ANALYSIS OF A LORENZ MODEL USING ADOMIAN DECOMPOSITION AND FRACTAL-FRACTIONAL OPERATORS

by

Tao YAN[a], Muflih ALHAZMI[b], Mukhtar Y. YOUSSIF[c], Amna E. ELHAG[d], Abdulrahman F. ALJOHANI[e], and Sayed SABER[f,g*]

[a] School of Computer Science, Chengdu University, Chengdu, China
[b] Mathematics Department, Faculty of Science, Northern Border University, Arar, Saudi Arabia
[c] Department of Mathematics and Statistics, College of Science, Taif University, Taif, Saudi Arabia
[d] Department of Mathematics, College of Science, Qassim University, Saudi Arabia
[e] Department of Mathematics, Faculty of Science, University of Tabuk, Tabuk, Saudi Arabia
[f] Department of Mathematics and Statistics, Faculty of Science, Beni-Suef University, Beni-Suef, Egypt
[g] Department of Mathematics, Al-Baha University, Baljurashi, Saudi Arabia

Original scientific paper
https://doi.org/10.2298/TSCI2406001Y

This paper extends the classical Lorenz system to incorporate fractal-fractional dynamics, providing a detailed numerical analysis of its chaotic behavior. By applying Caputo's fractal-fractional operators to the Lorenz system, the study explores the fractal and fractional nature of non-linear systems. Numerical methods are employed to solve the extended system, with suitable fractal and fractional orders chosen to demonstrate chaos and hyper-chaos. The results are presented graphically, highlighting the complex dynamic behavior of the system under different parameter conditions. This research advances the understanding of fractional calculus in modelling and controlling chaotic systems in various scientific fields.

Key-words: *fractional derivatives, non-linear equations, simulation, numerical results, iterative method, time varying control system, Lyapunov functions*

Introduction

During the 17th century, fractional calculus gained significant attention for its applications in engineering [1], physics [2], mathematical biology [3], as well as psychological and life sciences [4]. This specialized branch of calculus has transformed our capacity to model, examine, and interpret complex natural phenomena. Numerous interdisciplinary systems, such as those in viscoelasticity [5], dielectric polarization [6], electrode-electrolyte interactions [7], electromagnetic wave propagation [8], and quantum dynamics [9], are accurately represented by fractional differential equations. Fractional calculus is particularly effective in capturing chaotic behavior in dynamic systems, with examples including fractional-order models of the Lorenz [10-12], Chua's circuit [13], Rossler [14], Chen [15], and Liu systems [16, 17], Burke-Shaw [18], Newton-Leipnik [19]. The field primarily utilizes several types of fractional derivatives, namely, the Riemann-Liouville, Caputo, Caputo-Fabrizio, and Atangana-Baleanu operators, each linked to specific decay and memory characteristics like power laws, exponen-

[*] Corresponding author, e-mail: sayed011258@science.bsu.edu.eg

tial decay, and the Mittag-Leffler function. The role of fractal-fractional operators is especially impactful for real-world applications in fields like engineering, biology, physics, and medicine [20-26].

In this paper, we explore chaotic and hyper-chaotic behavior within the context of a unified family of chaotic systems, which generalizes the dynamics of three different types of Lorenz systems. These systems are distinguished by a key parameter, σ, that determines the specific chaotic behavior of the system. The traditional Lorenz system, governed by a single parameter, can be described:

$$\begin{aligned} \dot{u} &= (25\eta + 10)(v - u) \\ \dot{v} &= (28 - 35\eta)u - uw + (29\eta - 1)v \\ \dot{w} &= uv - \frac{(\alpha + 8)}{3} w \end{aligned} \tag{1}$$

where $\sigma \in [0, 0.8)$. In the fractal-fractional framework, this system (1) is extended:

$$\begin{aligned} {}^{FFC}\mathcal{D}_{0,t}^{p,q} u(t) &= (25\eta + 10)(v - u) \\ {}^{FFC}\mathcal{D}_{0,t}^{p,q} v(t) &= (28 - 35\eta)u - uw + (29\eta - 1)v \\ {}^{FFC}\mathcal{D}_{0,t}^{p,q} w(t) &= uv - \frac{(\alpha + 8)}{3} w \end{aligned} \tag{2}$$

where σ, v denote the fractional and fractal orders, respectively. This paper extends the analysis of fractal-fractional non-linear systems by translating them into linear equations and conducting numerical analyses. We construct a fractal-fractional Lorenz model, select appropriate parameters and initial conditions to demonstrate chaotic behavior, and apply Caputo's fractal-fractional operators. The numerical solutions to the fractal-fractional models are presented, along with graphical results, illustrating the system's behavior under various fractal and fractional order settings.

Existence and uniqueness

Our goal is to establish that the system presented in eq. (1) has a unique solution under specific conditions. Assume that, for all $t \in [0, T]$, the functions $u(t)$, $v(t)$, and $w(t)$ are bounded, such that $||u||_\infty \leq N$, $||v||_\infty \leq N$, and $||w||_\infty \leq N$. The system can be written in the form:

$$\begin{aligned} g_1(t,u,v,w) &= (25\eta + 10)(v - u) \\ g_2(t,u,v,w) &= (28 - 35\eta)u - uw + (29\eta - 1)v \\ g_3(t,u,v,w) &= uv - \frac{(\eta + 8)}{3} w \end{aligned} \tag{3}$$

Given that u, v, and w are bounded, the functions g_u, g_v, and g_w will also be bounded. Consequently, there exist constants N_u, N_v, and N_w such that:

$$\sup_{t\in\mathcal{U}} |u(t)| = \|u\|_\infty \leq N_u, \quad \sup_{t\in\mathcal{V}} |v(t)| = \|v\|_\infty \leq N_v, \quad \sup_{t\in\mathcal{W}} |w(t)| = \|w\|_\infty \leq N_w$$

We begin by showing that these functions satisfy the linear growth condition:

$$|g_u(t,u,v,w)| \leq (25\eta + 10)\left(\sup_{t\in\mathcal{U}}|u| + \sup_{t\in\mathcal{V}}|v|\right) \leq (25\eta + 10)(N_u + N_v) = N_{g_u} < \infty$$

$$|g_v(t,u,v,w)| \leq |28 - 35\eta| \sup_{t\in\mathcal{U}}|u| + |29\eta - 1| \sup_{t\in\mathcal{V}}|v| + \sup_{t\in\mathcal{U}}|u| \sup_{t\in\mathcal{W}}|w| \leq |28 - 35\eta| N_u + |29\eta - 1| N_v + N_u N_w = N_{g_v} < \infty$$

$$\left|g_w(t,u,v,w)\right| \le \sup_{t\in\mathcal{U}}|u|\sup_{t\in\mathcal{V}}|v| + \frac{(\eta+8)}{3}\sup_{t\in\mathcal{W}}|w| \le N_u N_v + \frac{(\eta+8)}{3}N_w = N_{g_w} < \infty$$

Furthermore, we demonstrate that the functions meet the Lipschitz condition:

$$\left|g_u(t,u_1,v,w) - g_u(t,u_2,v,w)\right| = \left|(25\eta+10)(u_1-u_2)\right| \le \frac{3}{2}\left|25\eta+10\right|\left|u_1-u_2\right|$$

$$\left|g_v(t,u,v_1,w) - g_v(t,u,v_2,w)\right| = \left|(29\eta-1)(v_1-v_2)\right| \le \frac{3}{2}\left|29\eta-1\right|\left|v_1-v_2\right|$$

$$\left|g_w(t,u,v,w_1) - g_w(t,u,v,w_2)\right| = |c|\left|w_1-w_2\right| \le \frac{3}{2}|c|\left|w_1-w_2\right|$$

The functions g_u, g_v, and g_w satisfy the Lipschitz condition provided that the maximum of $|a|$, $|b|$, and $|c|$ is less than 1.

