INFERENCES BASED ON TYPE-I CENSORING COMPETING RISKS DATA OF POWER HAZARD RATE MODELS IN THE PRESENCE OF PARTIALLY OBSERVED CAUSES OF FAILURE

by

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When population units fail for several reasons, the competing risks model is triggered. The failure time and associated reason of failure are noted in this model. It is possible to partially observe the reasons why the competing risks model fails. In this work, where the failure time is distributed with the power hazard rate distribution, we utilize the competing risks model under partially observed reasons of failure. We develop maximum likelihood estimators of the model parameters with related estimated confidence intervals based on the independent type-I censoring competing risks data. Two distinct approaches are used to construct the bootstrap point estimate and associated bootstrap confidence ranges. Analysis is done using actual type-I competing risks data that has some failure causes missing at random.

Key words: *power hazard rate distribution, maximun likelihood estimation, competing risks model type-I censoring scheme, bootstrap confidence interval*

Introduction

The hazard rate function (HRF) was employed in reliability or survival analysis to gauge the unit's likelihood estimation (LH) of failure or death. The HRF can be used to solve the problem of identifying the process of aging or classifying the lifetime distribution. The HRF is used as a density function describe the force of decrement in actuarial work or force of mortality. In addition, HRF was employed to present the instantaneous failure rate, see [1]. As shown in [2], the power of HRF is selected to characterize different lifetime distributions. For given the power HRF:

$$H(t)=\alpha t^{\beta} \tag{1}$$

The probability density function (PDF) under power HRF eq. (1) is called power failure rate (FR) distribution is formulated:

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$$f(x) = \alpha t^{\beta} \exp\left\{\frac{-\alpha}{\beta+1} t^{\beta+1}\right\}, \;\; x > 0; \;\; -1 < \beta < \infty, \;\; \alpha > 0 \tag{2}$$

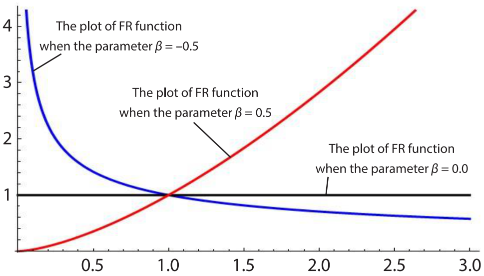

Figure 1. The FR functions of power FR distribution for $\alpha = 1$ and different values of β

where two parameters α and β known as scale and shape parameters. In literture, certain distributions can be categorized as a special case for power FR distribution. For example, we obtain the exponential distribution at $\beta = 0$, Rayleigh distribution at $\beta = 1$ and $\alpha = 1/\theta$, linear failure rate distribution at $\beta = 1$, Weibull distribution at $\beta = \theta - 1$. The power FR distribution has increasing FR function at $\beta > 0$, constant FR function at $\beta = 0$ and decreasing FR function at $\beta < 0$, see fig. 1.

There are a number of reasons why units in real-world populations do not function as intended. Additionally, using a model known as competing risks, we seek to evaluate one reason of failure in relation other causes. Time-to-failure and related failure reasons made up the observed data in this model. The competing risks concept has garnered a lot of interest in the literature, for further information, see [3-6]. This issue was just resolved in [7-9]. The failure time and associated failure causes are noted in the competing risks model. However, under certain limited and complex operating environments, the failure time may be noticed but the corresponding causes of failure cannot be identified; these are referred to as partially observed causes of failure [10, 11]. In this research, we construct the statistical inference for competing power FR distributions in the presence of partially observable reasons of failure. The competing risks model is developed when data is being collected while adhering to the type-I censoring strategy. As a result, we suggest that there are just two separate reasons for failure. The ML and bootstrap (BS) approaches are taken into consideration when formulating the classical estimate results. Analyzing an actual data set is where the developed results are discussed.

Model formulation

Suppose that, a random sample of size n, of units with identical independent lifetimes given by $T_1, T_2, \ldots, T_n$. Prior the experiment is running the ideal test time η is proposed. Under consideration, only two independent causes of failure are observed the observed failure time define by $T_i = \min(T_{i1}, T_{i2})$, $i = 1, 2, \ldots, n$. The competing risks data, (T_i, δ_i), $i = 1, 2, \ldots, r$, $r \le n$ are observed. Therefore, the observed values of competing failure time is denoted by $\mathbf{t} = \{(t_1, \delta_1), (t_2, \delta_2), \ldots, (t_r, \delta_r)\}$. The indicator δ_r under partially observed causes of failure $\delta \in \{1, 2, ^*\}$mean the first, second or unobserved causes of failure, respectively.

