NYSTROM METHODS AND COMBINATION FOR SOLVING THE FIRST-KIND BOUNDARY INTEGRAL EQUATION

by

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Original scientific paper
https://doi.org/10.2298/TSCI2404573L

Based on the single-layer potential theory, the Laplace equation can be converted into the problem of the first-kind boundary integral equation (BIE^{1st}). The kernel of BIE^{1st} is characterized by the logarithmic singularity. In this paper, we investigate the Nystrom method for solving the BIE^{1st}. The numerical solutions possess high accuracy orders O(h^3) and the combination of two kinds of Nystrom solutions has the same accuracy as the result with double grid. Furthermore, by the double power transformation, the proposed method can be used to deal with the problem on the non-smooth boundary and has the higher accuracy. The efficiency is illustrated by some examples.

Key words: *Nystrom method, BIE^{1st}, combination method*

Introduction

In this paper, we consider the Laplace equation:

$$\begin{aligned} \Delta u &= 0, \quad \text{in} \ \ \Omega \\ u &= g, \quad \text{on} \ \ \Gamma \end{aligned} \tag{1}$$

where Ω is the domain in R^2 with the boundary Γ.

As we know, eq. (1) can be transformed to the BIE by the single-layer potential theory, which can preserve symmetry properties. Now we consider the boundary integral equation:

$$g(y) = -\frac{1}{2\pi}\int_{\Gamma} p(x)\ln|x-y|\,\mathrm{d}s(x), \quad \forall y \in \Gamma \tag{2}$$

When the solution $p(x)$ from eq. (2) for the given known function g is obtained, then the BIE can be used to compute the values of the unknown function u in the interior:

$$u(y) = -\frac{1}{2\pi}\int_{\Gamma} p(x)\ln|x-y|\,\mathrm{d}s(x), \quad \forall y \in \frac{\Omega}{\Gamma} \tag{3}$$

The BIE^{1st} is usually known as an ill posed problem. The ill-posedness of the problem will be stronger when the kernel of BIE^{1st} becomes smoother. Because the fundamental solution of the Laplace equation in R^2 have the logarithmic singularity, the corresponding BIE^{1st} has the weak singularity and the condition number of the discrete matrix has the order $O(h^{-1})$. Sloan *et al*. [1] studied the well-posedness of BIE^{1st}, and proved that when the radius of gyration $c_\Gamma \neq 1$, eq. (2) has unique solution and eq. (3) is well-posed.

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Lemma 1. [1] When $c_\Gamma \in (0, 1)$

$$-\int_\Gamma\int_\Gamma \ln|s-t|\, y(t)y(s)\mathrm{d}l_t\mathrm{d}l_s \geq 0$$

and the inequality is equal if and only if $y = 0$.

Also, a Galerkin method to solution of BIE[1st] was proposed in [1]. Later, Yan [2] gave the collocation method to deal with the problem on polygonal domains, and showed that the collocation method has the super convergence estimate in interior points.

However, two aforementioned methods have some disadvantages. That is, the work of calculating discrete matrix is too large and the order of accuracy is lower. To overcome these shortcomings, Huang *et al.* [3] and Huang and Lu [4] introduced the Nystrom method, namely, the mechanical quadrature method, which can be co-operated with the splitting extrapolations and suitable for solving boundary integral equations. On the other hand, the splitting extrapolation method based on the multivariate asymptotic expansion of the error is a much efficient technique to solve large problems in parallel. Recently, the Nystrom method coupled with splitting extrapolation has been applied to many multidimensional problems and other more complicated problems, such as non-linear elasticity [5], axisymmetric Poisson's equation [6].

As an accelerating convergence technique, the combination method introduced firstly by Lin and Lu [7] also attracted more and more interest. Similar to the extrapolation method, the combination method combines several approximations to obtain an approximation of higher accuracy. In particular, the combination method is an efficient parallel method to obtain an approximation of high accuracy with a high degree of parallelism, since the loads of computing the approximations in this method are close to each other. This method has been used to solve the Volterra integral equations with weakly singular kernels [8]. In this paper, we propose the Nystrom method, *i.e.*, the mechanical quadrature method for solving the first kind boundary integral equation derived from the Laplace eq. (1), and used the combination method and the double power transformation method to deal with the problem which has the non-smooth boundary.

