MULTIPLE SOLITON SOLUTIONS OF THE MODEL mKdV WITH TIME-DEPENDENT VARIABLE COEFFICIENTS

by

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We investigate exact multiple soliton waves in the time-dependent variable coefficients modified KdV (mKdV) equation. Employing similarity transformations, as well as tanh and sech function methods, we present multiple kink wave, singular kink, and triangular function solutions for the time-dependent mKdV equation with variable coefficients. Specifically. We investigate the construction of multiple solitons by choosing a special form of the time variable. Our results demonstrate that the specific choice of these analogous temporal variables can effectively control the characteristics of numerous soliton kinks, offering potential applications in diverse fields

Key words: *mKdV equations, exact solutions, similarity transformation*

Introduction

The KdV model and the mKdV model are the most well-known mathematical representations of shallow water wave surfaces. Notably, it is specifically regarded as one of the solvable differential models, meaning that its solutions are precisely known [1-5]. That is an integrable model, and the scattering inverse transform method solves it. Current research focuses on the mathematical theory stemming from the KdV model, first proposed by Boussinesq in 1877 and subsequently rediscovered by Korteweg and De Vries in 1895. This theory is very important [6-8]. Furthermore, Korteweg and De-Vries used the solitary waves that Scott Russell initially suggested in their KdV model [9, 10]. Numerous natural phenomena are connected to the KdV and mKdV models [11-14]. Examples of these include plasma, soliton theory, and quantum mechanics [15-20].

The field of optics and soliton research is thriving due to its potential applications in the creation of new optical communication models and data transfers [3, 10, 15, 19, 20], involving chemical reactions, light-induced phase shifts in metals, and electron dynamics in

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semiconductors. A unique type of ultra-short pulse that allows for long-distance transmission while maintaining the pulse's structure and velocity is the optical soliton [21-23].

Over the past few decades, a wide array of linear and non-linear partial differential models has emerged to generate new analytical and numerical solutions. Different methodologies have been proposed to explore these models, offering diverse approaches to understanding their solutions. Key examples include the Hirota's method [6], tanh method [7, 24], the inverse scattering method, the Backlund transformation, Adomian decomposition technique [9], and Miura's transformation.

We can utilize various direct and indirect methods to derive solutions for models of non-linear differential equations, facilitated by symbolic arithmetic packages [7]. To obtain analytical solutions for numerous non-linear differential models involving sine, cosine, cotangent, tangent, and hyperbolic cotangent functions, techniques developed by Wazwaz [9] and Malfliet [24] have been employed and further refined. Additionally, many other methods have been introduced by mathematicians and physicists to analyze and discover new solutions for non-linear differential models, including the F-expansion method, spectral collocation method, sub-equation method, iterative variation technique, and Jacobi elliptic method [15-20].

It is important to remember that the equilibrium that results from the non-linear possessions and the dispersion is what leads to the classical soliton in independent differential non-linear equations with constant coefficients. However, because of their extremely intriguing qualities for possible future scientific applications, differential non-linear models with time-dependent coefficients own garnered a lot of thoughtfulness recently.

One of the primary objectives of this article is to present an effective way for directly verifying the existence of multiple solitons in the mKdV equation with time-dependent coefficients. Several analytical solutions are obtained using symbolic mathematics software, specifically MAPLE.

Model and technique of solution

We examine the following generalized mKdV equation with a time-dependent coefficient, which describes by the model:

$$U_t - 6\alpha(t)U^2 U_x + \beta(t)U_{xxx} = 0, \quad \alpha(t) = a\beta(t) \tag{1}$$

where $\alpha(t)$, $\beta(t)$ are the variable time coefficients, $U = U(t, x)$ – the scalar-valued function, t – the time variable, and x – the spatial variable co-ordinate, and it is a time dependent variable. When $\alpha(t) = \beta(t) = 1$, eq. (1) is become the standard mKdV equation. Assume that:

$$U(t, x) = u(T, x) \tag{2}$$

where $T = T(x)$ is the time real function. Selecting $\beta(t) = \partial T/\partial t$ substituting the transformation eq. (2) into eq. (1), we obtain that eq. (1) becomes:

$$u_T - 6au^2 u_x + u_{xxx} = 0 \tag{3}$$

Using Malfliet's technique, we derive multiple solitons of the KdV eq. (3), as proposed in [24]. This approach provides multiple solitons more easily and conveniently than the inverse scattering method [11]. Abdel-Rahman [12] explores the multiple solitons of the mKdV, Boussinesq, regularized long wave, and modified Boussinesq models using a slightly modified technique. Additionally, the combined KdV and mKdV equations have been investigated by Zhang *et al.* [13], while Hassanian *et al.* [14] discuss multiple Jacobi elliptic functions applied to various non-linear models and coupled systems in. In this manuscript, we detail Malfliet's procedure and calculate multiple soliton waves for the mKdV equation with time-dependent variables. We assume the solitary wave of the suggested non-linear model:

$$u(T,x)=u(\xi),\quad \xi=\varphi+k(x-\omega T) \tag{4}$$

where ξ is the traveling wave and k, ω are random constants, which will be specified later, denoting the wave number and velocity, respectively, and φ – an arbitrary constant.

Replacing eq. (4) into eq. (3), we have:

$$\omega u'-6au^2u'+k^2u'''=0 \tag{5}$$

Malfliet's procedure in [24] assumes that the solution can be expressed as the product of two functions:

$$u(\xi)=p(\xi)q(\xi) \tag{6}$$

where $p(\xi)$, $q(\xi)$ are unspecified functions. We have these relations:

$$u'=(q'p)+(q\leftrightarrow p),\quad u'''=(q'''p+3q'p'')+(q\leftrightarrow p) \tag{7}$$

The non-linear factor $-6auu'$ is transformed into:

$$-6au'u=-\alpha u'u-\beta u'u=-\alpha pqu'-\beta u(pq'+qp'),\quad 6a=\alpha+\beta \tag{8}$$

By replacing eqs. (7) and (8) into eq. (5), we get:

$$p\left[k^2q'''+3k^2\frac{p''}{p}q'-\alpha\left(u^2+\frac{\omega}{\alpha}\right)q'-\frac{\beta}{2}quu'\right]+q\left(p\leftrightarrow q\right)=0 \tag{9}$$

for the presence of the differential q' in eq. (9), we let

$$\frac{k^2p''}{p}=u^2+\frac{\omega}{\alpha}$$

with a such structures for q, therefore, we discovery that for $\omega=p$, q we obtain the Schrodinger model:

$$k^2\psi''-\left(u^2+\frac{\omega}{\alpha}\right)\psi=0 \tag{10}$$

where $-\omega/4k^2$ the eigenvalue and u^2/k^2 the scattering potential. Thus eq. (9) takes the form:

$$p\left[k^2q'''+(3-\alpha)\left(u^2+\frac{\omega}{\alpha}\right)q'-\frac{\beta}{2}quu'\right]+q\left(p\leftrightarrow q\right)=0 \tag{11}$$

differentiating eq. (10) w.r.t. ξ:

$$k^2\psi'''-\left(u^2+\frac{\omega}{\alpha}\right)\psi'-2\psi uu'=0$$

we deduce that the resulting equality is coincide with eq. (11) gives $3-\alpha=1$, $\beta/2=2$ thus $\alpha=4$ and $\beta=4$. This gives $6\alpha=8$, or $\alpha=4/3$. Since the two symbols p, q content the Schrodinger model:

$$k^2\psi''-\left(u^2+\frac{\omega}{4}\right)\psi=0 \tag{12}$$

Suppose the potential u^2/k^2 is attractive, *i.e.*, $u^2/k^2<0$ and we can find distinct N discrete eigenvalues:

$$-\frac{\omega_n}{4k_n^2},\quad n=1,2,...,N$$

related with it. So eq. (12) can be written:

$$k_n^2\psi_n''-\left(u^2+\frac{\omega_n}{4}\right)\psi_n=0,\ \frac{\mathrm{d}}{\mathrm{d}\xi},\ \xi_n=k_n(x-\omega_nT)+\varphi_n \tag{13}$$

Consequently, the universal solution of the mKdV equation with a time-dependent coefficient can be formulated in terms of the wave functions, ψ_n can be expressed. As the Schrodinger eq. (13) has the wave solutions at this point:

$$u(\xi)=\sum_{n=1}^{N}\psi_n^2(\xi_n)\,,\quad \xi_n=k_n(x-\omega_n T)+\varphi_n \tag{14}$$