Finally, we confirm that g_u, g_v, and g_w meet both the linear growth and Lipschitz conditions:

$$\left|g_u(t,u,v,w)\right|^2 \le 3\mathrm{e}^2 N_v^4\left(1+\frac{a^2|u|^2}{3\mathrm{e}^2 N_v^4}\right) \le C_u\left(1+|u|^2\right)$$

where

$$\frac{a^2}{3\mathrm{e}^2 N_v^4} < 1 \text{ and } C_u = \frac{a^2}{3\mathrm{e}^2 N_v^4}$$

$$\left|g_v(t,u,v,w)\right|^2 \le 3(28-35\eta)^2\,|u|^2 + 3(29\eta-1)^2\,|v|^2 + 3|u|^2|w|^2 \le$$

$$\le C_v\left(1+|v|^2\right), \text{ with } C_v = \frac{(29\eta-1)^2}{\left((28-35\eta)^2 N_u^2 + N_u^2 N_w^2\right)} < 1$$

$$\left|g_w(t,u,v,w)\right|^2 = \left|uv - \frac{\eta+8}{3}w\right|^2 \le C_w\left(1+|w|^2\right)$$

where

$$C_w = \frac{3(\eta+8)^2}{eN_v^4} < 1$$

Application of Adomian decomposition method

The non-linear terms in the system (2) are defined:

$$N_1(\overline{y}) = (25\eta+10)(v-u) = \sum_{j=0}^{\infty} A_{1j}$$

$$N_2(\overline{y}) = (28-35\eta)u - uw + (29\eta-1)v = \sum_{j=0}^{\infty} A_{2j} \tag{4}$$

$$N_3(\overline{y}) = uv - (\eta+8)w/3 = \sum_{j=0}^{\infty} A_{3j}$$

Following (4) are the terms used in the Adomian decomposition series:

$$u := 1 + 1.128379167 \times t^{0.5}$$

$$v := 3 + 1.114242509 \times t^{0.3} + 1.114242509 \times t^{0.3}((-2.228485018 + 32.31303276\eta) \times t^{3/10} -$$

$$-39.49327084 \times t^{0.5}\eta + 29.33785835 \times t^{0.5}) + 1.114242509 \times t^{0.9000000000}((-1276.147354 \times t^{1/5} +$$

$$+1044.132086)\eta^2 + (-107.5998186 + 992.0002592 \times t^{1/5})\eta - 37.71864103 \times t^{1/5} + 4.552300020)$$

$$w := 2 + 1.114242509 \times t^{0.3} + 1.114242509 \times t^{0.3}((-1.857070849 - 0.3714141697\eta) \times t^{3/10} +$$

$$+3.385137501 \times t^{0.5}) + 1.114242509 \times t^{0.9000000000}((0.1379484855 \times \eta^2 +$$

$$+(-45.26236922 \times t^{1/5} + 37.79788501)\eta + 23.88847266 \times t^{1/5} + 3.034866681)$$

Numerical approach

Under the fractal-fractional-Caputo operator, the numerical approach described by (2) is presented. Model (2) is reformulated into Volterra form, since the fractional integral is differentiable, so in Riemann-Liouville sense:

$$^{FFP}\mathcal{D}_{0,t}^{\alpha,\beta} g(t) = \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t (t-\tau)^{-\alpha} g(\tau)\mathrm{d}\tau \frac{1}{\tau t^{\beta-1}}$$