Nystrom method

Let Γ be a smooth simply closed curve described:

$$x(t) = \left(x_1(t), x_2(t)\right) : t \in [0, 2\pi]$$

Assume that there exist two positive constants μ_1 and μ_2, such that:

$$0 < \mu_1 \leq |x'(t)| \leq \mu_2, \ \forall t \in [0, 2\pi]$$

where

$$x_1(t), x_2(t) \in \tilde{C}^l[0, 2\pi] \text{ and}$$

$$\tilde{C}^l[0, 2\pi] = \left\{u(t) \in C[0, 2\pi] : \partial_t^\alpha u \text{ is periodic in } [0, 2\pi], \text{for } \alpha \leq l\right\}$$

For convenience, we denote:

$$(Kp)(y) = -\frac{1}{2\pi}\int_\Gamma p(x)\ln|x-y|\,\mathrm{d}s(x)$$

Then we can rewrite eq. (2):

$$g = Kp$$

By the parameter transformation, we have:

$$g(x(t)) = -\frac{1}{2\pi}\int_0^{2\pi} p(x(s))\ln\left|x(s)-x(t)\right|\left|x'(s)\right|\mathrm{d}s$$

Denote $\bar{g} = g(x(t))$, $\bar{p}(t) = p(x(t))$:

$$(\bar{K}\bar{p})(t) = -\frac{1}{2\pi}\int_0^{2\pi} p(x(s))\ln\left|x(s)-x(t)\right|\left|x'(s)\right|\mathrm{d}s \tag{4}$$

in which the kernel $G(s) = \ln|t-s|p(s)$. Israeli [9] proved that:

$$Q_n(G) = h\sum_{j=1,t\neq s_j}^{n} G(s_j) + \ln\left(\frac{h}{2\pi}\right)g(t)h \tag{5}$$

possess the high accuracy approximation of the integral

$$I(G) = \int_0^{2\pi} G(s)\mathrm{d}s$$

Also, the error $E_n(G)$ has the asymptotic expanding when $g \in \tilde{C}^{2N}[0, 2\pi]$:

$$E_n(G) = Q_n(G) - I(G) = 2\sum_{j=1}^{N-1}\frac{\zeta'(-2j)}{(2j)!}g^{(2j)}(t)h^{2j+1} + o(h^{2N}) \tag{6}$$

where $s_j = jh$, $h = 2\pi/n$. The $\zeta(t)$ and $\zeta'(t)$ are Riemann-Zeta function and its derivatives, respectively. The mechanical approximation operator can be defined as $t_j = s_j$:

$$\bar{g}(t) = \bar{K}\bar{p}(t) = -\frac{1}{2\pi}h\sum_{j=1,t\neq s_j}^{n}\ln\left|x(s_j)-x(t)\right|p(x(s_j))\left|x'(s_j)\right| - -\frac{1}{2\pi}\ln\left(\frac{h}{2\pi}\left|x'(t)\right|\right)p(x(t))\left|x'(t)\right|h$$

$$\bar{g}(t_i) = -\frac{1}{2\pi}h\sum_{j=1,t\neq s_j}^{n}\ln\left|x(s_j)-x(t_i)\right|p(x(s_j))\left|x'(s_j)\right| - -\frac{1}{2\pi}\ln\left(\frac{h}{2\pi}\left|x'(t_i)\right|\frac{h}{2\pi}h\right)p\left(x(t_i)\right)\left|x'(t_i)\right|h$$

for $i = 1,..., n$. The corresponding algebraic equation can be obtained:

$$\bar{g} = K_n\bar{p} \tag{7}$$

In [3, 4], authors give

Lemma 2. For the integral operator K

$$(Kp)(t) = -\frac{1}{2\pi}\int_{\Gamma} p(s)\ln\left|s-t\right|\mathrm{d}l_s$$

we have the approximate operators K_n

$$K_n(p) = -\frac{1}{2\pi}h\sum_{j=1,t=s_j}^{n} p(s_j)\ln\left|s_j - t\right| - \frac{1}{2\pi}\ln\left(\frac{h}{2\pi}\right)p(t)h$$

assembled compact converge to K.

By *Lemma 1* and *Lemma 2*, we can directly get the convergence of approximate solution. The corollary is obvious.

Corollary 3. Assume that there exists a big integer N, such that for all $n > N$, the operator K_n^{-1} exists and has a uniform bound. When

$$\left\|K_n^{-1}(K_n - K)\right\| < 1 \text{ we have } \left\|K^{-1}\right\| \le \frac{\left\|K_n^{-1}\right\|}{1-\left\|K_n^{-1}(K_n - K)\right\|}$$

and the approximate solution has a posteriori estimate

$$\left\|p - p_n\right\| \le \frac{\left\|K_n^{-1}\right\|(\left\|K_n - K\right\|\left\|p_n\right\| + \left\|g - g_n\right\|)}{1-\left\|K_n^{-1}(K_n - K)\right\|}$$

The proof can be obtained by the assembled compactness theory and some calculus. We omit details.