If there is no overlap between the functions ψ_n, eq. (13) can be expressed:

$$k_n^2\psi_n''-\left(\psi_n^4+\frac{\omega_n}{4}\right)\psi_n=0$$

To get the functions ψ_n satisfied the aforementioned equation, according to the tanh function method, we can equilibrium the non-linear term ψ_n^5 with the highest linear term ψ_n'', we may determine the series degree of the solution as $5s = s + 2$, so $s = 1/2$, therefore, we must take the transformation:

Then from eq. (15), we find that φ_n satisfies:

$$2k_n^2\varphi_n\,\varphi_n''-(\varphi_n')^2-4\varphi_n^4-\omega_n\varphi_n^2=0 \tag{15}$$

thus the solution take the form:

$$\varphi_n=a_0+a_1\tanh(\xi_n),\ \ [\tanh(\xi_n)]'=1-\tanh^2(\xi_n) \tag{16}$$

By substituting into eq. (15), we can derive an algebraic system by setting the coefficients of the distinct factors of $\tanh(\xi_n)$ to zero. Solving this system gives:

$$a_0=\frac{\sqrt{3}}{6},\ \ a_1=\frac{\sqrt{3}}{6},\ \ k_n=\frac{\sqrt{3}}{3},\ \ \omega_n=\frac{1}{3} \tag{17}$$

Then the functions φ_n take the form:

$$\varphi_n=\frac{\sqrt{3}}{6}\left(1+\tanh\left(\frac{\sqrt{3}}{3}\left(x-\frac{1}{3}T\right)\right)+\varphi_n)\right) \tag{18}$$

Thus we get the subsequent multiple kink wave for the mKdV model of time dependent variable:

$$u(\xi)=\frac{\sqrt{3}}{6}\left[\sum_{n=1}^{N}\left(1+\tanh\left(\frac{\sqrt{3}}{3}\left(x-\frac{1}{3}T\right)\right)+\varphi_n)\right)\right],\quad T=\int\beta(t)\,\mathrm{d}t \tag{19}$$

By seeking an alternative solution for the function φ_n defined in eq. (15) using the tan function method, we obtain:

$$\varphi_n=b_0+b_1\tan(\xi_n),\ \ [\tan(\xi_n)]'=1+\tan^2(\xi_n) \tag{20}$$

By replacing into eq. (20), we can create an algebraic system by setting the coefficients of the distinct factors of $\tanh(\xi_n)$ to zero. Solving this system of equations yields:

$$b_0=\frac{\sqrt{-3}}{6},\ \ b_1=\frac{\sqrt{3}}{6},\ \ k_n=\frac{\sqrt{3}}{3},\ \ \omega_n=-\frac{1}{3} \tag{21}$$

Then the functions φ_n take the form:

$$\varphi_n=\frac{\sqrt{3}}{6}i+\frac{\sqrt{3}}{6}\tan\left(\frac{\sqrt{3}}{3}\left(x+\frac{1}{3}T\right)+\varphi_n)\right) \tag{22}$$

Consequently, we derive the subsequent multiple bell wave solution of the mKdV model by utilizing a time-dependent variable:

$$u(\xi)=\frac{\sqrt{3}}{6}\left[\sum_{n=1}^{N}\left(i+\tan\left(\frac{\sqrt{3}}{3}\left(x+\frac{1}{3}T\right)+\varphi_n)\right)\right)\right],\quad T=\int\beta(t)\,\mathrm{d}t \tag{23}$$

By looking for another solution for eq. (15) using the coth function method, we obtain:

$$\varphi_n=c_0+c_1\coth(\xi_n),\ \ [\coth(\xi_n)]'=1-\coth^2(\xi_n) \tag{24}$$

Substituting into eq. (24), we can establish an algebraic system by equating the coefficients of the distinct factors of coth function zero. Resolving this system of equations yields:

$$a_0=\frac{\sqrt{3}}{6},\ \ a_1=\frac{\sqrt{3}}{6},\ \ k_n=\frac{\sqrt{3}}{3},\ \ \omega_n=\frac{1}{3} \tag{25}$$