From this, we derive the results:

$$^{RL}\mathcal{D}_{0,t}^{\alpha}(u(t)) = \beta t^{\beta-1}[(25\eta+10)(v-u)]$$

$$^{RL}\mathcal{D}_{0,t}^{\alpha}(v(t)) = \beta t^{\beta-1}[(28-35\eta)u - uv + (29\eta-1)v]$$

$$^{RL}\mathcal{D}_{0,t}^{\alpha}(w(t)) = \beta t^{\beta-1}[uv - (\eta+8)w/3]$$

Now, consider substituting the RL derivative with the Caputo derivative to leverage the integer-order initial conditions. On both sides, we apply the Riemann-Liouville fractional integral:

$$u(t) = u(0) + \frac{\beta}{\Gamma(\alpha)} \int_0^t \tau^{\beta-1}(t-\tau)^{\alpha-1} g_1(u,v,w,\tau)\mathrm{d}\tau$$

$$v(t) = v(0) + \frac{\beta}{\Gamma(\alpha)} \int_0^t \tau^{\beta-1}(t-\tau)^{\alpha-1} g_2(u,v,w,\tau)\mathrm{d}\tau \tag{5}$$

$$w(t) = w(0) + \frac{\beta}{\Gamma(\alpha)} \int_0^t \tau^{\beta-1}(t-\tau)^{\alpha-1} g_3(u,v,w,\tau)\mathrm{d}\tau$$

where g_1, g_2, and g_3 are defined in eq. (3). We now introduce a novel procedure for the aforementioned model (5) at t_{n+1}, which transforms our model into:

$$u^{n+1} = u^0 + \frac{\beta}{\Gamma(\alpha)}\int_0^t \tau^{\beta-1}(t_{n+1}-\tau)^{\alpha-1} g_1(u,v,w,\tau)\mathrm{d}\tau$$
$$v^{n+1} = v^0 + \frac{\beta}{\Gamma(\alpha)}\int_0^t \tau^{\beta-1}(t_{n+1}-\tau)^{\alpha-1} g_2(u,v,w,\tau)\mathrm{d}\tau \tag{6}$$
$$w^{n+1} = w^0 + \frac{\beta}{\Gamma(\alpha)}\int_0^t \tau^{\beta-1}(t_{n+1}-\tau)^{\alpha-1} g_3(u,v,w,\tau)\mathrm{d}\tau$$

Approximating the integrals (6) gives:

$$u^{n+1} = u^0 + \frac{\beta}{\Gamma(\alpha)}\sum_{j=0}^{n}\int_{t_j}^{t_{j+1}} \tau^{\beta-1}(t_{n+1}-\tau)^{\alpha-1} g_1(u,v,w,\tau)\mathrm{d}\tau$$
$$v^{n+1} = v^0 + \frac{\beta}{\Gamma(\alpha)}\sum_{j=0}^{n}\int_{t_j}^{t_{j+1}} \tau^{\beta-1}(t_{n+1}-\tau)^{\alpha-1} g_2(u,v,w,\tau)\mathrm{d}\tau \tag{7}$$
$$w^{n+1} = w^0 + \frac{\beta}{\Gamma(\alpha)}\sum_{j=0}^{n}\int_{t_j}^{t_{j+1}} \tau^{\beta-1}(t_{n+1}-\tau)^{\alpha-1} g_3(u,v,w,\tau)\mathrm{d}\tau$$

Now, approximating the function $\tau^{\beta-1}g_i\,(u, v, w, \tau)$ for $i = 1, 2, 3$ in the interval $[t_j, t_{j+1}]$ using Lagrange piece-wise interpolation yields:

$$G_j(\tau) = \frac{\tau - t_{j-1}}{t_j - t_{j-1}} t_j^{\beta-1} g_1\left(u^j, v^j, w^j, t_j\right) - \frac{\tau - t_j}{t_j - t_{j-1}} t_{j-1}^{\beta-1} g_1\left(u^{j-1}, v^{j-1}, w^{j-1}, t_{j-1}\right)$$
$$H_j(\tau) = \frac{\tau - t_{j-1}}{t_j - t_{j-1}} t_j^{\beta-1} g_2\left(u^j, v^j, w^j, t_j\right) - \frac{\tau - t_j}{t_j - t_{j-1}} t_{j-1}^{\beta-1} g_2\left(u^{j-1}, v^{j-1}, w^{j-1}, t_{j-1}\right) \tag{8}$$
$$J_j(\tau) = \frac{\tau - t_{j-1}}{t_j - t_{j-1}} t_j^{\beta-1} g_3\left(u^j, v^j, w^j, t_j\right) - \frac{\tau - t_j}{t_j - t_{j-1}} t_{j-1}^{\beta-1} g_3\left(u^{j-1}, v^{j-1}, w^{j-1}, t_{j-1}\right)$$

