A POWERFUL ANALYTICAL METHOD TO SOME NON-LINEAR WAVE EQUATIONS

by

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In the paper, the 1-D wave equation and non-linear diffusion equation are considered and the approximate solutions are obtained by using the variational iteration method. The obtained results show that the proposed method is efficient and simple.

Key words: *variational iteration method, 1-D wave equation, non-linear diffusion equation*

Introduction

Non-linear PDE are used to describe complex problems in the fields of machines, control process, ecological and economic system, chemical circulation system and epidemiology. However, it is very difficult for us to find their exact solutions. Recently, many effective and powerful methods have been presented to approximate the exact solution of non-linear PDE, such as Backlund transformation [1], Hirota's bilinear method [2], Darboux transformation [3] and the tanh method [4], and many others methods [5-12].

The variational iteration method (VIM) [13, 14] has been proved to be a useful mathematical tool for solving non-linear differential equations. In recent years, with the development of variational iteration method, this method has been successfully applied to many aspects such as: The Z-K equations [15], coupled Burger's equation [16] and various engineering problems [17], *etc.*

In the paper, 1-D wave equation and non-linear diffusion equation are studied by variational iteration method, and the approximate solutions of 1-D equation and non-linear diffusion equation are obtained successfully by using VIM. Finally, the effectiveness of VIM is proved by error comparison.

The variational iteration method

To illustrate this basic method, we consider the non-linear partial differential equation:

$$Lu(t,x)+Nu(t,x)=f(t,x) \tag{1}$$

where L is the linear operator, N – the non-linear operator, and $f(t, x)$ – the continuous function.

According to the VIM, we can construct an equation:

$$u_{n+1}(t,x)=u_n(t,x)+\int_0^t \lambda\left\{Lu_n(s,x)+N\tilde{u}_n(s,x)-f(s,x)\right\}\mathrm{d}s \tag{2}$$

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where λ is the general Lagrange multiplier which can be identified optimally via the variation theory, $\tilde{u}_n$ – the restricted variation, *i.e.*, $\delta\tilde{u}_n = 0$ and the subscript n denote the n^{th}-order approximation.

The 1-D wave equation

Considering the wave equation:

$$u_{tt} = c^2 u_{xx}, \ -\infty < x < \infty, \ t > 0 \tag{3}$$

with the initial conditions:

$$u(0,x) = f(x), \ u_t(0,x) = \mathrm{g}(x)$$

The correction functional of this equation can be written down:

$$u_{n+1} = u_n + \int_0^t \lambda \left\{ \frac{\partial^2 u_n}{\partial s^2} - c^2 \frac{\partial^2 \tilde{u}_n}{\partial x^2} \right\} \mathrm{d}s \tag{4}$$

Making the aforementioned correction functional stationary, and noticing that $\tilde{u}_n$ is considered as a restricted variation:

$$\delta u_{n+1} = \delta u_n + \delta \int_0^t \lambda \left\{ \frac{\partial^2 u_n}{\partial s^2} - c^2 \frac{\partial^2 u_n}{\partial x^2} \right\} \mathrm{d}s = \delta u_n + \lambda \delta u_n' \big|_{s=t} - \lambda' \delta u_n \big|_{s=t} + \int_0^t \lambda'' \delta u_n \mathrm{d}s = 0$$

generates the stationary conditions:

$$\delta u_n : 1 - \lambda_s(t,x) = 0, \ \delta u_{n_s} : \lambda(t,x) = 0, \ \delta u_n : \lambda_{ss}(s,x) = 0 \tag{5}$$

Therefore, the Lagrange multiplier is identified:

$$\lambda = s - t \tag{6}$$

Substituting this values of Lagrange multiplier $\lambda = s - t$ into the functional eq. (4) yields the iteration formula:

$$u_{n+1} = u_n + \int_0^t (s-t) \left\{ \frac{\partial^2 u_n}{\partial s^2} - c^2 \frac{\partial^2 u_n}{\partial x^2} \right\} \mathrm{d}s \tag{7}$$

Example 1. Consider the wave equation [18]:

$$u_{tt} = u_{xx}, \ -\infty < x < \infty, \ t > 0 \tag{8}$$

with the initial conditions:

$$u(0,x) = \sin(x), \ \ u_t(0,x) = \cos(x) \tag{9}$$

The correction functional of this equation is of the form:

$$u_{n+1} = u_n + \int_0^t (s-t) \left\{ \frac{\partial^2 u_n}{\partial s^2} - \frac{\partial^2 u_n}{\partial x^2} \right\} \mathrm{d}s \tag{10}$$

Taking into account the specified initial condition, we select $u_0 = \sin x + t\cos x$. Using this selection into eq. (10), we obtain the following successive approximations:

$$
\begin{aligned}
u_0 &= \sin x + t\cos x \\
u_1 &= \sin x\left(1-\frac{t^2}{2!}\right)+\cos x\left(t-\frac{t^3}{3!}\right) \\
u_2 &= \sin x\left(1-\frac{t^2}{2!}+\frac{t^4}{4!}\right)+\cos x\left(t-\frac{t^3}{3!}+\frac{t^5}{5!}\right) \\
&\vdots \\
u_n &= \sin x\left(1-\frac{t^2}{2!}+\frac{t^4}{4!}+(-1)^{n-1}\frac{t^{2n}}{2n!}\right)+\cos x\left(t-\frac{t^3}{3!}+\frac{t^5}{5!}+(-1)^{n-1}\frac{t^{2n+1}}{2n+1!}\right)
\end{aligned}
\tag{11}
$$

Letting $n \to \infty$, we get

$$u = \lim_{n\to\infty} u_n = \sin x \cos t + \cos x \sin t = \sin(x+t)$$

This gives the exact solution by $u = \sin(x + t)$.