Then the functions φ_n take the form:

$$\varphi_n=\frac{\sqrt{3}}{6}\left(1+\coth\left(\frac{\sqrt{3}}{3}\left(x-\frac{1}{3}T\right)\right)+\varphi_n\right) \tag{26}$$

Consequently, we derive the subsequent multiple singular kink wave solution of the mKdV model with a time-dependent variable:

$$u(\xi)=\frac{\sqrt{3}}{6}\left[\sum_{n=1}^{N}\left(1+\coth\left(\frac{\sqrt{3}}{3}\left(x-\frac{1}{3}T\right)+\varphi_n\right)\right)\right],\quad T=\int\beta(t)\,\mathrm{d}t \tag{27}$$

Looking for an alternative solution the function φ_n presented in eq. (15) using the rational tanh function, and applying the same steps, we find:

$$\varphi_n=\frac{1}{3}+\frac{2\sqrt{3}}{3}\left(\frac{\tanh\left(\frac{\sqrt{3}}{3}\left(x-\frac{4}{3}T\right)+\varphi_n\right)}{1\ \pm\ \tanh^2\left(\frac{\sqrt{3}}{3}\left(x-\frac{4}{3}T\right)+\varphi_n\right)}\right) \tag{28}$$

Thus we get the subsequent kink shaped wave of the mKdV model with time dependent variable:

$$u(\xi)=\sum_{n=1}^{N}\left(\frac{1}{3}+\frac{2\sqrt{3}}{3}\left(\frac{\tanh\left(\frac{\sqrt{3}}{3}\left(x-\frac{4}{3}T\right)+\varphi_n\right)}{1\ \pm\ \tanh^2\left(\frac{\sqrt{3}}{3}\left(x-\frac{4}{3}T\right)+\varphi_n\right)}\right)\right),\quad T=\int\beta(t)\,\mathrm{d}t \tag{29}$$

Based on these results, the multiple wave solutions for the mKdV model with the time-dependent eq. (1) are obtained directly. The single kink solution for $N=1$ takes the form:

$$u(\xi)=\frac{\sqrt{3}}{6}\left(1+\tanh\left(\frac{\sqrt{3}}{3}\left(x-\frac{1}{3}T\right)\right)+\varphi_1)\right),\quad T=\int\beta(t)\,\mathrm{d}t \tag{30}$$

The evolutional behavior of the solution eq. (19) at $N = 1$ characterized in fig. 1 demonstrations one soliton presentation with the following selection $\varphi_1 = 0$. Figure 1(a) with $\beta(t) = 1$, fig. 1(b) with $\beta(t) = 20t$, fig. 1(c) with $\beta(t) = \sinh t$, fig. 1(d) with $\beta(t) = \sin t$, fig. 1(e) with $\beta(t) = t\sin t$, fig. 1(f) with $\beta(t) = t^2\sin t$, fig. 1(g) with $\beta(t) = \sec^2 t$, fig. 1(h) with $\beta(t) = e^2$, and fig. 1(k) with $\beta(t) = \cos(t)e^{\sin(t)}$. We gain different types of one kink shaped soliton solution signified in fig. 1. The evolutional behavior of the solution eq. (19) when $N = 2$