Thus, using (8), system (7) becomes:

$$u^{n+1} = u^0 + \frac{\beta}{\Gamma(\alpha)}\sum_{j=0}^{n}\int_{t_j}^{t_{j+1}} \tau^{\beta-1}(t_{n+1}-\tau)^{\alpha-1} G_j(\tau)\mathrm{d}\tau$$
$$v^{n+1} = v^0 + \frac{\beta}{\Gamma(\alpha)}\sum_{j=0}^{n}\int_{t_j}^{t_{j+1}} \tau^{\beta-1}(t_{n+1}-\tau)^{\alpha-1} H_j(\tau)\mathrm{d}\tau \tag{9}$$
$$w^{n+1} = w^0 + \frac{\beta}{\Gamma(\alpha)}\sum_{j=0}^{n}\int_{t_j}^{t_{j+1}} \tau^{\beta-1}(t_{n+1}-\tau)^{\alpha-1} J_j(\tau)\mathrm{d}\tau$$

Thus (9) leads us to the formulations:

$$u^{n+1} = u^0 + \frac{\beta h^\alpha}{\Gamma(\alpha+2)} \sum_{j=1}^{n} \Big[t_j^{\beta-1} g_1\left(u^j, v^j, w^j, t_j\right) \times \left((n+1-j)^\alpha (n-j+2+\alpha) - (n-j)^\alpha (n-j+2+2\alpha)\right) -$$
$$- t_{j-1}^{\beta-1} g_1\left(u^{j-1}, v^{j-1}, w^{j-1}, t_{j-1}\right) \times \left((n-j)^\alpha (n-j+2+\alpha) - (n-j-1)^\alpha (n-j-1+2+2\alpha)\right)\Big]$$

$$v^{n+1} = v^0 + \frac{\beta h^\alpha}{\Gamma(\alpha+2)} \sum_{j=1}^{n} \Big[t_j^{\beta-1} g_2\left(u^j, v^j, w^j, t_j\right) \times \left((n+1-j)^\alpha (n-j+2+\alpha) - (n-j)^\alpha (n-j+2+2\alpha)\right) -$$
$$- t_{j-1}^{\beta-1} g_2\left(u^{j-1}, v^{j-1}, w^{j-1}, t_{j-1}\right) \times \left((n-j)^\alpha (n-j+2+\alpha) - (n-j-1)^\alpha (n-j-1+2+2\alpha)\right)\Big]$$

$$w^{n+1} = w^0 + \frac{\beta h^\alpha}{\Gamma(\alpha+2)} \sum_{j=1}^{n} \Big[t_j^{\beta-1} g_3\left(u^j, v^j, w^j, t_j\right) \times \left((n+1-j)^\alpha (n-j+2+\alpha) - (n-j)^\alpha (n-j+2+2\alpha)\right) -$$
$$- t_{j-1}^{\beta-1} g_3\left(u^{j-1}, v^{j-1}, w^{j-1}, t_{j-1}\right) \times \left((n-j)^\alpha (n-j+2+\alpha) - (n-j-1)^\alpha (n-j-1+2+2\alpha)\right)\Big]$$

Numerical simulation and discussions

The numerical simulations of the system (2) are presented in figs 1-6, for $\sigma = 1$, $\nu = 1$, $\sigma = 1$, $\nu = 0.95$, $\sigma = 1$, $\nu = 0.90$, respectively.