Non-linear diffuse equations

Consider the non-linear diffuse equations:

$$u_t = (u^m u_x)_x \tag{12}$$

with initial condition $u(0, x) = f(x)$.

By VIM, the correction function of eq. (12) reads of the form:

$$u_{n+1} = u_n + \int_0^t \lambda\left[\frac{\partial u_n}{\partial s} - \frac{\partial}{\partial x}\left(u_n^m \frac{\partial u_n}{\partial x}\right)\right]\mathrm{d}s \tag{13}$$

This yields the stationary conditions:

$$1+\lambda\big|_{s=t} = 0, \quad -\lambda'\big|_{s=t} = 0$$

This in turn gives: $\lambda = -1$.

This value of the Lagrange multiplier is substituted into eq. (13) to obtain the iterative formula:

$$u_{n+1} = u_n - \int_0^t \left[\frac{\partial u_n}{\partial s} - \frac{\partial}{\partial x}\left(u_n^m \frac{\partial u_n}{\partial x}\right)\right]\mathrm{d}s \tag{14}$$

Example 2. Solve the PDE:

$$u_t = \frac{\partial}{\partial x}\left(u\frac{\partial u}{\partial x}\right) \tag{15}$$

with initial condition: $u(0, x) = x^2/c$, where $c > 0$, and c is arbitrary constants.

The exact solution of this equation is $u = x^2/(c - 6t)$.

The correction functional for this equation is written:

$$u_{n+1} = u_n - \int_0^t \left[\frac{\partial u_n}{\partial s} - \frac{\partial}{\partial x}\left(u_n^m \frac{\partial u_n}{\partial x}\right)\right]\mathrm{d}s \tag{16}$$

The given initial conditions taken into account, selecting $u_0 = x^2/c$ give the successive approximations:

$$u_0 = \frac{x^2}{c},\ u_1 = \frac{x^2}{c} + \frac{6x^2 t}{c^2},\ u_2 = \frac{x^2}{c} + \frac{6x^2 t}{c^2} + \frac{36x^2 t^2}{c^3} + \frac{72x^2 t^3}{c^4}$$
$$\vdots \tag{17}$$
$$u_n = \frac{x^2}{c} + \frac{6x^2 t}{c^2} + \frac{36x^2 t^2}{c^3} + \frac{216x^2 t^3}{c^4} + \frac{864x^2 t^4}{c^5} + \cdots$$

Therefore, we obtain the solution of eq. (15) as $u_n = x^2/(c - 6t)$ in a closed form, which is an exact solution. In addition, fig. 1 and tab. 1 are used to compare u_4 with the exact solution of eq. (15), demonstrating the convergence of the VIM.

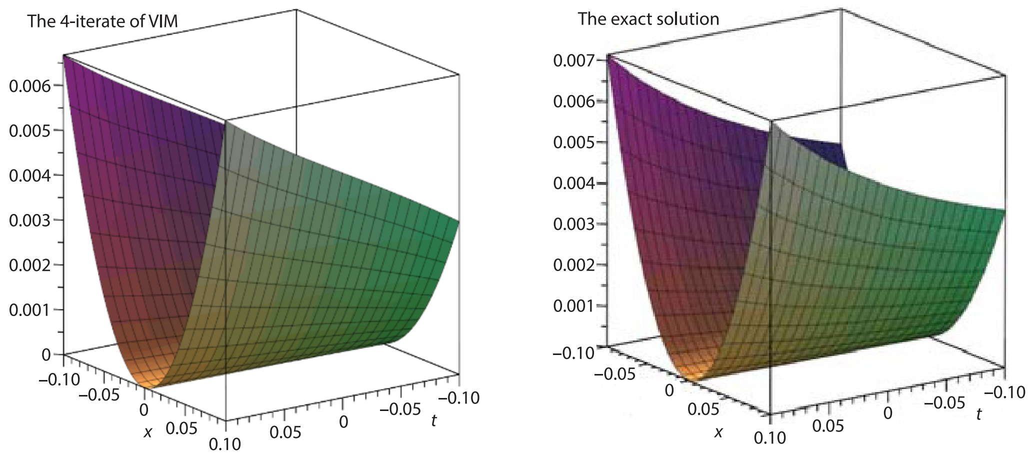

Figure 1. Comparison between the 4-iterate of VIM and the exact solution for *Example 2* where *c* = 2

Table 1. Absolute errors between the 4-iterate of VIM and the exact solution for *Example 2* where *c* = 2

t	x	u_4	u	Errors
0.03	0.02	$2.1816 \cdot 10^{-4}$	$2.1918 \cdot 10^{-4}$	$1.6213 \cdot 10^{-6}$
0.06	0.04	$9.4051 \cdot 10^{-4}$	$9.7561 \cdot 10^{-4}$	$2.6104 \cdot 10^{-5}$
0.09	0.06	$2.3310 \cdot 10^{-3}$	$2.4658 \cdot 10^{-3}$	$1.3476 \cdot 10^{-4}$
0.12	0.08	$4.5550 \cdot 10^{-3}$	$5.0000 \cdot 10^{-3}$	$4.4415 \cdot 10^{-4}$

It can be seen that the error between the fourth iteration solution of VIM method and the exact solution is relatively small. This shows that the approximate solution is efficient.

Conclusion

In this paper, the iterative models of 1-D wave equation, non-linear diffusion equation are developed by the VIM. The results show that the approximate solutions of the aforementioned equations can be easily obtained by using the VIM. By analyzing the approximate solution, it can be concluded that the VIM is an efficient and accurate method.

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