The joint LHF of type-I of the competing risks data $\mathbf{t} = \{(t_1, \delta_1), (t_2, \delta_2), \ldots, (t_r, \delta_r)\}$, is defined:

$$f_{1,2,\ldots,r}(\mathbf{t}|\Theta) = \frac{n!}{(n-r)!}\mathbf{S}(\eta)^{n-r} \prod_{i=1}^{r} \left[h_1(t_i)\right]^{\gamma(\delta_i=1)} \left[h_2(t_i)\right]^{\gamma(\delta_i=2)} S_1(t_i) S_2(t_i) \left[\mathbf{f}(t_i)\right]^{\gamma(\delta_i=*)} \tag{3}$$

where $S(.)$ and $h(.)$ are denoted to survival and hazrd failure rate functions:

$$\gamma(\delta_i = j) = \begin{cases} 1, & \delta_i = j \\ 0, & \delta_i \ne j \end{cases} \tag{4}$$

We consider the following assumptions:

- The latent variable T_{ij} ~ power FR(α_j, β), j = 1, 2 and T_i ~ power FR($\alpha_1 + \alpha_2$, β).
- The observed random variable $t_i = \min(t_{i1}, t_{i2})$ has power FR($\alpha_1 + \alpha_2$, β).
- The number of failure under the cause j is denoted by

$$m_j = \sum_{i=1}^{r} \gamma(\delta_i = j), \quad j = 1,2,3$$

- Therefore, the joint LHF eq. (3) under the last assumptions can be formulated:

$$L(\alpha_1, \alpha_2, \beta \mid \mathbf{t}) \propto \alpha_1^{m_1} \alpha_2^{m_2} \left(\alpha_1 + \alpha_2\right)^{m_3} \exp\left\{\beta \sum_{i=1}^{r} \log t_i - \frac{\left(\alpha_1 + \alpha_2\right)}{\beta + 1} \left\{(n-r)\eta^{\beta+1} + \sum_{i=1}^{r} t_i^{\beta+1}\right\}\right\} \tag{5}$$

Remark 1:

- In this model, we propose some of causes of failure are lost form some units under test and the observed failure time have power FR distribution with scale parameter $\alpha_1 + \alpha_2$ and shape parameter β.
- Bernoulli distribution is considered as discrete distribution of m_3 with probability of success $p \in (0, 1)$ (masking probability).
- Binomial distributions is considered as discrete distributions of m_1 and m_2 with probability of success $\alpha_1/(\alpha_1 + \alpha_2)$ and $\alpha_2/(\alpha_1 + \alpha_2)$, respectively, and number of trials $(r - m_3)$.

Maximum likelihood estimation

In this section, we discuss the point maximum likelihood estimation (MLE) of the model parameters (MP). Also, we formulate the approximate information matrix to formulate the approximate confidence intervals.

Maximum likelihood estimators

The joint LHF eq. (5) under taken the natural logarithms is reduced:

$$\ell(\alpha_1, \alpha_2, \beta \mid \mathbf{t}) = m_1 \log \alpha_1 + m_2 \log \alpha_2 + m_3 \log\left(\alpha_1 + \alpha_2\right) + \beta \sum_{i=1}^{r} \log t_i - \\ - \frac{\left(\alpha_1 + \alpha_2\right)}{\beta + 1} \left\{(n-r)\eta^{\beta+1} + \sum_{i=1}^{r} t_i^{\beta+1}\right\} \tag{6}$$