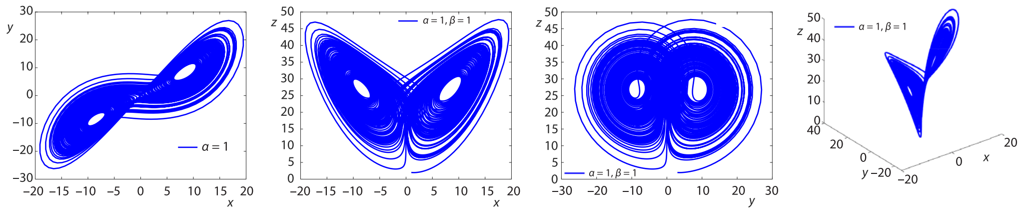

Figure 1. Numerical simulation for the system (1) at $\sigma = 1$, $\nu = 1$

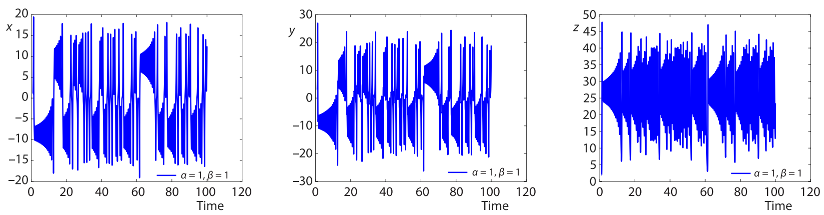

Figure 2. Time series solution of (1) at $\sigma = 1$, $\nu = 1$

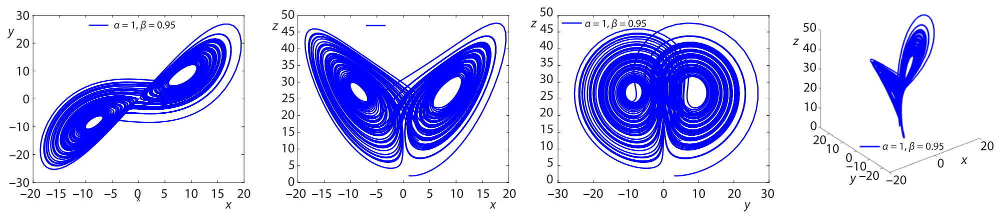

Figure 3. Numerical simulation for the system (1) at $\sigma = 1$, $\nu = 0.95$

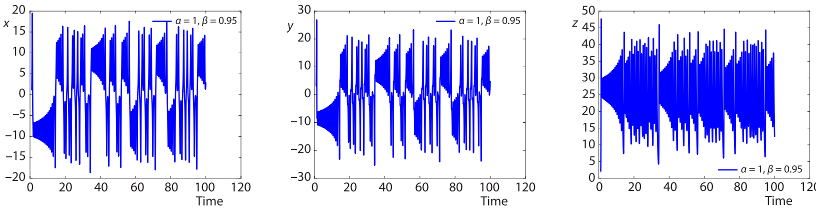

Figure 4. Time series solution of (1) at $\sigma = 1$, $\nu = 0.95$

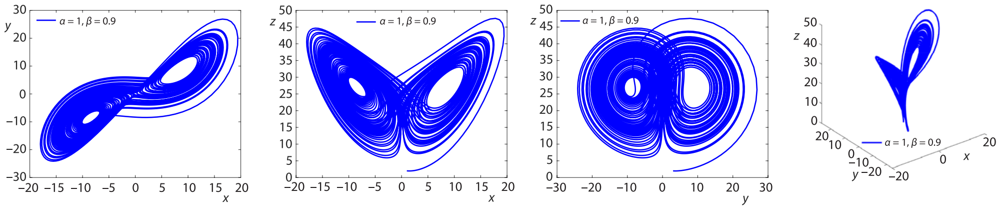

Figure 5. Numerical simulation for the system (1) at $\sigma = 1$, $\nu = 0.90$

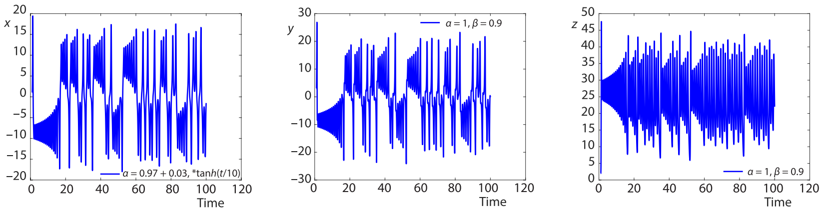

Figure 6. Time series solution of (1) at $\sigma = 1$, $\nu = 0.90$

Discussion

This section advocates for the use of fractal-fractional operators in modelling to incorporate memory and hereditary properties, thus enhancing the accuracy of chaotic systems such as the Lorenz system. The exploration of chaotic and hyper-chaotic behaviors was facilitated by varying fractal orders (σ, ν) under different parameter conditions. The fractal-fractional approach also shows promise for improving control in applications like electrical circuits and climate dynamics. Numerical simulations, figs. 1-6, illustrated how altering fractional orders, σ, and fractal dimensions, ν, influenced system behavior, particularly noting that decreasing ν led to progressively dampened chaotic oscillations and more stable dynamics. The findings underscore the system's sensitivity to small variations in fractal parameters, which could reflect noise or uncertainty. The effectiveness of the fractal-fractional ADM was discussed, highlighting its capability to break down non-linear terms for quicker and more efficient solution computation. Compared