$+ \eta \max\left\{\Psi_2(a^*, b^*, \rho), \frac{\Psi_2(b^*, \psi b^*, \rho)\Psi_2(a^*, \varphi a^*, \rho)}{\Psi_2(\varphi a^*, \psi b^*, \rho) + 1}\right\}$$

As a result:

$$\alpha\min\{\Psi_2(a^*,b^*,\rho),\Psi_2(a^*,b^*,\rho),\Psi_2(a^*,b^*,\rho)\}+\Psi_2(a^*,b^*,\rho)\le$$
$$\le\frac{\beta\Psi_2(b^*,b^*,\rho)\Psi_2(a^*,a^*,\rho)}{\Psi_2(b^*,b^*,\rho)+\Psi_2(b^*,a^*,\rho)}+$$
$$+\frac{\gamma[\Psi_2(b^*,b^*,\rho)]^q[\Psi_2(b^*,a^*,\rho)]^r[p+\Psi_2(a^*,a^*,\rho)]}{\mu\Psi_2(a^*,b^*,\rho)+\Psi_2(a^*,b^*,\rho)+\lambda\Psi_2(b^*,a^*,\rho)+1}+$$
$$+\frac{\delta\left[\sqrt{\Psi_2(a^*,b^*,\rho)\Psi_2(b^*,a^*,\rho)}+\sqrt{\Psi_2(a^*,b^*,\rho)\Psi_2(a^*,a^*,\rho)}+1\right]\Psi_2(a^*,b^*,\rho)}{1+\Psi_2(a,b^*,\rho)+\Psi_2(a^*,a^*,\rho)\Psi_2(a^*,b^*,\rho)\Psi_2(b^*,a^*,\rho)\Psi_2(b^*,b^*,\rho)}+$$
$$+\eta\max\left\{\Psi_2(a^*,b^*,\rho),\frac{\Psi_2(b^*,b^*,\rho)\Psi_2(a^*,a^*\rho)}{\Psi_2(a^*,b^*,\rho)+1}\right\}$$

Consequently:

$$\alpha\Psi_2(a^*,b^*,\rho)+\Psi_2(a^*,b^*,\rho)\le\eta\Psi_2(a^*,b^*,\rho)+\delta\Psi_2(a^*,b^*,\rho)$$
$$\Psi_2(a^*,b^*,\rho)\le(\delta-\alpha+\eta)\Psi_2(a^*,b^*,\rho)$$

But $(\delta + \eta - \alpha) < 1$. So we have, $\Psi_2(a^*, b^*, \rho) > \Psi_2(a^*, b^*, \rho)$. This gives rise to a paradox. Therefore, φ and ψ must have exactly one common invariant point in A.

Theorem 1. Let it be the case that (A, Ψ_1) and (A, Ψ_2) are BG-m space, let $Q = \{L_i : i \in I$, these to g positive integers$\}$ be a family of mappings on A so that the conditions holds:

- $\Psi_1(a, b, \rho) \le \Psi_2(a, b, \rho)\ \forall a, b, \rho \in A$,
- (A, Ψ_1) is a complete space,
- for all $L_j : A \to A \in Q$, $\exists\ L_i : A \to A \in Q$.

So that:

$$\Psi_2(L_i^m x,L_j^n b,\rho)+\alpha\min\{\Psi_2(L_i^m a,L_j^n b,\rho),\Psi_2(L_i^m b,L_j^n a,\rho),\Psi_2(a,b,\rho)\}\le$$
$$\le\frac{\beta\Psi_2(a,L_i^m\,\mathrm{a},\rho)\Psi_2(b,\mathrm{L}_j^n\,b,\rho)}{\Psi_2(b,L_j^n b,\rho)+\Psi_2(b,L_i^m a,\rho)}+\frac{\gamma[p+\Psi_2(a,L_i^m a,\rho)][\Psi_2(b,L_i^m b,\rho)]^q[\Psi_2(b,L_i^m b,\rho)]^r}{1+\lambda\Psi_2(b,L_i^m a,\rho)+\mu\Psi_2(a,L_j^n b,\rho)+\Psi_2(a,b,\rho)}+$$
$$+\frac{\delta\Psi_2(a,L_j^n b,\rho)[1+\sqrt{\Psi_2(a,b,\rho)\Psi_2(a,L_i^m a,\rho)}+\sqrt{\Psi_2(a,b,\rho)\Psi_2(b,L_i^m a,\rho)}]}{1+\Psi_2(a,b,\rho)+\Psi_2(a,L_i^m a,\rho)\Psi_2(a,L_j^n b,\rho)\Psi_2(b,T_i^m a,\rho)\Psi_2(b,L_i^m a,\rho)}+\tag{6}$$
$$+\eta\max\left\{G_2(a,b,\rho),\frac{G_2(a,L_i^m a,\rho)G_2(b,L_j^n b,\rho)}{1+G_2(L_i^m a,L_j^n b,\rho)}\right\}$$