The LH equations (LHE) of the MP $\Theta = \{\alpha_1, \alpha_2, \beta\}$ are obtain from eq. (6) by taken the zero value of partial derivatives:

$$\frac{\partial \ell(\alpha_1, \alpha_2, \beta \mid \mathbf{t})}{\beta} = \sum_{i=1}^{r} \log t_i + \frac{\left(\alpha_1 + \alpha_2\right)}{\left(\beta + 1\right)^2} \left\{(n-r)\eta^{\beta+1} + \sum_{i=1}^{r} t_i^{\beta+1}\right\} - \\ - \frac{\left(\alpha_1 + \alpha_2\right)}{\beta + 1} \left\{(n-r)\eta^{\beta+1} \log \eta + \sum_{i=1}^{r} t_i^{\beta+1} \log t_i\right\} = 0 \tag{7}$$

$$\frac{\partial \ell(\alpha_1, \alpha_2, \beta \mid \mathbf{t})}{\alpha_1} = \frac{m_1}{\alpha_1} + \frac{m_3}{\alpha_1 + \alpha_2} - \frac{1}{\beta + 1} \left\{(n-r)\eta^{\beta+1} + \sum_{i=1}^{r} t_i^{\beta+1}\right\} = 0 \tag{8}$$

and

$$\frac{\partial \ell(\alpha_1,\alpha_2,\beta|\mathbf{t})}{\alpha_2}=\frac{m_2}{\alpha_2}+\frac{m_3}{\alpha_1+\alpha_2}-\frac{1}{\beta+1}\left\{(n-r)\eta^{\beta+1}+\sum_{i=1}^{r}t_i^{\beta+1}\right\}=0 \tag{9}$$

From the LHE eqs. (8) and (9), we have:

$$\alpha_1+\alpha_2=\frac{r(\beta+1)}{\left\{(n-r)\eta^{\beta+1}+\sum_{i=1}^{r}t_i^{\beta+1}\right\}} \tag{10}$$

Therefore, the LHE are reduced:

$$\sum_{i=1}^{r}\log t_i+\frac{r}{(\beta+1)}-r\frac{\left\{(n-r)\eta^{\beta+1}\log\eta+\sum_{i=1}^{r}t_i^{\beta+1}\log t_i\right\}}{\left\{(n-r)\eta^{\beta+1}+\sum_{i=1}^{r}t_i^{\beta+1}\right\}}=0 \tag{11}$$

and

$$\alpha_j(\beta)=\frac{m_j r(\beta+1)}{(r-m_3)\left\{(n-r)\eta^{\beta+1}+\sum_{i=1}^{r}t_i^{\beta+1}\right\}} \tag{12}$$

The LHE are reduced to one non-linear eq. (11) can be solve by iteration method as the following theorem. The MLE of has α_1 and α_2 are acquired from eq. (12) by replacing β by $\hat{\beta}$.

Theorem: From eq. (11), the MLE of β, is obtained by fixed point iteration (FPI):

$$\beta=g(\beta) \tag{13}$$

where

$$g(\beta)=\frac{r\left\{(n-r)\eta^{\beta+1}+\sum_{i=1}^{r}t_i^{\beta+1}\right\}}{r\left\{(n-r)\eta^{\beta+1}\log\eta+\sum_{i=1}^{r}t_i^{\beta+1}\log t_i\right\}-\left\{(n-r)\eta^{\beta+1}+\sum_{i=1}^{r}t_i^{\beta+1}\right\}\sum_{i=1}^{r}\log t_i}-1 \tag{14}$$

The FPI with initial value obtained from eq. (6) after replace α_1 and α_2 from eq. (12). When the value of $|\beta^{i+1}-\beta^{i}|$ is sufficiently small the iteration is stopped.

Approximate information matrix and confidence intervals

The negative expectation of the second partly derivative of the log-LHF with respect to model inputs is the definition of the Fisher information matrix (FIM) in the literature. This expectation is more unachievable in more situations. Therefore, we replace FIM by approximate information matrix define on the parameter vector $\Theta=\{\alpha_1,\alpha_2,\beta\}$:

$$\text{AIM}(\Theta)=\left[-\frac{\partial^2\ell(\alpha_1,\alpha_2,\beta|\mathbf{t})}{\partial\Theta_i\partial\Theta_j}\right]_{i=1,2,3\text{ and }j=1,2,3} \tag{15}$$