to traditional methods, ADM demonstrated robustness for complex systems exhibiting fractional and fractal behavior. This model can be applied to physical systems where integer-order models fail to capture important details, emphasizing the relevance of long-term memory effects in fields like electrical engineering, climate science, and biology.

Conclusion

The study successfully integrated fractal-fractional dynamics into the classical Lorenz system, effectively capturing and analyzing complex chaotic and hyper-chaotic behaviors through the introduction of fractional and fractal orders. The ADM proved to be an efficient numerical technique for solving such non-linear systems, allowing for accurate computation of approximate solutions by decomposing non-linear terms. Numerical results indicated that fractional and fractal orders significantly influenced system dynamics, with lower fractal dimensions leading to reduced chaotic behavior. This suggests the potential of fractal-fractional calculus for controlling chaos in non-linear systems, applicable across various fields, including climate dynamics, engineering, and biology. The findings emphasize the utility of fractal-fractional operators in accurately modelling complex systems where memory and hereditary effects are critical. Future research could explore incorporating stochastic effects or time-varying parameters to further enhance the model's applicability to real-world chaotic phenomena. This study lays a solid foundation for future inquiries into fractal-fractional dynamic systems and chaos control.

Acknowledgment

The authors would like to acknowledge the Deanship of Graduates Studies and Scientific Research, Taif University, for funding this work.