The combination method

Now, inspired by the ideas in [2], we present the combination method to obtain the new solution which can achieve the higher accuracy. First, we construct two approximation operators:

$$\begin{aligned}\bar{K}_n^{(i)} p(t) = -\frac{1}{2\pi} h \sum_{j=1, t\ne s_j + \frac{h}{m}i}^{n} \ln\left| x\left(s_j + \frac{h}{m}i\right) - x(t)\right| p\left(x\left(s_j + \frac{h}{m}i\right)\right)\left| x'\left(s_j + \frac{h}{m}i\right)\right| - \\ -\frac{1}{2\pi} h \ln\left(\left|x'(t)\right|\frac{h}{2\pi}\right) p\big(x(t)\big)\left|x'(t)\right|, \quad i = 1,2\end{aligned} \tag{8}$$

and two approximation equations:

$$\bar{K}_n^{(i)} p_n^{(i)} = \bar{g}, \quad i = 1,2 \tag{9}$$

respectively. Then, we solve eq. (9) in parallel, and get the average values of solutions:

$$p_n^{comb} = \frac{1}{2}(p_n^{(1)} + p_n^{(2)}) \tag{10}$$

The numerical experiments show that p_n^{comb} and $p_n^{(i)}$ have the same order of accuracy (see the results in next section).

The problems in the non-smooth domains

If Γ is not a smooth boundary, the kernel of BIE$^{1\text{st}}$ has the singularity in the corner. Huang and Lu [4] gave a double power transformation deal with the situation. This transformation is suitable for the combination method. Recently, the similar method also used by Fermo and Laurita to mixed boundary value problems in domains with corners [10] and the BIE$^{1\text{st}}$ on polygonal regions [11].

For any Γ_i, let $\Gamma = \cup_{i=1}^{m} \Gamma$, and variables change in interval (0, 1). Then we consider the integral:

$$I = \int_0^1 f(x)\mathrm{d}x$$

Using the transformation:

$$S(t) = \frac{1}{2}\left[\tanh\left(\frac{\pi}{2}\sinh(t)\right) + 1\right]$$

one can obtain that

$$I=\int_{-\infty}^{+\infty} f(S(t))S'(t)\mathrm{d}t \tag{11}$$

Choose the positive integer N and the step size h. The cutoff of eq. (11) can be written:

$$I_h^{(N)} = h\sum_{i=-N}^{i=N} f(S(ih))S'(ih) \tag{12}$$

The error is:

$$I - I_h^{(N)} \sim \exp(-CN)/\log N)$$

with the constant $C > 0$. Clearly, it has the exponent rate of convergence. Generally, the integral eq. (12) with the range of integration ($-hN$, kN) can be seen as the period function because the kernel tends to zero as the rate of double power.

Numerical experiments

In this section, we will present some examples to verify the efficiency of the proposed method. For convenience, Method I denotes the scheme for $i = 1$, and Method II denotes the scheme for $i = 2$ in eq. (9), respectively. The combination method refers to scheme (10). Also, let error = $|u(x_i, y_i) - u^h(x_i, y_i)|$ be the error of point (x_i, y_i), where $u(x_i, y_i)$ is the exact value of $u(x, y)$ at point (x_i, y_i) and $u^h(x_i, y_i)$ is the corresponding numerical solution, respectively.

Example 1. We consider eq. (1) with a circle boundary:

$$\Delta u = 0 \quad (x,y)\in\Omega$$
$$u = g \quad (x,y)\in\Gamma$$

where Ω is the circle $(x/0.5)^2 + (y - 0.5)^2 < 1$, and the analytic solution is $u = \ln[(x - 10)^2 + y^2]$. Consider two points $(x, y) = (0.3, 0.1)$ and $(x, y) = (0.2, 03)$ in Ω. Then the Nystrom solutions with error results are shown in tab. 1.