The AIM at the ML estimate of the MP $\hat{\Theta}=\{\hat{\alpha}_1,\hat{\alpha}_2,\hat{\beta}\}$ is denoted by $\text{AIM}^{(0)}(\hat{\Theta})$. Under normal proporty of ML estimate $\hat{\Theta}=\{\hat{\alpha}_1,\hat{\alpha}_2,\hat{\beta}\}$. The MP estimate are distributed with bivariate normal distribution and defined:

$$\hat{\Theta} \to \mathrm{N}\left(\hat{\Theta},\ \mathrm{AIM}^{(0)}(\hat{\Theta})\right) \tag{16}$$

The corresponding $100(1-\gamma)\%$ approximate confidence intervals of the MP $\Theta = \{\alpha_1, \alpha_2, \beta\}$ are formulated:

$$\hat{\alpha}_1 \mp z_{\frac{\gamma}{2}}\sqrt{e_{11}},\ \ \hat{\alpha}_2 \mp z_{\frac{\gamma}{2}}\sqrt{e_{22}},\ \text{ and }\ \hat{\beta}_1 \mp z_{\frac{\gamma}{2}}\sqrt{e_{33}} \tag{17}$$

where the tabulated value $z_{(\gamma/2)}$ is computed from $N(0,1)$ with confidence level equal to $(1-\gamma)$. Also, the values e_{11}, e_{22} and e_{33} are the diagonal of matrix $\mathrm{AIM}^{(0)}(\hat{\Theta})$. The approximate confidence intervals define by eq. (17) have shown that, the lower bound of interval may be zero. Hence, in the negative lower bound case logarithmic transformation under delta method can be applied, see [12, 13] as follows.

The pivotal quantity $(\log\hat{\Theta}_i - \log\Theta_i)/\mathrm{Var}(\log\hat{\Theta}_i)$ has standard normal distribution. Hence the approximate interval estimate of the MP $\Theta = \{\alpha_1, \alpha_2, \beta\}$ is formulated:

$$\left(\frac{\hat{\Theta}_i}{\exp\left(z_{\frac{\gamma}{2}}\sqrt{\mathrm{Var}(\log\hat{\Theta}_i)}\right)}, \hat{\Theta}_i \exp\left(z_{\frac{\gamma}{2}}\sqrt{\mathrm{Var}(\log\hat{\Theta}_i)}\right)\right) \tag{18}$$

where the value $\mathrm{Var}(\log\hat{\Theta}_i) = \mathrm{Var}(\hat{\Theta}_i)/\hat{\Theta}_i$ and $i = 1, 2, 3$.

Bootstrap confidence intervals

The BS procedure is referred to as a similar method in statistical literature. This technique is used to estimate the bias and variance of an estimator, calibrate hypothesis tests, and estimate the confidence ranges of the MP. The methods that were covered early in [14-17] were both parametric and non-parameteric. The BC interval using percentile BS and BS-methods is covered in this section; refer to [18]. The percentile BS and BS-t confidence intervals are constructed using the following procedure.

Algorithms:

- From the observed original data set $\mathbf{t} = \{(t_1, \delta_1), (t_2, \delta_2),..., (t_r, \delta_r)\}$, compute the ML estimate of the MP $\hat{\Theta} = \{\hat{\alpha}_1, \hat{\alpha}_2, \hat{\beta}\}$.
- For given masking probability p generate m_3 from Bernoulli distribution and two integer numbers m_1 and m_2 generated from binomial distribution.
- For given η generate the BS random variable $\mathbf{t} = \{t_1^*, t_2^*, t_r^*\}$ from power FR distribution with scale parameter $\hat{\alpha}_1 + \hat{\alpha}_2$ and shape parameter $\hat{\beta}$.
- Based on$\mathbf{t} = \{t_1^*, t_2^*, t_r^*\}$ compute the BS sample estimates $\hat{\Theta}^* = \{\hat{\alpha}^*{}_1, \hat{\alpha}^*{}_2, \hat{\beta}^*\}$.
- Steps from 2 to 4 are repeated MB times.
- Put the BS sample estimate in aseding order $(\hat{\alpha}^*{}_{1(1)}, \hat{\alpha}^*{}_{1(2)}, \dots, \hat{\alpha}^*{}_{1(\mathrm{MB})})$, $(\hat{\alpha}^*{}_{2(1)}, \hat{\alpha}^*{}_{2(2)}, \dots, \hat{\alpha}^*{}_{2(\mathrm{MB})})$, and $(\hat{\beta}^*{}_{(1)}, (\hat{\beta}^*{}_{(2)}, ..., (\hat{\beta}^*{}_{(\mathrm{MB})})$.
- The BS point estimate is given:

$$\hat{\alpha}_{1\mathrm{Boot}} = \frac{1}{\mathrm{MB}}\sum_{i=1}^{\mathrm{MB}}\hat{\alpha}^*_{1(i)},\ \hat{\alpha}_{2\mathrm{Boot}} = \frac{1}{\mathrm{MB}}\sum_{i=1}^{\mathrm{MB}}\hat{\alpha}^*_{2(i)} \text{ and } \hat{\beta}_{\mathrm{Boot}} = \frac{1}{\mathrm{MB}}\sum_{i=1}^{\mathrm{MB}}\hat{\beta}^*_{(i)} \tag{19}$$

Confidence intervals under percentile bootstrap technique

Suppose that, the empirical cumulative distribution function of ordering BS sample estimate define by $\Psi(x) = P(\hat{\Theta}^*_k < x)$ of $\hat{\Theta}^*_k$, $k = 1, 2, 3$. Then, $\hat{\Theta}^*_{k\mathrm{boot}} = \Psi^{-1}(x)$. Hence, $100(1-\gamma)\%$ percentile BC of the MP given:

$$\left[\hat{\Theta}^*_{k\text{boot}}\left(\frac{\gamma}{2}\right),\ \hat{\Theta}^*_{k\text{boot}}\left(1-\frac{\gamma}{2}\right)\right] \tag{20}$$

Confidence intervals under BS-t technique

The ordered BS sample estimate transformed to the order statistics defined:

$$\Omega_k^{*[i]} = \frac{\Theta_k^{*[i]} - \hat{\Theta}_k}{\sqrt{\text{var}\left(\Theta_k^{*[i]}\right)}},\quad k=1,2,3,\quad i=1,2,\ldots,\text{MB} \tag{21}$$

where $\hat{\Theta}_1 = \hat{\alpha}_1$, $\hat{\Theta}_2 = \hat{\alpha}_2$, $\hat{\Theta}_3 = \hat{\beta}$, and $\Omega_k^{*[1]} < \Omega_k^{*[2]} < \ldots < \Omega_k^{*[\text{MB}]}$.

Suppose that, the empirical cumulative distribution function of ordering statistics sample define by $\Psi(x) = P(\hat{\Theta}^*_k < x)$ on Ω^*_k. Hence, we define for a given:

$$\hat{\Theta}_{k\text{boot-t}} = \hat{\Theta}_k + \sqrt{\text{Var}(\hat{\Theta}_k)}\mathbf{\Psi}^{-1}(x) \tag{22}$$

Hence, $100(1-\gamma)\%$ BC interval of the MP given:

$$\left(\hat{\Theta}_{k\text{boot}-t}\left(\frac{\gamma}{2}\right), \hat{\Theta}_{k\text{boot}-t}\left(1-\frac{\gamma}{2}\right)\right) \tag{23}$$

Data analysis

In this section, we consider the real data set presented by [19] of the lifetimes obtained under conventional laboratory environment. The real data sets obtained from radiation male mice at age of 5-6 weeks with 300 roentgens. This data is descussed by different authors [20-24]. Suppose thymic lymphoma is considered as the first cause. Hence, the data under first cause of failure {159, 189, 191, 198, 200, 207, 220, 235, 245, 250, 256, 261, 265, 266, 280, 343, 356, 383, 403, 414, 428, 432}. Other causes are considered as the second cause of failure {40, 42, 51, 62, 163, 179, 206, 222, 228, 252, 249, 282, 324, 333, 341, 366, 385, 407, 420, 431, 441, 461, 462, 482, 517, 517, 524, 564, 567, 586, 619, 620, 621, 622, 647, 651, 686, 761, 763}.

For testing the validity of data for power FR distribution, we plot the fitted survival functions and empirical survival functions, figs. 2 and 3. Also, the values of Kolmogorov-Smirnov (K-S) distances between the fitted distribution function and empirical distribution function which equal 0.2063 for the first cause and 0.1021 for the second cause. Hence, data have shown the power FR distributions is good fit for the model data. The ordered data under two choose of $\eta = \{4, 6\}$ are reported in tab. 1. The results of MLE and BS estimate for $\eta = 0.4$ and $\eta = 0.6$ are reported in tabs. 2 and 3.