References

[1] Arena, P., *et al*., *Non-Linear Non-Integer Order Circuits and Systems*, World Scientific, Singapore, Singapore, 2000
[2] Hilfer, R., *Applications of fractional Calculus in Physics*, World Scientific, Singapore, Singapore, 2000
[3] Ahmed, E., Elgazzar A. S., On Fractional Order Differential Equations Model for Non-Local Epidemics, *Physica A*., *379* (2007), 2, pp. 607-614
[4] Ahmad, W. M., El-Khazali R., Fractional-Order Dynamical Models of Love, *Chaos Soliton Fract*, *33* (2007), 4, pp. 1367-1375
[5] Bagley, R. L., Calico, R. A., Fractional Order State Equations for the Control of Viscoelastically Damped Structures, *J. Guid Control Dyn*, *14* (1991), May, pp. 304-311
[6] Sun, H. H., *et al*., Linear Approximation of Transfer Function with a Pole of Fractional Order, IEEE Trans Auto Contr, *29* (1984), 5, pp. 441-444
[7] Ichise, M., *et al*., An Analog Simulation of Non-Integer Order Transfer Functions for Analysis of Electrode process, *J. Electroanal Chem.*, *33* (1971), 2, pp. 253-265
[8] Heaviside, O., *Electromagnetic Theory*, Chelsea, New York, USA, 1971
[9] Kusnezov, D., *et al*., Quantum Levy Processes and Fractional Kinetics, *Phys. Re.v Lett*., *82* (1999), 1136
[10] Podlubny, I., Geometric and Physical Interpretation of Fractional Integration and Fractional Differentiation, *Fract. Calc. Appl. Anal*., *5* (2002), 4, pp. 367-386
[11] Grigorenko, I., Grigorenko E., Chaotic Dynamics of the Fractional Lorenz System, *Phys. Rev. Lett*., *91* (2003), 034101
[12] Almutairi, N., Saber, S., Existence of Chaos and the Approximate Solution of the Lorenz-Lu-Chen System with the Caputo Fractional Operator, *AIP Advances*, *14* (2024), 1, 015112
[13] Hartley, T. T., *et al*., Chaos in a Fractional Order Chua's system, *IEEE Trans. Circ. Syst. I*, *42* (1995), 8, pp. 485-490
[14] Li, C. G., Chen, G., Chaos and Hyperchaos in the Fractional-Order Rossler Equations, *Physica A*., *341* (2004), Oct., pp. 55-61
[15] Li, C. P., Peng, G. J., Chaos in Chen's System with a Fractional Order, *Chaos Soliton. Fract.*, *22* (2004), 2, pp. 443-450
[16] Wang, X. Y., Wang, M. J., Dynamic Analysis of the Fractional-Order Liu System and Its Synchronization, *Chaos*, *17* (2007), 033106
[17] Ahmed, K. I. A., *et al*., Analytical Solutions for a Class of Variable-Order Fractional Liu System under Time-Dependent Variable Coefficients, *Results in Physics*, *56* (2024), 107311
[18] Saber, S., Control of Chaos in the Burke-Shaw System of Fractal-Fractional Order in the Sense of Caputo-Fabrizio, *Journal of Applied Mathematics and Computational Mechanics*, *23* (2024), 1, pp. 83-96
[19] Almutairi, N., Saber, S., On Chaos Control of Non-Linear Fractional Newton-Leipnik System Via Fractional Caputo-Fabrizio Derivatives, *Sci. Rep*., *13* (2023), 22726
[20] Salem M. A., *et al*., Modelling COVID-19 Spread and Non-Pharmaceutical Interventions in South Africa: A Stochastic Approach, *Scientific African*, *24* (2024), e02155

[21] Ahmed, K. I. A., *et al*., Different Strategies for Diabetes by Mathematical Modelling: Applications of Fractal-Fractional Derivatives in the Sense of Atangana-Baleanu, *Results Phys*., *52* (2023), 106892
[22] Salem M. A., *et al.*, Numerical Simulation of an Influenza Epidemic, *Prediction with Fractional SEIR and the ARIMA Model*, *18* (2024), 1, pp. 1-12
[23] Rania *et al.*, Mathematical Modelling and Stability Analysis of the Novel Fractional Model in the Caputo Derivative Operator, *A Case Study Saadeh, Heliyon*, *10* (2024), 5, e26611
[24] Almutairi, N., Saber, S., Chaos Control and Numerical Solution of Time-Varying Fractional Newton-Leipnik System Using Fractional Atangana-Baleanu Derivatives, *AIMS Mathematics*, *8* (2023), 11, pp. 25863-25887
[25] Caputo, M., Fabrizio, M., A New Definition of Fractional Derivative without Singular Kernel, *Prog. Fract. Differ. Appl*., *1* (2015), 2, pp. 73-85
[26] Almutairi, N., Saber, S., Application of a Time-Fractal Fractional Derivative with a Power-Law Kernel to the Burke-Shaw System Based on Newton's Interpolation Polynomials, *MethodsX*, *12* (2024), 102510

Paper submitted: July 19, 2024
Paper revised: October 20, 2024
Paper accepted: November 13, 2024