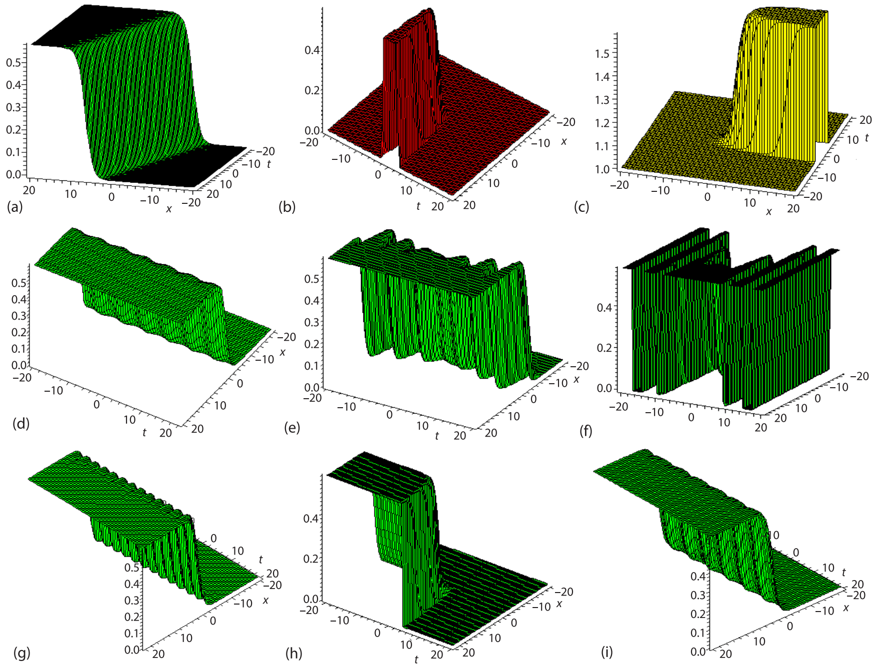

Figure 1. Structures of one soliton solution eq. (19) with $\varphi = 0$; (a) $\beta(t) = 1$, (b) $\beta(t) = 20t$, (c) $\beta(t) = \sinh t$, (d) $\beta(t) = \sin t$, (e) $\beta(t) = t\sin t$, (f) $\beta(t) = t^2\sin t$, (g) $\beta(t) = \sec^2 t$, (h) $\beta(t) = e^2$, and (k) $\beta(t) = \cos(t)e^{\sin(t)}$

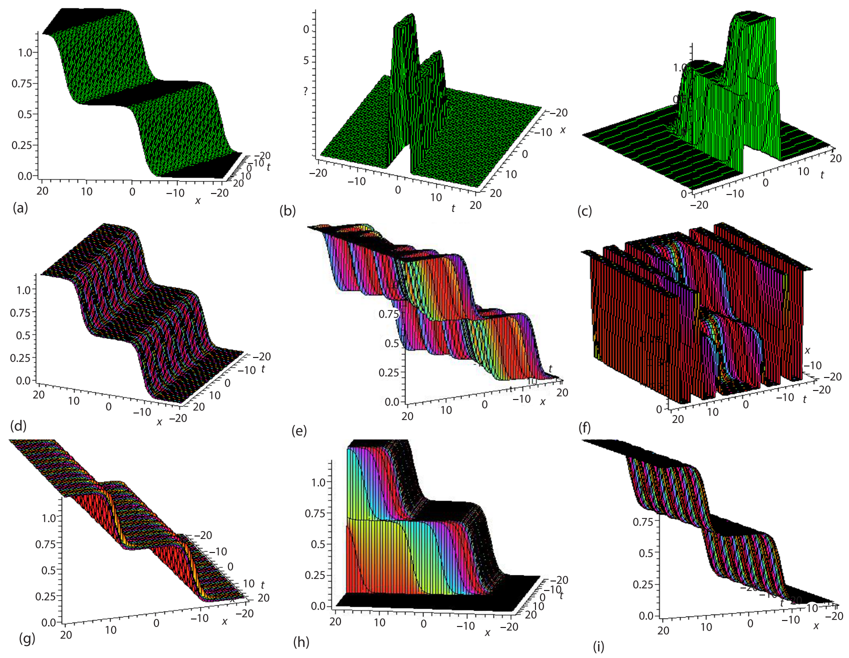

Figure 2. Structures of two soliton solution eq. (19) with $\varphi_1 = -5$ and $\varphi_2 = 5$; (a) $\beta(t) = 1$, (b) $\beta(t) = 20t$, (c) $\beta(t) = \sinh t$, (d) $\beta(t) = \sin t$, (e) $\beta(t) = t\sin t$, (f) $\beta(t) = t^2\sin t$, (g) $\beta(t) = \sec^2 t$, (h) $\beta(t) = e^2$, and (k) $\beta(t) = \cos(t)e^{\sin(t)}$