where p, q, r, λ, $\mu \in R^+$ also m, n are positive integers and α, β, δ, η, $\gamma \in [0, 1)$ are such that $2\delta t + \beta + \eta - \alpha < 1$, and

- Mappings L_i is continuous in $(A, \Psi_1)\ \forall i \in I$, then Q has a unique common invariant point.

The reasoning behind this proof follows closely the proof of the aforementioned theorem.

Concluding, observations, areas for future research and open problems

In this study, we have extended fixed-point theory from BB-m spaces to the broader class of BG-m spaces. By exploring these spaces, we aim to enhance our understanding of the existence and uniqueness of invariant point solutions, thereby broadening the scope of invariant point theorems. The results presented provide new insights into the behavior of mappings in these generalized spaces, contributing to the advancement of mathematical tools for both theoretical and applied contexts.

One promising avenue for future research is the application of fixed-point theory within G-m spaces to fractional calculus. Fractional calculus, which generalizes traditional integer-order derivatives and integrals to fractional orders, has gained significant attention due to its wide-ranging applications in fields such as physics, engineering, and biology. Invariant point techniques could play a key role in proving the existence and uniqueness of solutions to fractional differential equations, which are often non-linear and challenging to solve directly [22-26]. By extending the framework of G-m spaces to fractional calculus, we can explore new invariant point results that may simplify the analysis of fractional differential models. These techniques could help establish more general conditions for the existence of solutions, particularly in contexts where traditional methods face limitations. Additionally, further research into the stability and convergence of iterative methods for solving fractional equations using invariant point principles could lead to significant advancements in the numerical analysis of fractional calculus problems.

Future research could focus on generalizing BB-m spaces to higher-dimensional or more complex metric structures, potentially extending existing results to include more than two mappings or more intricate types of metric spaces. One promising direction is to apply the established common invariant point results to non-linear systems of equations or differential inclusions, where bi-metric spaces may offer valuable tools for analyzing the existence and uniqueness of solutions. Another important area for exploration is the development of computational algorithms to approximate the common invariant point of two mappings in BB-m spaces, including the analysis of error bounds and convergence properties for these numerical methods. Additionally, further investigation into more general conditions for the existence of common invariant points in B-m spaces is needed, considering weaker or alternative conditions such as contractive or non-expansive mappings. Lastly, exploring the connections between BB-m spaces and other mathematical structures, such as topological spaces and normed spaces, could provide insights into their potential applications in functional analysis and optimization.

Extending the framework to handle problems involving multiple sets and mappings could provide a more general approach to invariant point theory in complex systems. This would involve exploring the existence and uniqueness of common invariant points in systems where multiple distances or relationships exist between different components.

The relationship between BB-m spaces and graph theory stems from their shared ability to represent distances and relationships between elements using the structural framework of graphs. By integrating graph theory, we can extend invariant point theory and metric space concepts to more general contexts, especially those involving multiple interacting sets and complex relationships. This creates a rich framework for analyzing convergence, completeness, and fixed-point properties in both theoretical and applied settings. The concept of graphical B-m spaces opens new avenues for further exploration in invariant point theory and graphical analysis.

Future research could apply this innovative framework to real-world problems across various scientific fields, such as ecological networks (*e.g.*, habitat suitability), social networks (*e.g.*, affinity between individuals), and computational models (*e.g.*, distance measurements in dynamic networks). The symmetry of the underlying graph plays a crucial role in the results, and future studies could investigate how asymmetrical graph structures influence invariant point outcomes, exploring the broader implications of asymmetry in graphical metric spaces. As this area of study develops, graphical BB-m spaces have the potential to become a powerful tool in both theoretical and applied mathematics, unlocking exciting research opportunities across diverse disciplines.

Through these extensions, we aim to enhance the understanding of fixed- and invariant-point theory in non-traditional metric spaces and its relevance to both pure and applied mathematics.

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