Table 1. Errors of u in Ω at $(x, y) = (0.3, 0.1)$ and $(x, y) = (0.2, 0.3)$

N	3	6	12
u (0.3,0.1) = 4.5444	Error	Error	Error
Method I	0.2702	0.0257	0.0019
Method II	0.3116	0.0232	0.0014
Combination	0.0207	0.0012	$2.2448 \cdot 10^{-4}$
u (0.2,0.3) = 4.5657	Error	Error	Error
Method I	0.6830	0.1505	0.0073
Method II	0.9900	0.1352	0.0083
Combination	0.1535	0.0076	$-5.0326 \cdot 10^{-4}$

As seen from tab. 1, Methods I and II have almost the same error, and as the number of nodes increases, the error also decreases accordingly, which can meet the accuracy requirements. However, with the same number of node, the errors of numerical solutions of the combination method are significantly decreasing, while errors also meet the same convergence order.

Example 2. Consider the boundary with corner for eq. (1). The Γ is described:

$$\Gamma=\left\{x_1(t)=\frac{2}{\sqrt{3}\sin\left(\dfrac{t}{2}\right)},\ x_2(t)=-\sin t,\ 0\le t\le 2\pi\right\}$$

with a corner at $t = 0$. The corner angle is $2\pi/3$. Then, eq. (1) with such a boundary has the analytic solution:

$$u(r,\theta) = r^{2/3}\cos\left(\frac{3\theta}{2}\right)$$

in polar co-ordinate. Choose points $(x, y) = (0.5, -0.7)$ and $(x, y) = (0.2, 0.2)$, respectively. We present the Nystrom solutions with error results in tab. 2.

Table 2. Errors of u in Ω at $(x, y) = (0.5, -0.7)$ and $(x, y) = (0.2, 0.2)$

N	12	24	48
u (0.5,-0.7) = 0.1153	Error	Error	Error
Method I	0.0104	0.0034	$4.1880 \cdot 10^{-4}$
Method II	0.0411	0.0042	$6.6109 \cdot 10^{-4}$
Combination method	0.0153	$3.9537 \cdot 10^{-4}$	$1.2115 \cdot 10^{-4}$
u (0.2,0.2)= 0.0576	Error	Error	Error
Method I	0.0153	0.0015	$2.8122 \cdot 10^{-5}$
Method II	0.0235	0.0022	$1.3878 \cdot 10^{-4}$
Combination method	0.0041	$3.7217 \cdot 10^{-4}$	$5.5331 \cdot 10^{-5}$

Example 3. Consider eq. (1) with the rectangle boundary. The Ω is a rectangle with the bottom left point $(x_1, y_1) = (1, 0)$ and the top right point $(x_2, y_2) = (2, 2)$. The analytic solution is also set to be $u = \ln [(x - 10)^2 + y^2]$. Choose points $(x, y) = (2, 1.5)$ and $(x, y) = (1.9, 1.5)$, respectively. Then the Nystrom solutions with error results are shown in tab. 3.

Table 3. Errors of$u$ in Ω at $(x, y) = (2, 1.5)$ and $(x, y) = (2, 1.5)$

N	12	24	48
u (2,1.5) = 4.1934	Error	Error	Error
Method I	0.0564	0.1413	0.0670
Method II	0.2502	0.1041	0.0725
Combination method	0.0969	0.0186	0.0028
u (1.9,1.5) = 4.2174	Error	Error	Error
Method I	0.0196	$3.2675 \cdot 10^{-4}$	$1.1595 \cdot 10^{-5}$
Method II	0.0208	$5.1067 \cdot 10^{-4}$	$1.2058 \cdot 10^{-5}$
Combination method	$5.9892 \cdot 10^{-4}$	$9.1958 \cdot 10^{-5}$	$1.1827 \cdot 10^{-5}$

From tab. 3, we can see that the accuracy of numerical solution at point $(x, y) = (2, 1.5)$ is not obvious for Methods I and II, because this point on the boundary has the singularity. Surprisingly, the performance of the combination method still is much better than two methods. Although the distance of the point $(x, y) = (1.9, 1.5)$ to the boundary is very close, it still exhibits high accuracy.

Conclusion

In this paper we use Nystrom methods and the combination method to solve the first kind boundary integral equation, which is important to many applications. High accuracy and high parallelism are two features of this method. Our numerical results also confirm the efficiency of proposed methods.

Acknowledgment

This work is supported by the Fundamental Research Funds for the Central Universities of Civil Aviation Flight University of China (No. PHD2023-047).

Nomenclature

u – displacement, [m]
x, y – co-ordinates, [m]

Greek symbols

Γ – boundary of Ω, [m]
Ω – domain, [m]

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Paper submitted: June 1, 2023
Paper revised: July 20, 2023
Paper accepted: January 14, 2024