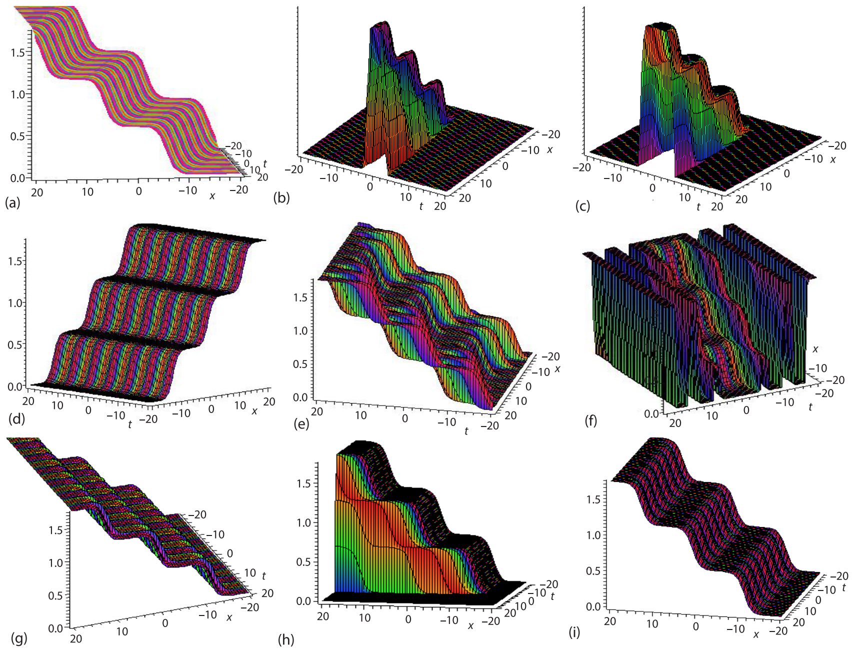

Figure 3. Structures of three soliton solution eq. (19) with $\varphi_1 = -7$ and $\varphi_2 = 7$; (a) $\beta(t) = 1$, (b) $\beta(t) = 20t$, (c) $\beta(t) = \sinh t$, (d) $\beta(t) = \sin t$, (e) $\beta(t) = t\sin t$, (f) $\beta(t) = t^2\sin t$, (g) $\beta(t) = \sec^2 t$, (h) $\beta(t) = e^2$, and (k) $\beta(t) = \cos(t)e^{\sin(t)}$

embodied in fig. 2 displays two soliton behaviors with the following selection $\varphi_1 = -5$ and $\varphi_2 = 5$. Figure 2(a) with $\beta(t) = 1$, fig. 2(b) with $\beta(t) = 20t$, fig. 2(c) with $\beta(t) = \sinh t$, fig. 2(d) with $\beta(t) = \sin t$, fig. 2(e) with $\beta(t) = \sin t$, fig. 2(f) with $\beta(t) = t^2\sin t$, fig. 2(g) with $\beta(t) = \sec^2 t$, fig. 2(h) with $\beta(t) = e^2$ and fig. 2(k) with $\beta(t) = \cos(t)e^{\sin(t)}$. We obtain different types of double kink shaped soliton solution characterized in fig. 2. The evolutional behavior of the solution eq. (19) when $N = 3$ denoted in fig. 3 displays three soliton behaviors with the following selection $\varphi_1 = -7$, $\varphi_2 = 7$, and $\varphi_3 = 0$. Figure 3(a) at $\beta(t) = 1$, fig. 3(b) at $\beta(t) = 20t$, fig. 3(c) at $\beta(t) = \sinh t$, fig. (3-d) at $\beta(t) = \sin t$, fig. 3(e) at $\beta(t) = t\sin t$, fig. 3(f) at $\beta(t) = t^2\sin t$, fig. 3(g) at $\beta(t) = \sec^2 t$, fig. 3(h) at $\beta(t) = e^2$, and fig. 3(k) at $\beta(t) = \cos(t)e^{\sin(t)}$. We gain different types of triple kink shaped soliton solution characterized in fig. 3.

Discussion

We have presented the time-dependent coefficient mKdV model with numerous soliton solutions in this leave. Using the tanh and tan function approaches in conjunction with the similarity transformation, we describe different kinds of multiple kink soliton solutions of the target model. Specifically, we investigate the developing multiple soliton constructions by a particular choice of similar time variable. The formally developed multiple soliton solutions demonstrate that the specific shape of these time-dependent variables can effectively influence the profiles of the multiple kink solitons. We explore different choices for the time-dependent coefficient and illustrate the obtained soliton solutions with graphs in figs. 1-3. The results can contribute to the discussion of additional integrable models for further insights.

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