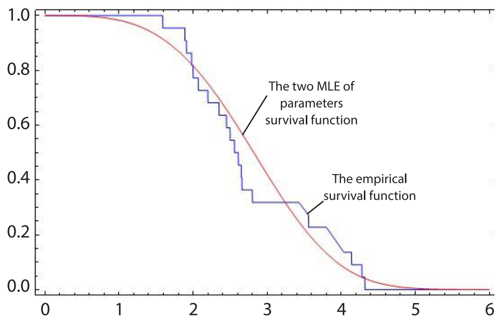

Figure 2. The fit data under first cause

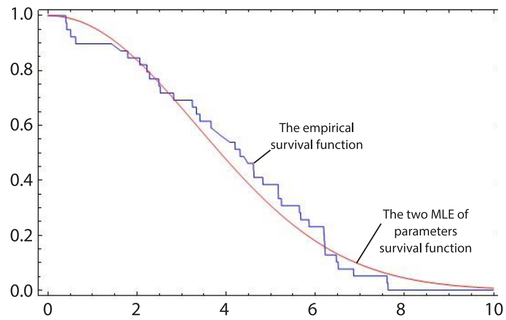

Figure 3. The fit data under second cause

Table 1. The Autopsy data of 61 male mice at age 5-6 weeks unde 300r radiation radiation

0.4	0.42	0.51	0.62	1.59	1.63	1.79	1.89	1.91	1.98	2.0	2.06	2.07	2.2	2.22
2	2	2	2	1	2	2	1	1	1	1	2	1	1	2
2.28	2.35	2.45	2.49	2.5	2.52	2.56	2.61	2.65	2.66	2.8	2.82	3.24	3.33	3.41
2	1	1	2	1	*	1	1	1	1	1	*	2	2	2
3.43	3.56	3.66	3.83	3.85	4.03	4.07	4.14	4.2	4.28	4.31	4.32	4.41	4.61	4.62
1	1	*	*	2	1	*	1	2	1	2	1	2	2	*
4.82	5.17	5.17	5.24	5.64	5.67	5.86								
2	*	2	2	*	2	*								

Table 2. The estimate values when $\tau = 0.4$

Parameter	$(.)_{ML}$	$(.)_{boot}$	95% ACI	95% ACI(boot-p)	95% ACI(boot-t)
α_1	0.0648	0.0782	(0.0244, 0.1052)	(0.0215, 0.1352)	(0.0258, 0.1011)
α_2	0.0533	0.0654	(0.0180, 0.0886)	(0.0104, 0.0975)	(0.0197, 0.0844)
β	0.9015	0.9333	(0.3264, 1.4766)	(0.3335, 1.4987)	(0.3211, 1.4714)

Table 3. The estimate values when $\tau = 0.6$

Parameter	$(.)_{ML}$	$(.)_{boot}$	95% ACI	95% ACI(boot-p)	95% ACI(boot-t)
α_1	0.0581	0.0847	(0.0231, 0.0931)	(0.0452, 0.2361)	(0.0365, 0.0874)
α_2	0.0609	0.0929	(0.0246, 0.0971)	(0.0421, 0.1345)	(0.0331, 0.0905)
β	0.9348	0.9873	(0.4919, 1.3775)	(0.3529, 1.547)	(0.4954, 1.3554)

Conclusions

Reliability studies frequently examine failure under various failure sources. Here, we create the statistical inference of competing power FR models in the case of partially observed failure causes. Additionally, real-world data is analyzed using the suggested model. The suggested model performs well under the type-I censoring scheme for competing hazards, according to the numerical results. Compared to approximate confidence intervals and BS-p confidence intervals, the results of the confidence interval under BS-t are more useful. Additionally, real-world data is analyzed using the suggested model. The suggested model performs well under the type-I censoring scheme for competing hazards, according to the numerical results. Percentile BC intervals, BC-t intervals, and confidence interval outcomes under ML estimation are useful. Additionally, take note of the following points from an actual data set.

- Compared to percentile BS and approximate confidence intervals, BC intervals are more useful.
- The outcome is better for extending the optimal test duration τ.

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